On the Theory of Screws in Elliptic Space (Third Note).

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[Read June 10th, 1886.]

This Note is a continuation of two previous Notes of mine on the same subject which have appeared in these *Proceedings* (Vol. xiv., p. 83, and Vol. xvi., p. 15). In the last two sections of the second Note, I gave some formulæ relating to infinitesimal motions, applicable to the three kinds of uniform space of three dimensions. In the present Note I consider finite motions.

Starting with Prof. Cayley's expression for an orthogonal matrix in terms of a skew matrix, I show how this is connected with the screw defining the motion. Then, transforming the matrix to its canonical form, I obtain formulæ relating to the distances and angles, through which points, lines, and planes are moved by a given screw. In this part of the paper I make use of Grassmann's methods, and of the theory of matrices, as presented in my paper on the subject in these *Proceedings* (Vol. xvi., p. 63). In the remaining part of the paper I make use of biquaternions, referring to my paper in the *American Journal of Mathematics* (Vol. vii., Pt. 4), and obtain the following theorem:—Any screw motion is represented by a biquaternion, in such wise that, if Q is a biquaternion, and ρ any bivector, $Q\rho Q^{-1}$ is the bivector into which ρ is transformed by the motion defined by Q; and, if Q is brought to the form

$$1+\alpha+\omega\,(S\alpha\alpha'+\alpha'),$$

the motion defined by Q is the motion defined by the screw $\alpha+\omega\alpha'$.

I.

In any kind of space a *motion* is a linear transformation which leaves the absolute unaltered. If we refer the absolute to a self-conjugate tetrahedron, and reduce its equation to the form

$$x^2+y^2+z^2+w^2=0,$$

we see that a motion is a linear transformation by which $x^2+y^2+z^2+w^2$ is unaltered,* that is to say, it is an orthogonal transformation, and

* To a scalar factor *près*; but we can always suppose this factor to be unity.

the condition that a matrix ϕ may be orthogonal is $\phi\phi' = 1$, if ϕ' is the conjugate of ϕ.

Now, let ϕ be any orthogonal matrix, and let

$$\psi = \frac{1-\phi}{1+\phi}.$$

Then

$$\psi' = \frac{1-\phi'}{1+\phi'}$$
$$= \frac{1-\phi^{-1}}{1+\phi^{-1}}$$
$$= -\frac{1-\phi}{1+\phi}$$
$$= -\psi.$$

That is, ψ is a skew symmetric matrix, and therefore any orthogonal matrix may be written in the form

$$\phi = \frac{1-\psi}{1+\psi},$$

where ψ is a skew symmetric matrix. For actual calculation, it is more convenient to write $\phi = \frac{2}{1+\psi} - 1.$

Since ϕ is a function of ψ, we see that the latent points of ϕ are the same as those of ψ, and that, if λ is a latent root of ψ, the corresponding latent root of ϕ will be $\frac{1-\lambda}{1+\lambda}$

I proceed to consider the latent points and roots of a skew symmetric matrix, ψ. Let e_1, e_2 ... be the latent points, λ_1, λ_2 ... the latent roots. I shall assume that ψ is not a derogatory matrix, so that λ_1, &c. are all unequal. We have

$$Se_i\,\psi e_k = \lambda_k\,Se_i e_k.$$

But

$$Se_i\,\psi e_k = Se_k\,\psi' e_i$$
$$= -Se_k\,\psi e_i,$$

because ψ is a skew symmetric matrix, $= -\lambda_i\,Se_i\,e_k$.

Therefore, either $\lambda_k + \lambda_i = 0$ or $Se_i e_k = 0$. Moreover, by taking $i=k$, we see that either $\lambda_i = 0$ or $r^2 e_i = 0$. If, then, we attend only to those latent points for which the corresponding latent root does not vanish, we see that all these points are on the absolute, that they group themselves in pairs for which the sum of the latent roots is zero, and that any two points not forming a pair are on a generator of the absolute.*

I shall now confine myself to matrices of the fourth order. Let

$$\psi = \begin{pmatrix} 0 & h & -g & a \\ -h & 0 & f & b \\ g & -f & 0 & c \\ -a & -b & -c & 0 \end{pmatrix}.$$

