zero. But the proof in (5) avails when $Ee' = \lambda e$ is not zero to show that for this special $e_1, \ldots, e_r$ the equation $| E+\rho | = 0$ has no root but zero; thus, besides the equation $Ee' = \lambda^{\rho}$, we have $(r-2)$ equations $Ef' = 0$, together with $Ee = 0$. In other words, the first invariant factor of $| E+\rho | = 0$ is of exponent 2, and the remaining ones are linear, and the matrix E satisfies the equation $E^2 = 0$. For a group known to be necessarily integrable, such that $| E+\rho | = 0$ reduces to $\rho^r = 0$ for all values of $e_1, \ldots, e_r$, it can be shown that a set $e_1, \ldots, e_r$ exists, not identically zero, for which $E = 0$.

Overlapping Intervals. By W. H. Young.

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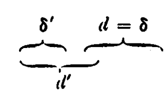
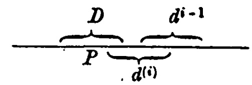
1. Given any set of overlapping intervals, we will show how to determine a countable set from among them which by themselves determine the most important properties of the given set.

Take, first, any one of the intervals, and let us denote it by d or δ. Then either there is no interval of the given set which abuts or overlaps with d on the left, or else there is such an interval. In the latter case we denote by δ' the part of this interval which extends beyond d to the left, and by d' the interval itself, which coincides with δ' if d' abuts with d, and which otherwise contains δ' as a part, and has its left-hand end point coincident with that of δ'.



Proceeding in this way towards the left, we must ultimately either come to an interval of the given set having no interval abutting or overlapping with it on the left, or else the parts of intervals $\delta', \delta'', \delta''', \ldots$ must get smaller and smaller without limit, and define a limiting point P external to all of them, and therefore external to the intervals $d', d'', d''', \ldots$ of the given set. Such a point P may, however, be internal to some other interval of the given set; in this case we choose out any one of the intervals containing P, say D. There will only be a finite number of the intervals $d, d', d'', \ldots$

which do not overlap with D. Let $d^{(i)}$ be the first which overlaps with D; then we select the intervals $d, d', d'', \ldots, d^{(i)}, D$, and omit from consideration all the intervals $d^{i+1}, d^{i+2}, \ldots$. Proceeding on these lines we can only be stopped (1) by coming to an interval of the given set having no interval of the given set abutting or overlapping with it on the left, (whose left-hand end point P is therefore neither internal to any interval of the given set nor a right-hand end point of any one of the intervals), or (2) by the parts of intervals $\delta', \delta'', \delta''', \ldots$ becoming smaller and smaller and defining a limiting point Q on the left of all of them, such that no interval of the given set contains Q as internal point. Q would be external to every one of the given intervals, unless it were a right-hand end point of one or more of them. Such a point, P or Q, may be properly called *an external* or *semi-external point of the given set*, though it should be remarked that a semi-external point of a set of overlapping intervals is not necessarily, as in the case of non-overlapping intervals, an end point of only *one* of the intervals.



Having proceeded in like manner on the right of d, we start afresh in each of the one or two segments left over, and take again any one of the intervals and treat it as we did d. Continuing this process, we get a set of *non-overlapping* intervals $\delta, \delta', \delta'', \ldots$, &c., which, by the theory of sets of non-overlapping intervals, can be arranged in *countable* order $\delta_1, \delta_2, \delta_3, \ldots$, and, corresponding to these, a *countable* set of the given intervals, $d_1, d_2, d_3, \ldots$, such that δ_i coincides with the whole or a part of d_i, and has at least one end point common with it. Any other of the given set of intervals lies entirely within one of the d_i's or else lies entirely within a set of the d_i's which overlap or abut all along.

Such a countable set $d_1, d_2, \ldots$ chosen in the manner indicated from among the intervals of the given set serves to classify the points of the straight line with reference to the given set in a manner analogous to that used in the theory of non-overlapping intervals. Any internal point of $d_1, d_2, \ldots$ is internal to the given set (of course), and the converse is also true, except that an internal point of the given set *may* be an end point of two intervals d_i, d_j which abut. Any external or semi-external point of $d_1, d_2, \ldots$ is external or semi-external to the given set, and *vice versa*.

2. We notice that the set $\delta_1, \delta_2, \ldots$ is not unique; but, since the ex-

ternal and semi-external points of the set then determined are the same as before, all such sets have the same derived set $\delta_1', \delta_2', \delta_3', \ldots$, and we may very properly call this *the derived set of the given set*, since it is obtained by amalgamating all abutting and *overlapping* intervals of the given set.

