where the meanings of the symbols have been already explained. A formula substantially equivalent to this, but in a different (and scarcely properly explained) notation, is given, " Aoust, Théorie des coordonnées curvilignes quelconques," *Annali di Matem.*, t. ii. (1868), pp. 39—64 ; and I was, in fact, led thereby to the foregoing further investigation.

As to the definition of the radius of geodesic curvature, I remark that, for a curve on a given surface, if PQ be an infinitesimal arc of the curve, then if from Q we let fall the perpendicular QM on the tangent plane at P (the point M being thus a point on the tangent PT of the curve), and if from M, in the tangent plane and at right angles to the tangent, we draw MN to meet the osculating plane of the curve in N, then MN is in fact equal to the infinitesimal arc QQ' mentioned near the beginning of the present paper, and the radius of geodesic curvature ρ is thus a length such that $2\rho . MN = \overline{PQ}^2$.

On certain Tetrahedra specially related to four Spheres meeting in a Point. By SAMUEL ROBERTS, *F.R.S.*

At our last meeting I mentioned an elementary theorem relating to a tetrahedron. Namely, if on each of the edges we take an arbitrary point, and describe a sphere through each vertex, and the three arbitrary points taken on the three adjacent edges, the four spheres meet in a point.

In this shape, the result is easily shewn by means of inversion. The needful preliminaries are as follows :—

(*a*) The analogue in plane space is this,—If an arbitrary point be taken on each side of a triangle, the three circles passing each through a vertex, and the two arbitrary points on adjacent sides, intersect in one point. This is a known result.

If we invert with regard to any point outside the plane, we get the following theorem :—

(*b*) If three circles on a sphere meet in a point forming by their intersections two and two together a triangular figure whose sides are circular, then, if we take any point on each of the circles and draw another circle through each simple intersection and the two arbitrary points taken on the circles to which the intersection belongs, these three circles last drawn meet in a point. This is the analogue on the sphere of the plane theorem. The resulting figure is, it will be ob-

served, perfectly symmetrical, consisting of six circles intersecting three together in eight points, and two together in six points.

Now, take a tetrahedron $ABCD$, and the arbitrary points a, a', a'', b, c, d on each of the edges. Suppose a sphere B passes through $B, a, c. d$; a sphere C through C, a', b, d; and a sphere D through $D, a'', b, c.$

Let the circle $rcpK$, meeting the side ADB in r and the side BCD in p, be the intersection of the spheres B and D. Similarly, let the circle $qbpK$, meeting the side ACD in q and the side BCD in p, be the intersection of the spheres C and D. Then K, p are the triple-intersections of the spheres B, C, D. If now a sphere be described through $Aaa'a''$, the points r, q, and a corresponding point s in the side ABC, become triple points of intersection of the spheres $(A, B, D), (A, C, D),$ and (A, B, C) respectively. Then, by the analogue (b), the three circles $(ars), (a''rq), (a'qs)$ meet in a point which must be the point K, and lies on the sphere A; that is to say, the four spheres A, B, C, D meet in a point.

It is perhaps worth while to regard the theorem from another point of view, taking as our data four spheres intersecting in a point. I had not worked out the question in this form when I presented the theorem which now becomes porismatic as in the corresponding plane case.

Consider the section of the three spheres B, C, D by any plane BCD, passing through p, a triple intersection of the spheres. Let BCD be any triangle so drawn that each side passes through an intersection of two of the section circles, and each vertex is on the section circle passing through the intersections on the adjacent sides.

Through BC, CD, DB respectively describe planes passing also through s, q, r respectively, the three other triple intersections of the spheres, s being the intersection of the spheres A, B, C; q that of the spheres A, C, D; and r that of the spheres A, B, D. Suppose that these planes meet in A'. And, as before, let the spheres B, C, D meet BA', CA', DA' in a, a', a'' respectively. Then the sphere through A', a, a', a'' passes through K, the other triple-intersection of the spheres B, C, D, and through q, r, s. This sphere therefore remains the same when the triangle BCD is porismatically varied, and is, in fact, the fourth given sphere A.

The series of triangles is singly infinite, and we shall see that the locus of A' is not the sphere A, but a circle thereon. But BCD may be any plane through the point p, and a system of planes through a point is doubly infinite. Hence the series of tetrahedra completely taken is trebly infinite.