Then we find that the latent roots are given by

$$\lambda^4 - \lambda^2 (a^2+b^2+c^2+f^2+g^2+h^2) + (af+bg+ch)^2 = 0.$$

Now, write α^2 for $a^2+b^2+c^2+f^2+g^2+h^2$, and let

$$\sin\phi = \frac{2(af+bg+ch)}{\alpha^2}$$

then we have
$$\lambda^4 - \lambda^2\alpha^2 + \frac{\alpha^4\sin^2\phi}{4} = 0,$$

and, writing λ^2, μ^2 for the roots of this equation, we get

$$\lambda^2 = -\alpha^2 \sin^2\frac{\phi}{2},$$

$$\mu^2 = -\alpha^2 \cos^2\frac{\phi}{2}.$$

We see that the latent roots of ψ are

$$\pm\alpha i \sin\frac{\phi}{2}, \quad \pm\alpha i \cos\frac{\phi}{2},$$

and that, therefore, the latent roots of

$$\frac{1-\psi}{1+\psi}$$

are
$$\frac{1+\alpha i\sin\frac{\phi}{2}}{1-\alpha i\sin\frac{\phi}{2}}, \quad \frac{1+\alpha i\cos\frac{\phi}{2}}{1-\alpha i\cos\frac{\phi}{2}}, \quad \frac{1-\alpha i\sin\frac{\phi}{2}}{1+\alpha i\sin\frac{\phi}{2}}, \quad \frac{1-\alpha i\cos\frac{\phi}{2}}{1+\alpha i\cos\frac{\phi}{2}}.$$

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* For if α, β are any two points, $T^2(\lambda\alpha+\mu\beta) = \lambda^2T^2\alpha + 2\lambda\mu S\alpha\beta + \mu^2T^2\beta$, and therefore vanishes for all values of λ, μ if $T^2\alpha = S\alpha\beta = T^2\beta = 0$; and therefore, if these three conditions are fulfilled, the line $[\alpha\beta]$ is a generator of the absolute.

It will be worth while to give the actual value of

$$\Phi = \frac{1-\psi}{1+\psi}.$$

If $\beta = af+bg+ch$, $\Delta = 1+a^2+\beta^2$, we find (*Crelle*, t. 32, 1846)

$$\Delta\Phi = \begin{pmatrix} 1+f^2-g^2-h^2+b^2+c^2-a^2-\beta^2, & 2(-h-ab+fg-c\beta), & 2(g-ca+hf+b\beta), & 2(-a+bh-cg-f\beta) \\ 2(h-ab+fg+c\beta), & 1+g^2-h^2-f^2+c^2+a^2-b^2-\beta^2, & 2(-f-bc+gh-a\beta), & 2(-b+cf-ah-g\beta) \\ 2(-g-ca+hf-b\beta), & 2(f-bc+gh+a\beta), & 1+h^2-f^2-g^2-c^2+a^2+b^2-\beta^2, & 2(-c+ag-bf-h\beta) \\ 2(a+bh-cg+f\beta), & 2(b+cf-ah+g\beta), & 2(c+ag-bf+h\beta), & 1-a^2-b^2-c^2+f^2+g^2+h^2-\beta^2 \end{pmatrix}$$

We have also
$$\Delta = 1+a^2+\frac{a^4\sin^2\phi}{4}$$
$$= \left(1+a^2\cos^2\frac{\phi}{2}\right)\left(1+a^2\sin^2\frac{\phi}{2}\right).$$

II.

Let $a = ae_2e_3+be_3e_1+ce_1e_2+fe_1e_4+ge_2e_4+he_3e_4$ be any screw, $x = xe_1+ye_2+ze_3+we_4$ any point.* Then, if $xa = l$, we have

$$(lmnp) = (\psi \between xyzw),\dagger$$

where ψ is the same matrix as in (1). We see, therefore, that to a given screw a appertains a certain skew symmetric matrix, and it follows from what was proved in (1) that the latent roots of the matrix are

$$\pm iTa\sin\frac{\phi}{2},\quad \pm iTa\cos\frac{\phi}{2},$$

where Ta, ϕ are the tensor and the pitch of a respectively.

It appears, therefore, that the connexion between a motion and a screw is as follows: the motion is defined by an orthogonal matrix,

* The general formula for any kind of space can be got from this by writing ea, eb, ec, for a, b, c in the value of Δ, and in the three first columns of the matrix.

† It will be convenient to denote screws, points, and planes by their first coordinates: thus, "the plane l" means the plane $(lmnp)$.

to this corresponds a skew symmetric matrix, and this skew matrix appertains to a certain screw. And conversely, given a screw, we can find the skew symmetric matrix appertaining to it, and then an orthogonal matrix corresponding to this skew symmetric matrix, and this orthogonal matrix defines a motion given by the screw.

