

# ON CONFORMAL RATIONAL TRANSFORMATIONS IN A PLANE.

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From the standpoint of the theory of functions the theory of conformal rational transformations in a plane or of rational correspondences between complex variables is at present well established.

Less attention has been paid to the purely geometrical side of such transformations.

Probably the first important contribution to the subject was made by CAUCHY <sup>1)</sup>, who for analytic purposes studied the properties of the surface

$$\zeta = \text{mod} \left[ \frac{f(\xi + i\eta)}{g(\xi + i\eta)} \right],$$

where  $\xi$ ,  $\eta$ ,  $\zeta$  are Cartesian co-ordinates, and  $f$  and  $g$  polynomials in  $z = \xi + i\eta$ . He proved that at a  $k$ -fold point of the curve obtained for some definite constant value of  $\zeta$ , the  $k$  branches of the curve form  $2k$  equal angles, each angle measuring  $\frac{\pi}{k}$  radians. These curves and their orthogonal trajectories have been investigated by DARBOUX <sup>2)</sup>.

Great interest has also been attached to those algebraic curves which satisfy LAPLACE's differential equation and which are called *stelloids* by F. LUCAS <sup>3)</sup> and *potential curves* by E. KASNER <sup>4)</sup>.

<sup>1)</sup> *Œuvres complètes d'AUGUSTIN CAUCHY* (Paris, Gauthier-Villars), 1<sup>ère</sup> série, t. IV, p. 59. In the third line from the top  $\frac{dy}{dx} = 0$  must be replaced by  $\tan \theta$ . — See also: BRILL und NOETHER, *Die Entwicklung der Theorie der algebraischen Functionen in älterer und neuerer Zeit* [Jahresbericht der Deutschen Mathematiker-Vereinigung, Bd. III (1892-93), pp. I-XXXIII, 109-566], 187-189.

<sup>2)</sup> G. DARBOUX, *Sur une classe remarquable de courbes et de surfaces algébriques* (Paris, Hermann, 1896), pp. 66-83.

<sup>3)</sup> F. LUCAS, *Géométrie des polynômes* [Journal de l'École Polytechnique, XLVI<sup>e</sup> Cahier (1879), pp. 1-33].

<sup>4)</sup> E. KASNER, *On the Algebraic Potential Curves* [Bulletin of the American Mathematical Society, Vol. VII (1901), pp. 392-399]. — For further references see: G. LORIA, *Spezielle algebraische und transzendente ebene Kurven* (2. Aufl.) (Leipzig, Teubner), Vol. I (1910), pp. 439-453.

The following investigation is concerned chiefly with geometrical properties of rational transformations in the complex plane. It is naturally connected with the geometry of the polynomial as interpreted by F. LUCAS, loc. cit.

Designating by  $f(\zeta)$  and  $g(\zeta)$  two polynomials which are supposed to be prime and of degree  $m$  and  $n$  respectively, a general rational transformation may be written in the form

$$(1) \quad \zeta' = \frac{f(\zeta)}{g(\zeta)};$$

or separating in both,  $f$  and  $g$ , real from imaginary:

$$f(\zeta) = u + iv, \quad g(\zeta) = r + is,$$

there is

$$(2) \quad \zeta' = \frac{u + iv}{r + is}.$$

From this

$$(3) \quad \begin{cases} x' = \frac{ru + sv}{r^2 + s^2}, \\ y' = \frac{rv - su}{r^2 + s^2}. \end{cases}$$

The intersections of the curves  $u = 0$  and  $v = 0$  give the zeros, those of  $r = 0$  and  $s = 0$  the poles of the function. Thus it is apparent that the curves to be considered in connection with the given transformation are represented by the equations  $u = 0, v = 0, r = 0, s = 0, ru + sv = 0, rv - su = 0, u^2 + v^2 = 0, r^2 + s^2 = 0$ . I shall first establish the conception of «associate» point groups connected with a polynomial <sup>5)</sup>. It will be seen that by means of these a pencil of stelloids can be defined in a very simple manner.

