Structural Precedence under Effective Irreversibility: A Robust Viability and Hamilton–Jacobi Framework for Non-Ideal Systems

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Abstract

We formalize the Structural Precedence principle for non-ideal dynamical systems operating under effective irreversibility and worst-case uncertainty. We show that robust safety guarantees are equivalent to membership in the robust viability kernel of an extended structural system including physical state, structural modes, and information state. We further connect this characterization with Hamilton–Jacobi–Isaacs (HJI) reachability theory and viscosity solution frameworks. The result provides a unified geometric interpretation of robust safety and clarifies the structural limitations of adaptive and reflexive control mechanisms.

1 Introduction

Real-world engineered systems operate under irreversibility, bounded resources, and uncertainty. In such environments, safety guarantees cannot generally be derived solely from adaptive control or policy refinement. Instead, safety is constrained by the structural geometry of the system state space.

The goal of this work is to formalize this statement using standard tools from viability theory, differential games, and Hamilton–Jacobi reachability.

2 Extended Structural System Model

We define the extended state space:

$$X_E = X_P \times X_M \times X_I$$

where:

- X_P : Physical state
- X_M : Discrete structural modes (interlocks, guards, architecture constraints)
- X_I : Information / observation state (estimator state, sensor health, etc.)

Dynamics:

$$\dot{x}_E = f_E(x_E, u, w)$$

Assumption 1.

- *f_E is continuous and locally Lipschitz in x_E*
- *Control set U and disturbance set W are compact*
- *Solutions exist and are unique*

For example, degradation of sensing reliability (modeled within X_I) can move the system outside $\mathcal{V}(K)$ even if the physical state remains nominal, illustrating that structural safety depends on the full extended state, not only on physical variables.

3 Effective Irreversibility

Definition 1 (Effective Irreversibility)**.** *A set $X_{\text{irr}} \subset X_E$ is effectively irreversible if exit from the set cannot be guaranteed within admissible resources, time horizon, or control authority.*

Safe set:

$$K = X_E \setminus X_{\text{irr}}$$

4 Robust Viability and Adversarial Reachability

Definition 2 (Robust Viability Kernel)**.**

$$\mathcal{V}(K) = \{x_0 \in K : \exists u(\cdot)\ \forall w(\cdot)\ x(t) \in K\ \forall t \geq 0\}$$

Notation. We denote the viability kernel of the admissible set K by $\mathcal{V}(K)$. When introducing Hamilton–Jacobi reachability formulations, we use $V(x)$ to denote the associated value function. The kernel is then recovered as a superlevel set of the value function, e.g.

$$\mathcal{V}(K) = \{x \in X_E\ :\ V(x) > 0\}.$$

Definition 3 (Adversarial Backward Reachable Set)**.**

$$\mathcal{B}^* = \{x_0 : \forall u(\cdot)\ \exists w(\cdot)\ \exists t : x(t) \in X_{\text{irr}}\}$$

5 Structural Precedence Equivalence

Theorem 1 (Structural Precedence Equivalence)**.** *Under Assumption 1 and effective irreversibility:*

$$\mathcal{V}(K) = X_E \setminus \mathcal{B}^*$$

Robust safety guarantees exist if and only if:

$$x_0 \in \mathcal{V}(K)$$

Remark 1. *This equivalence implies that robust safety is a geometric property of the extended structural state space.*

6 Hamilton–Jacobi–Isaacs Characterization

This formulation is naturally interpreted as a differential game between control inputs and disturbances, where structural precedence corresponds to the existence of robust invariant sets under worst-case disturbances.

Define Hamiltonian:

$$H(x,p) = \min_{u \in U} \max_{w \in W} p \cdot f_E(x,u,w)$$

Under Isaacs condition:

$$\min_u \max_w = \max_w \min_u$$

Value function satisfies HJI PDE:

$$\partial_t V + H(x, \nabla V) = 0$$

Let $\ell(x)$ be an implicit surface function encoding the effectively irreversible set X_{irr}, e.g. a signed distance or level-set representation. The value function is initialized using

$$V(0,x) = \ell(x),$$

and evolves according to the Hamilton–Jacobi equation. Under standard assumptions, the safe set is recovered as

$$\mathcal{V}(K) = \{x : V(x) > 0\}.$$

7 Viscosity Solution Framework

Definition 4. *A viscosity solution satisfies PDE inequalities using smooth test functions touching from above and below.*

Theorem 2 (Comparison Principle)**.** *Under standard regularity assumptions, viscosity solutions are unique.*

8 Set Characterization via Value Function

$$\mathcal{V}(K) = \{x : V(x) > 0\}$$

$$\mathcal{B}^* = \{x : V(x) \leq 0\}$$

9 Structural Margin

Definition 5. *Structural margin:*

$$m(x) = V(x)$$

Provides continuous measure of safety robustness.

10 Interpretation

Robust safety guarantees are structural properties of the extended system.

Adaptive or reflexive policies operate only inside structural safety margin.

11 Scope of Validity

Results apply under:

- Worst-case uncertainty
- Effective irreversibility
- Fixed structural architecture

12 Conclusion

Structural Precedence is equivalent to robust viability kernel characterization and HJI reachability geometry in non-ideal systems.

References

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