We have now to reduce these matrices to their canonical forms, and to see what our metric functions become.

III.

The latent roots of the orthogonal matrix Φ have been given in (1), and we see that they are of the form $\alpha, \beta, \alpha^{-1}, \beta^{-1}$. Call the latent points $e_1 e_2 e_3 e_4$, as before; then $e_1 e_2$, $e_1 e_3$, $e_2 e_3$, $e_3 e_4$ are generators of the absolute, and we see that, since the edges of the tetrahedron of reference are generators, the equation of the absolute must be of the form $xz - \lambda yw = 0$. It will be necessary to work out the three transformed equations of the absolute.

If e_1, e_2, e_3, e_4 are four points on the absolute, such that $e_1 e_2$, $e_1 e_4$, $e_2 e_3$, $e_3 e_4$ are generators, it can be verified without difficulty that the four points

$$\eta_1 = e_1 + e_3, \quad \eta_2 = e_2 + e_4, \quad \eta_3 = e_1 - e_3, \quad \eta_4 = e_1 - e_4$$

are the vertices of a self-conjugate tetrahedron.

Now, for *any* self-conjugate tetrahedron, the fundamental metric functions are $x^2+y^2+z^2+w^2$, $l^2+m^2+n^2+p^2$, $a^2+b^2+c^2+f^2+g^2+h^2$ for points, planes, and lines, respectively; and we can therefore suppose that the tetrahedron $\eta_1 \eta_2 \eta_3 \eta_4$ was originally taken as the tetrahedron of reference.

It will be convenient to begin by considering the transformation of line coordinates. Now, a condition that has to be satisfied is that $af+bg+ch$ shall be transformed into itself, and not into a multiple of itself.

This condition is not satisfied if we take the values of η_1, &c., just given; but it is satisfied if we take

$$\eta_1 = \frac{1}{\sqrt{2}}(e_1+e_3), \quad \eta_2 = \frac{i}{\sqrt{2}}(e_2+e_4), \quad \eta_3 = \frac{-i}{\sqrt{2}}(e_1-e_3),$$

$$\eta_4 = \frac{1}{\sqrt{2}}(e_2-e_4),$$

values which give

$$e_1 = \frac{\eta_1 + i\eta_3}{\sqrt{2}}, \quad e_2 = \frac{\eta_2 + i\eta_4}{\sqrt{-2}}, \quad e_3 = \frac{\eta_1 - i\eta_3}{\sqrt{2}}, \quad e_4 = \frac{\eta_2 - i\eta_4}{\sqrt{-2}}.$$

We find

$$2\,(e_2e_3,\ e_3e_1,\ e_1e_2,\ e_1e_4,\ e_2e_4,\ e_3e_4)$$

$$= \left(\begin{vmatrix} -1, & 0, & i, & -1, & 0, & i \\ 0, & -2i, & 0, & 0, & 0, & 0 \\ -1, & 0, & -i, & 1, & 0, & i \\ -1, & 0, & -i, & -1, & 0, & -i \\ 0, & 0, & 0, & 0, & 2i, & 0 \\ 1, & 0, & -i, & -1, & 0, & i \end{vmatrix} \between \eta_2\eta_3,\ \eta_3\eta_1,\ \eta_1\eta_2,\ \eta_1\eta_4,\ \eta_2\eta_4,\ \eta_3\eta_4\right).$$

Now, let the coordinates of any line be $(abcfgh)$ with respect to $e_1e_2e_3e_4$, and $(ABCFGH)$ with respect to $\eta_1\eta_2\eta_3\eta_4$; then we must have

$$ae_2e_3+be_3e_1+ce_1e_2+fe_1e_4+ge_2e_4+he_3e_4$$
$$= A\eta_2\eta_3+B\eta_3\eta_1+C\eta_1\eta_2+F\eta_1\eta_4+G\eta_2\eta_4+H\eta_3\eta_4.$$