## § 1.

### The intersections of $u = 0$ and $v = 0$ .

Let  $u + iv = a_0 \prod_{i=1}^m (\zeta - \zeta_i) = 0$  be an equation of the  $m^{\text{th}}$  degree in  $\zeta$  with the roots  $\zeta_i$  ( $i = 1, 2, 3, \dots, m$ ). This equation is of course satisfied by the simultaneous solutions of  $u = 0$  and  $v = 0$ . As both  $u$  and  $v$  are of the  $m^{\text{th}}$  degree also, there are  $m^2$  common solutions in the Cartesian plane  $xy$ , of which  $m$  are real and are represented by the  $m$  points  $\zeta_i$ , so that there are  $m^2 - m = m(m - 1)$  imaginary intersections between  $u = 0, v = 0$ .

<sup>5)</sup> The term *points associates* was first used by DARBOUX, loc. cit. <sup>2)</sup>, p. 61, and appears implicitly also in KASNER's investigations. See also: E. STUDY, *Vorlesungen über ausgewählte Gegenstände der Geometrie*, I. Heft: *Ebene analytische Kurven und zu ihnen gehörige Abbildungen* (Leipzig, Teubner, 1911), pp. 8-19.

A geometrical interpretation for these points is easily found as follows:

If  $p$  and  $iq$  designate the real and imaginary parts of the left side of the quadratic equation

$$a_0 z^2 + a_1 z + a_2 = 0,$$

or its identical

$$a_0(z - z_1)(z - z_2) = 0,$$

then the simultaneous solutions of  $p = 0$  and  $q = 0$  in a Cartesian plane are represented by the intersection of two orthogonal equilateral hyperbolas passing through the points  $z_1$  and  $z_2$  ( $A_1$  and  $A_2$ ) and having the midpoint of  $z_1$  and  $z_2$  as a common center.

The complete quadruple  $A_1 A_2 A_3 A_4$  determined by these intersections has the circular points  $I_1$  and  $I_2$  and the common center  $M$  of the hyperbolas as diagonal points. The imaginary pair  $A_3 A_4$ , or the associates of  $A_1 A_2$ , are also obtained as the finite intersections of the zero-circles around  $z_1$  and  $z_2$ .

Returning to the original equation of degree  $m$  in  $z$ , any two linear factors give rise to such a quadruple. Among the  $m$  linear factors  $\frac{m(m-1)}{2}$  different products of two may be formed, so that the theorem results:

*The  $m^2$  simultaneous solutions consist of the  $m$  real points represented by  $z_1, z_2, \dots, z_m$  and their associates.*

As is well known the equation  $u + \lambda v = 0$  with the variable parameter  $\lambda$  represents a pencil of stelloids. Any two consecutive asymptotes of a stelloid include an angle equal to  $\frac{\pi}{m}$  and there are  $m$  real asymptotes, assuming  $m$  distinct roots.

Hence the theorem:

*In a Cartesian plane a pencil of stelloids of the  $m^{\text{th}}$  order is fully determined by  $m$  arbitrary real points and their associates as base-points.*

## § 2.

**The curves  $u + v = 0$ ,  $r + s = 0$ ,  $ru + sv = 0$ ,  $rv - su = 0$   
and their intersections.**

Writing  $r + is = \prod_1^n (z - z_k)$ , there is obtained for  $r^2 + s^2 = 0$  the equivalent equation.

$$\prod_1^n [(x - x_k)^2 + (y - y_k)^2] = 0.$$

Geometrically it is represented by the  $n$  zero-circles around the  $z_k$ 's, so that each of the circular points is  $n$ -fold. The only real part of the curve consists of the  $n$  isolated points  $z_k$ .

The equations  $ru + sv = 0$  and  $rv - su = 0$  have  $(m + n)^2 = m^2 + 2mn + n^2$

common solutions. They are both satisfied by the  $m^2$  and  $n^2$  common solutions of  $u=0$ ,  $v=0$  and  $r=0$ ,  $s=0$ . The  $2mn$  remaining solutions satisfy simultaneously  $u^2 + v^2 = 0$ ,  $r^2 + s^2 = 0$ , and are consequently all imaginary.