If we consider the plane BCD as fixed, but the base triangle BCD variable, the edge AB (for instance) meets the two circles $arsK$, $Bcpd$, and the straight line sd. Hence in its different positions AB forms a

system of generators of a hyperboloid of one sheet, opposite to sd, which is also a generator. Moreover, the circles $arsK$, $Bcdp$ are circular sections of opposite systems, and the sphere A meets the hyperboloid in another circle through A parallel to the circle $Bcpd$.

In like manner, the simultaneous movements of AC, AD generate two other hyperboloids, and the circle through A parallel to the base plane is common to them all. For a given position of the triangle, the same point A is the intersection of the corresponding generators through B, C, D respectively.

This follows at once from the remark, that if in the plane BCD, or in any parallel plane, we take a circle, and from any point thereof draw lines parallel to pB, pC, pD respectively, and through their remaining intersections with the circle draw lines parallel to the corresponding sides of the triangle, these last lines meet in a point on the circle. This theorem (c) is obvious when the figure is drawn.

It is easy to frame a line model exemplifying the foregoing conclusion, if we take for our base any three circles meeting in a point, and in a parallel plane any other circle.[*]

If now we move the plane BCD about p, the movement of AB generates a system of hyperboloids having in common a fixed circular section $arsK$ and a common generator pK (in fact, the radical axis of the spheres B, C, D) of the same system as AB. This series of hyperboloids is doubly infinite, linear in two parameters.

Suppose now the base BCD and the circles thereon are given, also the circle through A, parallel to the base and the generator pK common to three hyperboloids, obtained as above, indicated. It is plain that, if through *any* point K of Kp, we take three circular sections of the hyperboloids opposite to the sections in the base respectively, we have the same singly infinite series of tetrahedra, but different sets of spheres, the three corresponding to B, C, D having the same radical axis.

When the plane BCD is given, the maximum tetrahedron is that one whose base is a triangle having its sides respectively perpendicular to pb, pc, pd.

And for any plane through p, that is to say, for four given spheres, generally the maximum tetrahedron has its sides perpendicular to the lines drawn from the quadruple intersection to the triple intersections; or, what amounts really to the same thing, the vertices of the tetrahedron are determined by the extremities of the right lines drawn through the quadruple intersection and the centres of the four spheres.

The minimum tetrahedron is represented by the line Kp common to the series of hyperboloids.

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[*] The author exhibited a model of this kind.

The plane analogue (*a*) is immediately extended by general projection to the system of three conics having three triple intersections,[*] and it is natural to infer a similar generalization of the solid theorem. We have to consider the system of four conicoids, having a common plane section, a common quadruple intersection, and four triple intersections. With the assistance of the system of hyperboloids, a proof of the spherical case can be established which admits of immediate extension to the generalised form.

Thus (in outline) the four spheres being given, and any plane *BCD* through *p* a triple intersection, construct the hyperboloid generated by lines meeting the circles *Bcdp*, *Krs*, and the line *Kp*, and the other two hyperboloids similarly related to the circles *Cbdp*, *Dbcp*, &c. These three hyperboloids intersect again in a circle through *A* parallel to the plane *BCD*. The spheres on which this circle and the opposite sections through *K* lie, coalesce in one sphere determined by the point *K*.

For the generalised case, we have, instead of circles, conics meeting one and the same conic on a given plane, namely, the common plane section of the four conicoids; instead of opposite circular sections of the hyperboloids, we have conics passing through the points (*a*, *β*), (*γ*, *δ*) respectively, *a*, *β*, *γ*, *δ* denoting the four points in which the hyperboloid in question meets the common conic. Moreover, the theorem (*c*) is similarly extended by general projection. The reasoning in the case of spheres can now be immediately transferred to the generalised system of conicoids.

Observe that the conics in the base plane, and the conic through the vertex corresponding to the circle through *A* parallel to the base, are not parallel for finite positions of the common plane section.

The extended result can be also shewn by the theory of homologous figures in space of three dimensions.

Historical Note on Dr. Graves's Theorem on Confocal Conics.
By Samuel Roberts, F.R.S.

[*Read April 14th*, 1881.]

Dr. Graves's theorem for plane conics (1841) is as follows:—If two tangents be drawn to an ellipse from any point of a confocal ellipse,

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[*] Or, if two conics *a*, *β* intersect in the points *a*, *b*, *c*, *d*, and through *d* we draw a transversal meeting *a* in *k*, and *β* in *l*, and if *p* is a fixed point on *a* and *q* a fixed point on *β*, we see at once by anharmonic ratios, taking four positions, that the intersection of the lines *kp*, *lq* moves on a conic through *a*, *b*, *c*, *p*, *q*.