Substituting for e_2e_3, &c., their values in terms of $\eta_2\eta_3$, &c., and comparing coefficients of $\eta_2\eta_3$, &c., we get

$$2A = -(a+f)+(h-c),$$
$$2B = -2ib,$$
$$2C = i\,\{(a-f)-(h+c)\},$$
$$2F = -(a+f)-(h-c),$$
$$2G = 2ig,$$
$$2H = i\,\{(a-f)+(h+c)\};$$

and therefore

$$4\,(A^2+B^2+C^2+F^2+G^2+H^2)$$
$$= (a+f)^2+(h-c)^2-4b^2-(a-f)^2-(h+c)^2$$
$$+(a+f)^2+(h-c)^2-4g^2-(a-f)^2-(h+c)^2$$
$$= 8\,(af-ch)-4\,(b^2+g^2),$$

or $$A^2+B^2+C^2+F^2+G^2+H^2 = 2\,(af-ch)-(b^2+g^2).$$

It can be easily verified that we have

$$AF+BG+CH = af+bg+ch.$$

If we consider the transformation of point coordinates, we shall find, if $(xyzw)$ are the coordinates of a point with respect to $e_1e_2e_3e_4$, and

$(XYZW)$ are its coordinates referred to $\eta_1\eta_2\eta_3\eta_4$,

$$X^2+Y^2+Z^2+W^2 = 2(xz-yw).$$

IV.

We have now to consider the canonical form of the screw itself on which our transformations depend. I use ω as in my second note on the theory of screws, to denote the operation of taking the conjugate with respect to the absolute; so that, whatever x may be, ωx is its conjugate with respect to the absolute. The points of reference e_i were determined as the latent points of a certain matrix, which appertained to a certain screw; call this screw a, then the equation determining the latent points is

$$e_i a = \lambda_i\, \omega e_i.$$

Now, let $ABCFGH$ be the coordinates of a referred to the tetrahedron $e_1e_2e_3e_4$. Then we have

$$e_1 a = Ae_1e_2e_3+Ge_1e_2e_4+He_1e_3e_4.$$

But
$$\omega e_1 = e_1e_2e_4.$$

Therefore
$$G=\lambda_1,$$

$$A=H=0.$$

Again,
$$e_2 a = Be_2e_3e_1+Fe_2e_1e_4,$$

$$\omega e_2 = -e_3e_2e_1 = e_2e_3e_1.$$

Therefore
$$F=0,\quad B=\lambda_2.$$

The equation $e_3 a = -\lambda_1\omega e_3$ gives $C=0$, and we get, as the canonical form,

$$\lambda_2 e_3 e_1+\lambda_1 e_2 e_4,$$

or
$$iTa\left(\pm\sin\frac{\phi}{2}\, e_3e_1 \pm\cos\frac{\phi}{2}\, e_2e_4\right),$$

where there is nothing so far to determine the signs; but if we remember that we must have

$$2\lambda_1\lambda_2 = T^2 a\sin\phi,$$

it is obvious that the two signs must be different, and we can take

$$a = iTa\left(\sin\frac{\phi}{2}\, e_3e_1-\cos\frac{\phi}{2}\, e_2e_4\right),*$$

* This transformation can also be effected by supposing the screw to be referred to $\eta_1\eta_2\eta_3\eta_4$, and then using the equations defining $e_1e_2e_3e_4$, and their expressions in terms of $\eta_1\eta_2\eta_3\eta_4$.

and we see at once that the axes of a are e_3e_1, e_2e_4. It should be noticed that we have now fixed the correspondence between the latent roots of Φ, the matrix defining the motion, and the points $e_1e_2e_3e_4$, viz., the order of the latent roots is

$$\frac{1+iTa\cos\frac{\phi}{2}}{1-iTa\cos\frac{\phi}{2}},\quad \frac{1-iTa\sin\frac{\phi}{2}}{1+iTa\sin\frac{\phi}{2}},\quad \frac{1-iTa\cos\frac{\phi}{2}}{1+iTa\cos\frac{\phi}{2}},\quad \frac{1+iTa\sin\frac{\phi}{2}}{1-iTa\sin\frac{\phi}{2}}.$$

I shall denote them as α, β, α^{-1}, β^{-1}.

V.

It will be worth while to give a few results connected with the transformations in the last two sections. I shall suppose that we take $e_1e_2e_3e_4$ as the tetrahedron of reference.