Hence the theorem:

*If  $u$ ,  $v$  and  $r$ ,  $s$  are two pairs of irreducible polynomials in  $x$  and  $y$  with real coefficients of degree  $m$  and  $n$  respectively,*

$$ru + sv = 0$$

and

$$rv - su = 0$$

*have at most  $m^2 + n^2$  real solutions.*

This appears also directly from the identity

$$(ru + sv)^2 + (rv - su)^2 = (u^2 + v^2)(r^2 + s^2).$$

If furthermore  $u$ ,  $v$ ,  $r$ ,  $s$ , satisfy LAPLACE'S differential equation, then the two equations have  $(m+n)$  real and  $(m+n)(m+n-1)$  imaginary solutions. Of the latter  $2mn$  belong to the  $4mn$  imaginary solutions of  $u^2 + v^2 = 0$  and  $r^2 + s^2 = 0$ , which geometrically are obtained as the intersections of the zero-circles around the  $mz'_i$ 's and the  $nz'_k$ 's. Thus the circular points count for  $2mn$  common solutions of  $u^2 + v^2 = 0$  and  $r^2 + s^2 = 0$ .

The question is whether the circular points are multiple points of  $ru + sv = 0$  and  $rv - su = 0$ . For this purpose consider the product

$$(u + iv)(r - is) = (\alpha_0 + i\beta_0) \prod_1^m (x + iy - z_i) \prod_1^n (x - iy - z_k^*)$$

where  $z_k \cdot z_k^* = |z_k|^2$ . Assuming  $m > n$  we get

$$ru + sv + i(rv - su) = (\alpha_0 + i\beta_0) \prod_1^n (x + iy - z_i) \prod_1^n (x - iy - z_k^*) \prod_{n+1}^m (x + iy - z_i),$$

and after some reductions

$$(4) \quad \begin{cases} ru + sv = (x^2 + y^2)^n \psi(x, y) + \chi(x, y), \\ rv - su = (x^2 + y^2)^n \psi_1(x, y) + \chi_1(x, y), \end{cases}$$

where  $\psi$  and  $\psi_1$  are each of degree  $(m-n)$  and  $\chi$  and  $\chi_1$  each of degree  $(m+n-1)$ .

In case of  $n > m$ ,  $x^2 + y^2$  in (4) appears with the exponent  $m$ . For  $m = n$ ,  $\psi$  and  $\psi_1$  reduce to constants. Hence the theorem:

*Each of the curves*

$$ru + sv = 0,$$

$$rv - su = 0$$

*contains the circular points as multiple points. The degree of multiplicity is identical with that of the polynomial of lowest degree among  $u$ ,  $v$ ,  $r$ ,  $s$ .*

## § 3.

Correspondence between the  $\zeta'$ - and  $\zeta$ -plane.

According to the transformation (1),  $\zeta' = F(\zeta)$ , to every point in the  $\zeta$ -plane corresponds uniformly a point in the  $\zeta'$ -plane. Assuming the degree  $n$  in the denominator of  $F(\zeta)$ ,  $n \geq m$ , then, to every point in the  $\zeta'$ -plane correspond generally  $n$  distinct points in the  $\zeta$ -plane. These may be represented in a well known manner on a RIEMANN surface of  $n$  sheets. The interpretation in the corresponding Cartesian planes is as follows:

To any two distinct lines

$$\begin{aligned} a_1 x' + b_1 y' - a_1 x_1 - b_1 y_1 &= 0, \\ a_2 x' + b_2 y' - a_2 x_1 - b_2 y_1 &= 0 \end{aligned}$$

in the  $x'y'$ -plane correspond in the  $xy$ -plane the curves

$$(5) \quad \begin{cases} a_1(ru + sv) + b_1(rv - su) - (a_1 x_1 + b_1 y_1)(r^2 + s^2) = 0, \\ a_2(ru + sv) + b_2(rv - su) - (a_2 x_1 + b_2 y_1)(r^2 + s^2) = 0, \end{cases}$$

which intersect each other in a system of points, among which a certain number corresponds to the point  $(x_1, y_1)$  in the given transformation.