3. This gives us a direct proof of the Heine-Borel theorem, and shows us how to determine a finite number of the intervals such as are asserted to exist in the theorem. The enunciation of the theorem is as follows:—*Given a set of intervals such that every point of a given segment (A, B) is internal to some interval of the set, we can choose out a finite number of the intervals having the same property.*

For in this case there is no point of the segment external or semi-external to the intervals δ_i; hence they are finite in number, and therefore the same is true of the intervals d_i. Let these be $d_1, d_2, \ldots, d_k$, and let the end points of these intervals inside (A, B) be $P_1, P_2, \ldots$. Then we can choose out $(k-1)$ or less of the given intervals containing as internal points those of the points $P_1, P_2, \ldots$ which are not already internal to $d_1, d_2, \ldots, d_k$, and get in this way at most $(2k-1)$ of the intervals having the desired property.

4. Content.—It is convenient to *define* the content of the set as being the content I_δ of the derived set. This agrees with the definition already given in the case where the intervals do not overlap, and enables us to extend the theorems given in the first part of this paper concerning external and semi-external points and their connexion with the content to sets of intervals of the most general character.

5. The intervals of the derived set are such that every internal point of the given intervals is internal to some interval of the derived set, and no external or semi-external point of the given intervals is internal to the derived set; but it is not necessarily true that every internal point of the derived set is internal to some interval of the given set. If, however, any point P, internal to an interval δ', is not internal to any interval of the given set, then, however, we construct $d_1, d_2, \ldots$. P must be an end point of two abutting intervals δ_i and δ_j; and, since it is not internal to either of the corresponding d_i or d_j, it must also be a common end point of d_i and d_j, and these must therefore abut. Thus, if there be such a point P, there will be two or more intervals abutting at P,

and no interval containing P as internal point. Hence the number of these points P is at most countably infinite, and they are completely determined by the given intervals. We can call these points properly *isolated end points*. If we plot them down in the derived intervals, *we get a new set of intervals*, (non-overlapping), $D_1, D_2, \ldots$, *uniquely determined by the given set,** *and such that every internal point of the given intervals is internal to this set, and* vice versa.

6. From this theorem we can, if we please, deduce as a special case the so-called counterpart of the Heine-Borel theorem. The enunciation as given in Schoenflies's *Bericht*, p. 109, is as follows :—

"If to every point x of a set of points $X = (x)$, dense everywhere in a segment (A, B), there corresponds an interval ϵ containing x as internal point, and if every point of (A, B) is not internal to one of these intervals, then the intervals ϵ determine a finite or a closed set of points Q such that no point of Q is internal to the intervals ϵ, while this is the case for every point of the set complementary to Q."

It will be observed that the apparent restriction as to the relation of the set of points X to the intervals ϵ and to the continuum is superfluous, and only tends to complicate the issue. If we remove this restriction, the theorem is synonymous with the theorem stated at the end of § 5.

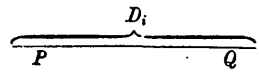
7. We can now prove the following *extension of the Heine-Borel theorem :—Given any closed set of points on a straight line and a set of intervals so that every point of the closed set of points is an internal point of at least one of the intervals, then there exist a finite number of the given intervals having the same property.*

For let us construct the equivalent set of non-overlapping intervals $D_1, D_2, \ldots$ (§ 5).

Then, by the fundamental property of this set, each point of the closed set of points is internal to one of the intervals D_i. If there be not a finite number of the intervals D_i having the same property, there must be a limiting point P of those intervals D_i, each of which contains at least one point of the closed set. P could not be a point of the closed set, since it is not internal to any one of these intervals D_i; but it is clearly a limiting point of the given set of points, and therefore belongs to the set, since it is a closed set.

* But not necessarily contained in the given set.

This is a contradiction, and therefore there can only be a finite number of the intervals D_i which contain points of the given set. Let these be $D_1, D_2, \ldots, D_k$. In any one of these, say D_i, the points of the closed set form a closed set lying entirely internal to D_i. Hence we can assign two points P and Q internal to D_i such that the points of the closed set which lie in D_i form a closed set lying betwen P and Q (not inclusive).



Now between P and Q, (both inclusive), there are only internal points of the given set of intervals (§ 5); hence, by the Heine-Borel theorem, we can assign a *finite* number of the given intervals so as to entirely cover up the segment (P, Q). In this finite number of the given intervals every point of the closed set of points in D_i must lie. Performing this for all values of i from 1 to k, we get a finite number of the original intervals having the required property.

The Continuations of certain Fundamental Power Series. By

M. J. M. Hill, M.A., Sc.D., F.R.S., Astor Professor of Mathematics at University College, London. Received, in revised form, February 6th, 1903.*

1. The theory of the continuation of power series has not, so far as I can ascertain, been hitherto illustrated by applications to simple cases where the work is unartificial.

By means of some well known formulæ, proved rigidly for the first time by Abel, in his famous memoir on *The Binomial Series*, I have succeeded in working out the continuations of the binomial series, the logarithmic series, and the series for arc tan x along arbitrary circuits. Using conjugate functions I have also succeeded in finding the continuation of the series for arc sin x.

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* The paper has been condensed from two papers communicated at the meetings of November 13th, 1902, and December 11th, 1902; the latter dealt exclusively with the case of arc sin x.