Let $a \equiv (abcfgh)$ be any screw; then we have

$$T^2a = 2(af-ch)-(b^2+g^2),$$

$$Saa' = af'+a'f-ch'-c'h-bb'-gg',$$

and therefore

$$\omega a \equiv (a,\ -g,\ -c,\ f,\ -b,\ -h),$$

$$2\xi a = (2a,\ b-g,\ 0,\ 2f,\ g-b,\ 0),$$

$$2\eta a = (0,\ b+g,\ 2c,\ 0,\ b+g,\ 2h),$$

$$\xi a = 0 \quad \text{if} \quad a = f = b-g = 0,$$

$$\eta a = 0 \quad \text{if} \quad c = h = b+g = 0.$$

Two screws a, a' are ξ-parallel if

$$\frac{a}{a'} = \frac{b-g}{b'-g'} = \frac{f}{f'};$$

they are η-parallel if

$$\frac{c}{c'} = \frac{b+g}{b'+g'} = \frac{h}{h'}.$$

The coordinates of a ξ-generator are

$$(0,\ \lambda,\ -1,\ 0,\ \lambda,\ \lambda^2).$$

The coordinates of an η-generator are

$$(1,\ \lambda,\ 0,\ \lambda^2,\ -\lambda,\ 0).$$

The equations of a ξ-generator are

$$z - \lambda y = 0,$$
$$w - \lambda x = 0.$$

The equations of an η-generator are

$$z - \lambda w = 0,$$
$$y - \lambda x = 0.$$

VI.

We have seen that the matrix Φ defining the motion can be reduced to the canonical form $\Phi = \dfrac{\alpha e_1, \beta e_2, \alpha^{-1} e_3, \beta^{-1} e_4}{e_1, \ e_2, \ e_3, \ e_4}$.

It follows that, if the coordinates of a point referred to the tetrahedron $e_1 e_2 e_3 e_4$ are $(xyzw)$, those of its new position referred to the same tetrahedron will be

$$\alpha x, \quad \beta y, \quad \alpha^{-1} z, \quad \beta^{-1} w.$$

Let P, P' be the two positions of the point. The distance between the points $(xyzw)$ and $(x'y'z'w')$ is given by

$$\cos PP' = \frac{xz' + x'z - yw' - y'w}{\sqrt{2(xz - yw)}\sqrt{2(x'z' - y'w')}},$$

and, therefore, for the two positions of P,

$$\cos PP' = (\alpha + \alpha^{-1}) \frac{xz}{2(xz - yw)} - (\beta + \beta^{-1}) \frac{yw}{2(xz - yw)}.$$

Now, $$Pe_3 e_1 = (xe_1 + ye_2 + ze_3 + we_4)\, e_3 e_1$$
$$= -ye_3 e_2 e_1 + we_3 e_1 e_4.$$

And, therefore, $$T^2 (Pe_3 e_1) = 2yw.*$$

Moreover, $T^2 (e_3 e_1) = -1$, and therefore, if θ is the distance of P from $e_3 e_1$, we have $$\sin^2\theta = -\frac{yw}{xz - yw},$$

and $$\cos^2\theta = \frac{xz}{xz - yw}.$$

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* If $le_2 e_3 e_4 + me_3 e_1 e_4 + ne_1 e_2 e_4 + pe_3 e_2 e_1 \equiv l$ is any plane, we have, in the present system of coordinates, $T^2 l = 2(lm - np)$.

We have, therefore,

$$\cos PP' = \cos D_1 \cos^2\theta + \cos D_2 \sin^2\theta,$$

where
$$\cos D_1 = \frac{\alpha+\alpha^{-1}}{2},$$

$$\cos D_2 = \frac{\beta+\beta^{-1}}{2},$$

and $D_1 D_2$ are obviously the distances through which points on e_3e_1, e_2e_4 are moved. As regards these points, we see at once that points on either of these axes remain on the same axis. I proceed to find the simplest expressions for D_1, D_2. We have

$$\cos D_1 = \frac{\alpha+\alpha^{-1}}{2}$$

$$= \tfrac{1}{2}\,\frac{1+iTa\cos\frac{\phi}{2}}{1-iTa\cos\frac{\phi}{2}} + \tfrac{1}{2}\,\frac{1-iTa\cos\frac{\phi}{2}}{1+iTa\cos\frac{\phi}{2}} = \frac{1-T^2a\cos^2\frac{\phi}{2}}{1+T^2a\cos^2\frac{\phi}{2}},$$

and therefore
$$\tan\frac{D_1}{2} = Ta\cos\frac{\phi}{2}.$$

In the same way, we find

$$\tan\frac{D_2}{2} = Ta\sin\frac{\phi}{2}.$$

We have
$$\frac{\tan\frac{D_2}{2}}{\tan\frac{D_1}{2}} = \tan\frac{\phi}{2},$$

and $\tan\frac{\phi}{2}$ is what Sir Robert Ball calls the pitch of the screw.