Now both equations (5) are of degree  $2n$ , so that they have  $4n^2$  common solutions. But each curve of (5) has each circular point as an  $n$ -fold point, so that of the  $4n^2$  points  $2n^2$  are consumed by the circular points, and as  $(r=0, s=0)$  satisfies (5),  $n^2$  of the remaining  $2n^2$  points lie in the intersection of  $r=0$  and  $s=0$ . Thus in the  $xy$ -plane there are  $n^2$  points corresponding to an arbitrary point  $(x_1, y_1)$  in the  $x'y'$ -plane. Of these  $n$  are real, corresponding to the  $n$  roots of  $f(\zeta) - g(\zeta)\zeta' = 0$ . The remaining  $n^2 - n$  imaginary points are the associates of the real roots. This follows also directly from (1). For a definite value of  $\zeta'$  (1) may be written in the form

$$\zeta' g(\zeta) - f(\zeta) = 0$$

which, when  $n \geq m$  is an equation of degree  $n$ , giving rise to the above  $n$  real points (roots) and their  $n^2 - n$  associates.

In the Cartesian plane, using (2), these points are obtained as the intersections of the curves, corresponding to the real and imaginary parts of (2):

$$(6) \quad \begin{cases} rx' - sy' - u = 0, \\ sx' + ry' - v = 0. \end{cases}$$

For every set of arbitrary values of  $x'$  and  $y'$  these equations determine a pencil of stelloids of the  $n^{\text{th}}$  order whose base-points are the same as those depending upon  $x', y'$  in (5). In case of  $m > n$  these stelloids are evidently of the  $m^{\text{th}}$  order.

When in (1)  $\zeta' = \zeta$ , and  $n \geq m$  an equation of degree  $(n+1)$

$$(7) \quad \Phi(\zeta) = 0$$

is obtained, whose  $(n+1)$  roots generally determine the  $(n+1)$  real double-points. Their  $n(n+1)$  associates are the imaginary double points and are obtained as the imaginary intersections of the curves  $U=0$  and  $V=0$  defined by  $\Phi(z) = U + iV$ .

It is also known that if  $z'_0$  is a  $k$ -fold branch-point of the function  $z$ , as defined by  $z' = f(z)$ , so that for  $z' = z'_0$ ,  $k$  values of  $z$  become equal to  $z_0$ , then to any curve passing singly through  $z'_0$  in the  $z'$ -plane corresponds a curve in the  $z$ -plane which passes  $k$  times through  $z_0$  and whose tangents at this point divide the full angle into  $2k$  equal parts.

For a branch point the otherwise orthogonal trajectories bisect the angles formed by the first.

As the discriminant of

$$(8) \quad z'g(z) - f(z) = 0$$

is of degree  $(2n-2)$  in  $z'$ , there are generally  $(2n-2)$  branch-points in the transformation.

#### § 4.

### On the transformations which leave $(n+1)$ arbitrary points and their associates invariant.

All conformal transformations of this kind may evidently be written in the form

$$(9) \quad z' = z - \frac{\Phi(z)}{\Psi(z)},$$

in which  $\Phi(z)$  and  $\Psi(z)$  are polynomials without a common factor and the roots of  $\Phi(z) = 0$  of degree  $(n+1)$  represent the given points. For  $\Psi(z)$  any polynomial prime to  $\Phi(z)$  may be chosen. There are therefore an unlimited number of such transformations possible, and it is evident that any rational transformation of a complex variable may be written in this form. I shall first consider the case, where in (1) the degrees of  $f(z)$  and  $g(z)$  are both equal to  $n$ , so that  $\Phi(z) = 0$  is equivalent with

$$(10) \quad zg(z) - f(z) = 0,$$

which explicitly, may be written in the form

$$(11) \quad z^{n+1} + a_0 z^n + a_1 z^{n-1} + \dots + a_n = 0.$$

In the form of (9), transformation (1) thus becomes

$$(12) \quad z' = z - \frac{z^{n+1} + a_0 z^n + a_1 z^{n-1} + \dots + a_n}{z^n + b_0 z^n + b_1 z^{n-1} + \dots + b_n},$$

or putting in evidence the real and imaginary parts of the rational fraction

$$(13) \quad z' = z - \frac{u + iv}{r + is}.$$

The points in the  $z$ -plane corresponding to an arbitrary value of  $z'$  are obtained as the  $n$  real intersections of the stelloids of the  $n^{\text{th}}$  order

$$(14) \quad rx - sy - rx' + sy' - u = 0,$$

$$(15) \quad sx + ry - sx' - ry' - v = 0,$$

as obtained from (13). Apparently of degree  $(n + 1)$ , these equations reduce to the  $n^{\text{th}}$  degree, as is easily seen from (12).

To any point  $(x', y')$  in the Cartesian plane correspond generally  $n^2$  points  $(x, y)$ , the intersections of the stelloids (14) and (15).  $u = 0$  and  $v = 0$  determine by their intersections  $(n + 1)^2$  double points of the transformation, consisting of  $(n + 1)$  real points and their associates. Designating by  $\zeta$  and  $\zeta'$  coordinates making polynomials in  $x, y$  and  $x', y'$  homogenous, the equations of the first polars of an arbitrary point  $(x', y', \zeta')$  with respect to  $u = 0$  and  $v = 0$  are

$$(16) \quad x' \frac{\partial u}{\partial x} + y' \frac{\partial u}{\partial y} + \zeta' \frac{\partial u}{\partial \zeta} = 0,$$

$$(17) \quad x' \frac{\partial v}{\partial x} + y' \frac{\partial v}{\partial y} + \zeta' \frac{\partial v}{\partial \zeta} = 0.$$

The first polars of the pencil  $u + \lambda v = 0$ , form also a pencil with the equation

$$(18) \quad x' \left( \frac{\partial u}{\partial x} + \lambda \frac{\partial v}{\partial x} \right) + y' \left( \lambda \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \right) + \zeta' \left( \frac{\partial u}{\partial \zeta} + \lambda \frac{\partial v}{\partial \zeta} \right) = 0,$$

and consequently pass through  $n^2$  base-points. The question is, under what conditions these points be identical with those corresponding to  $(x', y')$  in the transformation. This will be the case when the pencil of stelloids determined by (14) and (15) is identical with that of (18). For this purpose the pencil determined by (14) and (15) may be written in the form

$$(19) \quad \zeta x'(r + \lambda s) + \zeta y'(\lambda r - s) - \zeta' x(r + \lambda s) - \zeta' y(\lambda r - s) + \zeta'(u + \lambda v) = 0.$$

Writing

$$r = \frac{1}{n + 1} \cdot \frac{\partial u}{\partial x}, \quad s = \frac{1}{n + 1} \cdot \frac{\partial v}{\partial x},$$

(19) becomes

$$\begin{aligned} & \zeta x' \left( \frac{\partial u}{\partial x} + \lambda \frac{\partial v}{\partial x} \right) + \zeta y' \left( \lambda \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \right) - \zeta' x \frac{\partial u}{\partial x} - \lambda \zeta' x \frac{\partial v}{\partial x} \\ & - \lambda \zeta' y \frac{\partial v}{\partial y} - \zeta' y \frac{\partial u}{\partial y} + (n + 1)(u + \lambda v)\zeta' = 0, \end{aligned}$$

or

$$\begin{aligned} & \zeta x' \left( \frac{\partial u}{\partial x} + \lambda \frac{\partial v}{\partial x} \right) + \zeta y' \left( \lambda \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \right) - \zeta' \left( x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + \zeta \frac{\partial u}{\partial \zeta} \right) \\ & - \lambda \zeta' \left( x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} + \zeta \frac{\partial v}{\partial \zeta} \right) + \zeta \zeta' \frac{\partial u}{\partial \zeta} + \lambda \zeta \zeta' \frac{\partial v}{\partial \zeta} + (n + 1)(u + \lambda v)\zeta' = 0. \end{aligned}$$

This reduces to

$$(20) \quad x' \left( \frac{\partial u}{\partial x} + \lambda \frac{\partial v}{\partial x} \right) + y' \left( \lambda \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \right) + \zeta' \left( \frac{\partial u}{\partial \zeta} + \lambda \frac{\partial v}{\partial \zeta} \right) = 0.$$

Clearly, (20) and (18) are identical.

In this case

$$\Psi(\zeta) = r + is = \frac{1}{n+1} \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) = \frac{1}{n+1} f'(\zeta).$$

From this it is seen that the first polars and, as a consequence, all successive polars of a pencil of stelloids form themselves pencils of stelloids <sup>6)</sup>.

The results may be stated in the theorem:

$(n+1)$  distinct real points ( $A$ ) in a general position and their associates in a Cartesian plane completely determine a pencil of stelloids of order  $(n+1)$ .