We found

$$\cos PP' = \cos D = \cos D_1 \cos^2\theta + \sin D_1 \sin^2\theta.$$

Now, if P is taken on a fixed line, we have

$$\sin^2\theta = \sin^2\theta_1 \cos^2\delta + \sin^2\theta_2 \sin^2\delta,$$

$$\cos^2\theta = \cos^2\theta_1 \cos^2\delta + \cos^2\theta_2 \sin^2\delta,$$

if θ_1, θ_2 are the shortest distances between e_2e_3 and the line, and δ is the distance of P from the point where one of the perpendiculars cuts the line.

We have, therefore,

$$\cos D = \cos D_1 (\sin^2\theta \cos^2\delta + \sin^2\theta_2 \sin^2\delta)$$
$$+ \cos D_2 (\cos^2\theta_1 \cos^2\delta + \cos^2\theta_2 \sin^2\delta)$$
$$= \cos\Delta_1 \cos^2\delta + \cos\Delta_2 \sin^2\delta,$$

where
$$\cos\Delta_1 = \cos D_1 \sin^2\theta_1 + \cos D_2 \cos^2\theta_1,$$
$$\cos\Delta_2 = \cos D_1 \sin^2\theta_2 + \cos D_2 \cos^2\theta_2,$$

and Δ_1, Δ_2 are obviously the displacements of the points where the two shortest distances cut the line.

VII.

In this section I investigate the application of biquaternions to the representation of motions in any kind of space. In this application we have to consider a motion as a linear transformation of line coordinates; that is to say, we consider space as made up of screws.

I use the notations explained in my paper on biquaternions in Vol. VII. of the *American Journal of Mathematics.*

If a linear tranformation of line coordinates is to represent a motion, it must transform a line into a line, and it must leave the angle between two lines unaltered. It is obvious that if these conditions are to be satisfied, $af+bg+ch$ and $e^2(a^2+b^2+c^2)+f^2+g^2+h^2$ must be unaltered by the linear transformation. But if we represent the line by a bivector ρ, the two expressions just written are $\Omega N\rho / 2$ and $UN\rho$ respectively, and we see that a linear transformation represents a line if, and only if, it leaves $N\rho$ unaltered. Now consider the bivector ϖ given by the equation

$$\varpi = Q\rho Q^{-1},$$

where Q is a biquaternion; ϖ is obviously got by operating on ρ with a certain linear transformation, and since all quaternion identities hold for biquaternions, we see that $N\varpi = N\rho$. It follows that the operator $Q(\ )\,Q^{-1}$, operating on a bivector, represents a motion. Moreover, we easily see that, if $\rho = VQ$, $\varpi = \rho$, and therefore the motion is specially related to the screw VQ.

In § 3 of the paper on biquaternions already referred to, I have shown how we can find the axis of a given biquaternion by dividing it by a certain biscalar. Now if we have $\varpi = Q\rho Q^{-1}$, it is obvious that we can divide Q by any biscalar without altering ϖ, and we can therefore suppose Q to be a special biquaternion. This remark simplifies the biquaternion formulæ, and is important as reducing the disposable constants in the value of ϖ from seven to six.

Now, if $Q = \Delta + \text{A}$ is any quaternion, and P a vector, we have

$$\Pi = Q\text{P}Q^{-1} = \frac{1}{NQ} Q\text{P}KQ$$
$$= \frac{\text{P}(\Delta^2 + \text{A}^2) + 2\Delta V\text{AP} - 2\text{A}S\text{AP}}{\Delta^2 - \text{A}^2}.$$

We have to use this formula, remembering that all the quantities involved are biquaternions. I write

$$\Pi = \varpi + \omega\varpi', \quad \text{P} = \rho + \omega\rho', \quad \Delta = \delta + \omega\delta', \quad \text{A} = a + \omega a'.$$

I shall also suppose that Q is a special biquaternion. This condition gives

$$\delta\delta' = Saa',$$
$$NQ = \delta^2 + e^2\delta'^2 - a^2 - e^2a'^2.$$

Our object is to determine δ, δ' so that the motion represented by $Q\,(\;)\,Q^{-1}$ may be the same as that represented by the screw $a + \omega a'$ according to the principles used in the first part of this paper. To do this I suppose our coordinates chosen in such wise that

$$a + \omega a' = gj + b\omega j.$$

Then we must have* $a = c = f = h = 0$ in the value of Φ given in (1). I shall only consider one edge of the tetrahedron of reference.