The first polars of an arbitrary point  $P(xy)$  with respect to this pencil form a pencil of stelloids of order  $n$ , whose  $n^2$  base-points [ $n$  real points  $P(xy)$  and their associates] define with  $P$  a  $(1, n^2)$  correspondence in the Cartesian plane. Writing  $x + iy = \zeta$ ,  $x' + iy' = \zeta'$  the real part  $(1, n)$  of this correspondence between  $P'$  and  $P$  is also realized in a superposed complex plane by the transformation

$$(21) \quad \zeta' = \zeta - \frac{(n+1)f(\zeta)}{f'(\zeta)},$$

in which the roots of  $f(\zeta) = 0$  represent the points ( $A$ ), and where  $f'(\zeta)$  is the derivative of  $f(\zeta)$ .

**General case.** — In the original transformation (1) assume  $f(\zeta)$  and  $g(\zeta)$  without a common factor and in such a manner that

$$\zeta' g(\zeta) - f(\zeta) = F(\zeta) \text{ is of degree } n.$$

It is proposed to determine a transformation of type (21) in such a manner, that in

$$(22) \quad Z' = \zeta - \frac{(n+1)\Phi(\zeta)}{\Phi'(\zeta)} = \frac{\zeta \Phi'(\zeta) - (n+1)\Phi(\zeta)}{\Phi'(\zeta)},$$

$$\zeta \Phi'(\zeta) - (n+1)\Phi(\zeta) = F(\zeta).$$

Hence  $\Phi(\zeta)$  must be determined from the linear differential equation of the first order

$$(23) \quad \frac{d\Phi}{d\zeta} = \frac{F(\zeta)}{\zeta} + \frac{n+1}{\zeta} \Phi.$$

The general solution is easily found as

$$(24) \quad \Phi(\zeta) = c\zeta^{n+1} + \zeta^{n+1} \int \frac{\zeta' g(\zeta) - f(\zeta)}{\zeta^{n+2}} d\zeta$$

and is evidently a polynomial of degree  $(n+1)$ . Substituting this expression for  $\Phi(\zeta)$  in (22), a transformation is obtained in which to every point  $Z'$  correspond the base-

<sup>6)</sup> E. KASNER, loc. cit. <sup>4)</sup>.



points of the pencil of first polars of  $Z'$  with respect to the pencil of stelloids defined by  $\Phi(\zeta) = 0$ .

In particular, to the origin  $Z' = 0$ , correspond the base-points of the first polar pencil which are also obtained as the roots and their associates of the equation

$$(25) \quad \zeta' g(\zeta) - f(\zeta) = 0.$$

In other words:

*The  $n$  points and their associates corresponding to an arbitrary value  $\zeta'$  in the general transformation (1) are identical with the base-points of the first polar pencil of the origin with respect to the pencil of stelloids defined for every value of  $\zeta'$  by  $\Phi(\zeta) = 0$ .*

## § 5.

### Projectivity between a pencil of stelloids and a pencil of first polars.

Considering a pencil of curves of the  $(n+1)''$  order  $u + \lambda v = 0$  with  $(n+1)^2$  distinct base-points and a corresponding pencil of polars  $u_1 + \lambda v_1 = 0$  of an arbitrary point  $(x' y')$ , the two pencils are projective for the variable parameter, and generate a curve

$$(26) \quad u v_1 - u_1 v = 0.$$

In case of a pencil of first polars this is a curve of order  $(2n+1)$ ; in case of a pencil of last or straight polars, a curve of order  $(n+2)$ . I shall limit myself to the first case. It is seen that, generally, to every point in the plane is attached a certain curve as represented in (26). It passes through the  $(n+1)^2$  base-points of  $u + \lambda v = 0$ , the  $n^2$  base points of  $u_1 + \lambda v_1 = 0$ , the  $2n(n+1)$  intersections of  $u = 0$  and  $u_1 = 0$  and of  $v = 0$  and  $v_1 = 0$ . As there are in general  $3(n+1-1)^2 = 3n^2$  curves with a double point in a pencil like  $u + \lambda v = 0$ , the curve (26) also passes through these double points. All curves of this kind therefore pass through

$$(n+1)^2 + 3n^2 = 4n^2 + 2n + 1$$

fixed points, independent of the particular position of the pole  $(x', y')$ .