I take $\text{P} = i$; we get

$$NQ\,.\,\Pi = i\,(\delta^2 + \delta'^2 - g^2 - e^2b^2) - 2k\,(\delta g + e^2\delta' b) - 4bg\omega i - 2\,(\delta b + \delta' g)\,\omega k.$$

But, using the value of Φ given in (1), we get

$$\Delta\,.\,\Phi e_1 = (1 + e^2b^2)\,\{(1 - g^2)\,e_1 - 2ge_3\},$$
$$\Delta\,.\,\Phi e_4 = (1 + g^2)\,\{(1 - e^2b^2)\,e_4 - 2be_2\}.$$

And therefore, since

$$\Delta = (1 + g^2)(1 + e^2b^2),$$

$$\Delta\,.\,\Phi\,(e_1e_4) = -4bg\,e_2e_3 - 2b\,(1 - g^2)\,e_1e_2$$
$$+ (1 - g^2)(1 - e^2b^2)\,e_1e_4 - 2g\,(1 - e^2b^2)\,e_3e_4.$$

Now $$NQ = \delta^2 + \delta'^2 + g^2 + e^2b^2,$$

and $$e_2e_3 = \omega i, \quad e_1e_2 = \omega k, \quad e_1e_4 = i, \quad e_3e_4 = k.$$

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* The screw $(abcfgh)$ is here represented by $(fi + gj + hk) + \omega\,(ai + bj + ck)$ and not as in my paper on biquaternions.

We see that the two expressions agree if we take

$$\delta = 1,$$
$$\delta' = -bg,$$

and therefore we have the theorem.

If $Q = 1 + a + \omega\,(Saa' + a')$, then the operator $Q\,(\ )\,Q^{-1}$, operating on a bivector, represents the motion due to the screw $a + \omega a'$.

VIII.

In this and the following section I use the results of the last section to investigate formulæ for the angles and distances through which lines are moved, and for the composition of motions. The biquaternion Q will be supposed to be any special biquaternion, and will be taken as defining the motion. We have

$$\Pi\,(\Delta^2 - \mathrm{A}^2) = \mathrm{P}\,(\Delta^2 + \mathrm{A}^2) + 2\Delta V\mathrm{AP} - 2\mathrm{A}S\mathrm{AP}.$$

This gives

$$S\Pi\mathrm{P}\,.\,(\Delta^2 - \mathrm{A}^2) = \mathrm{P}^2\,(\Delta^2 + \mathrm{A}^2) - 2S^2\mathrm{AP}.$$

This equation gives us expressions for the distance and angle through which P is moved. In the special case in which P is a line cutting the axis of A at right angles, we have

$$\Omega\mathrm{P}^2 = 0,$$
$$S\mathrm{AP} = 0,$$

and we get
$$\cos\,(\mathrm{P}\Pi) = \frac{\delta^2 + e^2\delta'^2 + a^2 + e^2a'^2}{\delta^2 + e^2\delta'^2 - a^2 - e^2a'^2},$$

and therefore

$$\cos^2 \tfrac{1}{2}\,(\mathrm{P}\Pi) = \frac{\delta^2 + e^2\delta'^2}{\delta^2 + e^2\delta'^2 - a^2 - e^2a'^2}.$$

We have also
$$\mathrm{es}\,[\mathrm{P}\Pi] = \frac{4\delta\delta'}{\delta^2 + e^2\delta'^2 - a^2 - e^2a'^2}.$$

It will be convenient to introduce the following notations: Q being any special biquaternion, let

$$\cos^2(Q) = \frac{\mho\,(SQ)^2}{NQ} = \frac{T^2\,(SQ)}{NQ},\,*$$

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* I use the notation of my paper on biquaternions: $NQ = QKQ$, $T^2Q = \mho NQ$. It is convenient to write $\mho^2Q$, Ω^2Q for $(\mho Q)^2$, $(\Omega Q)^2$ respectively.

so that $$\sin^2(Q) = \frac{-\mho(VQ)^2}{NQ} = \frac{T^2VQ}{NQ},$$

and let $$\text{es}\{Q\} = \frac{4\delta\delta'}{NQ} = \frac{2\Omega(SQ)^2}{NQ}.$$

We see, then, that for a line cutting VQ at right angles, we have

$$(\text{P}\Pi) = 2(Q),$$
$$[\text{P}\Pi] = \{Q\}.$$

It is not hard to get expressions for $(\text{P}\Pi)$ and $[\text{P}\Pi]$ in the case of any screw P.

We have $$NQ\,.\,S\Pi\text{P} = \text{P}^2(\Delta^2+\text{A}^2) - 2S^2\text{AP},$$

which breaks up into

$$NQ\,.\,\mho S\Pi\text{P} = \mho\text{P}^2\,.\,\mho(\Delta^2+\text{A}^2) + e^2\Omega\text{P}^2\Omega(\Delta^2+\Delta^2)$$
$$-2\mho^2 S\text{AP} - 2e^2\Omega^2 S\text{AP},$$

$$NQ\,.\,\Omega S\Pi\text{P} = \mho\text{P}^2\,.\,\Omega(\Delta^2+\text{A}^2) + e^2\Omega\text{P}^2\mho(\Delta^2+\text{A}^2) - 4\Omega S\text{AP}\,.\,\mho S\text{AP},$$

and therefore, since $T\Pi = T\text{P}$, $S\text{AP} = SQ\text{P}$,

$$\cos(\Pi\text{P}) = \cos 2(Q) + \sin^2 e[\text{P}]\,\text{es}\,\{Q\} - 2\cos^2(Q\text{P}) - 2\sin^2 e[Q\text{P}],$$

$$\text{es}\,[\Pi\text{P}] = \text{es}\,\{Q\} + \sin^2 e[\text{P}]\cos 2(Q) - 4\cos(Q\text{P})\,\text{es}\,[Q\text{P}].$$

IX.

If we have $$\varpi = Q\rho Q^{-1},$$

we have $$Q'\varpi Q^{-1} = Q'Q\,.\,\rho\,(Q'Q)^{-1},$$

and therefore to compound two motions we multiply the corresponding biquaternions together.

It is important to prove that the product of two special biquaternions is a special biquaternion. A special biquaternion is defined by the equation

$$\Omega NQ = 0.$$

Now let Q, Q' be two biquaternions; we have

$$\Omega N(QQ') = \Omega NQNQ' = \Omega NQ\mho NQ' + \mho NQ\Omega NQ',$$

and therefore $\Omega N(QQ')$ vanishes if ΩNQ, $\Omega NQ'$ both vanish.

We have, if $Q'' = QQ'$,

$$\cos^2(Q'') = \frac{\mho\, SQQ' + e^2\Omega^2 SQQ'}{T^2QT^2Q'} = \cos^2(QKQ') + e^2\,\text{es}^2\,[QKQ'].$$

Moreover,

$$\text{es}\{Q''\} = 4\frac{\mho SQQ'\Omega SQQ'}{T^2QT^2Q'} = 4\cos(QKQ')\,\text{es}[QKQ'].$$

If Q, Q' are given in the standard form, so that we have

$$Q = 1+a+\omega(Saa'+a') = 1+\omega Saa'+\text{A},$$
$$Q' = 1+a_1+\omega(Sa_1a_1'+a_1') = 1+\omega Sa_1a_1'+\text{A}_1,$$

we have

$$\mho SQQ' = 1+e^2Saa'Sa_1a_1'+\mho S\text{AA}_1$$
$$= 1+\frac{e^2}{4}T^2\text{A}T^2\text{A}_1\,\text{es}[\text{A}]\,\text{es}[\text{A}_1]-T\text{A}T\text{A}_1\cos(\text{AA}_1),$$
$$\Omega SQQ' = Saa'+Sa_1a_1'+\Omega S\text{AA}_1$$
$$= -\frac{T^2\text{A}\,\text{es}[\text{A}]}{2}-\frac{T^2\text{A}_1'\,\text{es}[\text{A}_1]}{2}-T\text{A}T\text{A}_1\,\text{es}[\text{AA}_1],$$

and we can, if we choose, substitute these values in the values of $\cos^2(Q'')$, es $\{Q''\}$ given above.

As regards the connexion of the angles (Q), $\{Q'\}$ with the pitch and tensor of the screw defining the motion, I add the following formulæ:—

$$\tan^2(Q) = \frac{T^2\text{A}}{1+e^2S^2aa'}$$
$$= \frac{T^2\text{A}}{1+\frac{e^2}{4}T^4\text{A}\,\text{es}^2[\text{A}]},$$
$$\text{es}\{Q\} = \frac{4Saa_1}{1+e^2S^2aa_1+T^2\text{A}}$$
$$= -\frac{4T^2\text{A}\,\text{es}[\text{A}]}{(1+T^2\text{A}\,\text{ec}^2\tfrac{1}{2}[\text{A}])(1+e^2T^2\text{A}\,\text{es}^2\tfrac{1}{2}[\text{A}]}.$$

Lastly, I remark that, if $\mho SQQ'$ does not vanish, the screw resulting from two biquaternions is

$$\frac{VQQ'}{\mho SQQ'}.$$