Now the pencil  $u + \lambda v = 0$  cuts out on every line through the pole  $(x', y')$  an involution of degree  $(n+1)$ , with  $2(n+1) - 2 = 2n$  double points. The pole itself belongs to the ensemble of these double points, as is seen by assuming the curve of the pencil through the pole and drawing the tangent at this point. The double points on each line are, of course, also the points of tangency of curves of the pencil. The locus of these double points is a curve of order  $(2n+1)$  and is identical with the curve (26). To prove this, consider two corresponding curves  $u + \lambda v = 0$  and  $u_1 + \lambda v_1 = 0$ . The second cuts the first in the points of tangency of the tangents from the pole to the first curve. Hence, every point of (26) is a point of tangency of a tangent from the pole to a curve of the pencil  $u + \lambda v = 0$ , and consequently also one of the double points of the involution cut out on this tangent. Thus, the theorem may be stated:

Associated with the transformation which is defined as the correspondence between an arbitrary pole and the  $n^2$  base-points of the pencil of first polars with respect to a pencil of curves of order  $(n+1)$  is a system of curves of order  $(2n+1)$ , each curve being produced by the pencil of curves and the projective pencil of first polars. Each curve of the system is also the locus of the double-points of the involution cut out by the original pencil on every ray through the pole. All curves of the system pass through  $4n^2+2n+1$  fixed points.

In the transformation of an arbitrary pole into the  $n^2$  base-points of the pencil of first polars, the  $(n+1)^2$  base-points of the original pencil of curves are invariant. To a curve of the  $m^{\text{th}}$  order in the  $x'y'$ -plane corresponds a curve of the  $2mn^{\text{th}}$  order in the  $xy$ -plane; to a straight line in particular a curve of order  $2n$ . Considering a curve of order  $\mu$  in the  $xy$ -plane, the curve of order  $2n$  cuts this curve in  $2n\mu$  points. Transforming the curves of order  $\mu$  and  $2n$  back into the  $x'y'$ -plane, we find that a straight line cuts the transformed of the curve of order  $\mu$  in  $2n\mu$  points; it is therefore of order  $2n\mu$ . This order is reduced when the curve of order  $\mu$  passes through certain fixed points of the transformation. Applying these results to the case of a pencil of stelloids and the corresponding special transformation (12), we find for the product of the pencil of stelloids

$$u + \lambda v = 0$$

and the pencil of first polars with respect to the pole  $(x', y')$  from (14) and (15)

$$(rx - sy - rx' + sy' - u) + \lambda(sx + ry - sx' - ry' - v) = 0$$

the equation:

$$(27) \quad (x - x')(rv - su) - (y - y')(ru + sv) = 0.$$

In accordance with (26), this is a curve of order  $(2n+1)$ , and apparently contains each of the circular points as an  $n$ -fold point. Its order in (27) appears as  $(2n+2)$  but reduces to that of  $(2n+1)$ . For an arbitrary point  $P(x, y)$  on the curve (27), there is

$$(28) \quad \frac{y - y'}{x - x'} = \frac{rv - su}{ru + sv}.$$

To  $P(x, y)$  corresponds in the transformation (12) a point  $P'$ , with the co-ordinates  $\xi, \eta$ , so that

$$(29) \quad \begin{cases} \xi = x - \frac{ru + sv}{r^2 + s^2}, \\ \eta = y - \frac{rv - su}{r^2 + s^2}. \end{cases}$$

From (29) and (28), as in both  $x$  and  $y$  have the same value, we conclude

$$(30) \quad \frac{y - \eta}{x - \xi} = \frac{y - y'}{x - x'};$$

i. e.

Any point of the curve (27) and its transformed are collinear with the pole  $(x, y)$ . Every ray  $g$  through the pole cuts (27) in  $2n$  points  $P$  so that the  $2n$  correspon-

ding points  $P'$  on the transformed curve are collinear with the  $P'$ s. To the  $n^2$  base-points of the pencil of polars corresponds in every case the pole itself. From this it follows that the transformed curve has the lines from the base-points of the polar pencil as tangents at the pole. In other words, the transformed curve has an  $n$  fold point at the pole and  $n$  of its real branches pass through this point.

The transformed curve is evidently of order  $(n^2 + 2n)$ . To prove this directly, assume in the  $x'y'$ -plane an arbitrary straight line with the equation

$$ax' + by' + c = 0.$$

To this line corresponds in the  $xy$ -plane the curve

$$(31) \quad (r^2 + s^2)(ax + by + c) - a(ru + sv) - b(rv - su) = 0,$$

which easily reduces to the degree  $2n$  and which contains the circular points as  $n$ -fold points. It has the  $n^2$  intersections of  $r=0$ ,  $s=0$  in common with (27), and as each, (27) and (31) contain the circular points  $n$ -fold, these count as  $2n^2$  points of intersection, so that there are only  $2n(2n + 1) - n - 2n^2 = n^2 + 2n$  points of intersection of (27) and (31), variable with  $a$ ,  $b$ ,  $c$ , left.

Hence, transforming (31) and (27) into the  $x'y'$ -plane it is found that a straight line cuts the transformed of (27) in  $(n^2 + 2n)$  points as stated above. Summing up, the theorem may be stated:

*A pencil of stelloids of the  $(n + 1)^{\text{th}}$  order and a projective pencil of first polars with respect to an arbitrary pole produce a curve  $C$  of order  $(2n + 1)$  with the circular points at infinity both as  $n$ -fold points.*

*In the transformation defined by the poles and the base-points of the corresponding pencils of first polars, to the curve  $C$  corresponds a curve  $C'$  of order  $(n^2 + 2n)$ , having the circular points as  $n$ -fold and the pole  $O$  as  $n^2$ -fold points.*

*Corresponding points of  $C$  and  $C'$  are collinear with the pole  $O$ .*

## § 6.

### Examples.

**1. Involutoric circular transformation.** — Taking as double points of the transformation the roots of the equation  $c\chi^2 - 2a\chi - b = 0$ , transformation (21) assumes the form

$$\chi' = \chi - \frac{2(c\chi^2 - 2a\chi - b)}{2c\chi - 2a}$$

which reduces to

$$\chi' = \frac{a\chi + b}{c\chi - a}.$$

The pencil of stelloids  $u + \lambda v = 0$  consists now of a pencil of equilateral hyperbolas through the roots of  $c\chi^2 - 2a\chi - b = 0$ , represented by the two points  $A_1$ ,  $A_2$  and their associates  $A_3$ ,  $A_4$ . To an arbitrary pole  $O$  corresponds the point of con-

currence  $O$  of the straight polars, and conversely. From this it is seen that *every involutoric circular transformation is identical with a Steinerian transformation based upon a pencil of equilateral hyperbolas.*

The curves  $C$  and  $C'$  are ( $n = 1$ ) both circular cubics and are identical. Based on this transformation the theory of circular cubics admits of a very simple treatment, as will be shown elsewhere.

2. *The transformation having the cube-roots of unity as double points.* — The transformation becomes

$$z' = z - \frac{3(z^3 - 1)}{3z^2}$$

which reduces to

$$z' = \frac{1}{z^2}.$$

The pencil of stelloids (cubics) through the three cube-roots of unity and their associates is given by

$$x^3 - 3xy^2 - 1 + \lambda(3x^2y - y^3) = 0,$$

and the projective pencil of first polars, consisting also of stelloids, and consequently of equilateral hyperbolas, by

$$(x^2 - y^2)x' - 2xyy' - 1 + \lambda[2xyx' + (x^2 - y^2)y'] = 0.$$

The product of the two pencils is easily found as the curve of the 5<sup>th</sup> order.

$$(x^4y + 2x^2y^3 + 2xy + y^5)x' - (x^5 + 2x^3y^2 - x^2 + xy^4 + y^2)y' - 3x^2y + y^3 = 0,$$

which is a bicircular quintic through the pole  $(x', y')$ . Its transformed is found as a curve of the eighth order ( $n = 2$ ,  $n^2 + 2n = 8$ ) which, with  $x, y$  as current coordinates, has the equation

$$(x^2 + y^2)[(x' - x)^2 + (y' - y)^2 - 2(x^2 + y^2)(yx' - xy')^2] - [x(x' - x)^2 + 2y(x' - x)(y' - y) - x(y' - y)^2]^2 = 0.$$

The point  $(x', y')$  is a quadruple point of the curve with two real and two imaginary branches. As can easily be verified, the tangents at this point pass through the base points of the pencil of equilateral hyperbolas.

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