

V.—LOGICAL THEORY OF THE IMAGINARY.

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IN his address before the meeting of the British Association at Southport Prof. Cayley, having referred to the amount of discussion which the notion of negative magnitudes has occasioned in philosophy, said: "But it is far otherwise with the notion which is really the fundamental one (and I cannot too strongly emphasise the assertion) underlying and pervading the whole of modern analysis and geometry, that of imaginary magnitude in analysis and of imaginary space (or space as a *locus in quo* of imaginary points and figures) in geometry: I use in each case the word imaginary as including real. This has not been, so far as I am aware, a subject of philosophical discussion or inquiry." Prof. Cayley then proceeded to say "considering the prominent position which the notion occupies—say even that the conclusion were that the notion belongs to mere technical mathematics, or has reference to nonentities in regard to which no science is possible, still it seems to me that (as a subject of philosophical discussion) the notion ought not to be thus ignored; it should at least be shown that there is a right to ignore it."

It is evident from the tone of the passage I have quoted that Prof. Cayley was not satisfied with the attitude adopted towards this notion by the majority of those who have treated the subject. Most writers seem to have adopted the view that however useful such an expression as $\sqrt{-1}$ may be in technical mathematics, still, even in pure algebra, the expression is essentially devoid of meaning. The dissatisfaction implied in the above quotation must be my excuse for venturing to dissent from the eminent writers who have held this view and for attempting to evolve from the logical standpoint an interpretation for the imaginary of pure algebra.

Prof. A. Macfarlane in his pamphlet on *The Imaginary of Algebra* has divided analysts into three classes with respect to the theory and use of $\sqrt{-1}$: "first, those who have considered it as *undefined* and *uninterpreted*, and consequently

make use of it only in a tentative manner; *second*, those who have considered it as *undefinable* and *uninterpretable*, and build upon this supposed fact a special theory of reasoning; *third*, those who, viewing it as capable of definition, have sought for the definition in the ideas of geometry".

As an example of the first class Prof. Macfarlane instances the astronomer Airy, and as an example of the second the view put forward by Boole in his *Laws of Thought* (p. 68), who bases on the non-interpretability of the symbol $\sqrt{-1}$ in mathematics, a claim to dispense with the interpretability of the intermediate results in other processes of reasoning.

Prof. Macfarlane does not adduce expressly any instances of the third class. It may I suppose be considered as representing the common opinion on the subject.

The following attempt at a logical interpretation of the mathematical symbol was suggested to me by the consideration of the function evolved by Boole in pursuance of the view referred to above. Here, however, the process is reversed and an attempt made to explain an uninterpretable symbol by an intelligible logical relation.

With the doubtful exception of Carnot, whose discussion of the subject in his *Geométrie de Position* touches very closely the view here advocated, the greatest names in mathematics are identified *exclusively* with attempts at finding a geometrical interpretation for the imaginary roots of unity. De Morgan who assigns a geometrical meaning to "double" and "triple" algebra says expressly of "single algebra" or what I have above called pure algebra that the symbol $\sqrt{-1}$ is in it unmeaning. The same view seems to have been held by Clifford. It is expressly stated in the *Common Sense of the Exact Sciences*. More recently this view has been reasserted by Mr. Russell in his *Foundations of Geometry*, who however adds that he is "unacquainted with any satisfactory philosophy of imaginaries in pure algebra". Essentially the same standpoint is adopted by Mr. Whitehead in his recently published *Universal Algebra*.

This position seems to me to be essentially paradoxical, and the difficulties inherent in it very great. Whoever adopts this view is obliged to hold that in pure, or, to use Dr. Morgan's term, "single" algebra impossible or imaginary quantities are an anomaly, and that they receive whatever meaning they have as something tacked on from the outside by this application to a particular subject-matter. This would simply be an unaccountable process in any logical theory of the movement of thought. Moreover it evades precisely the point which has to be explained, *vis.*, how an

imaginary expression which arises quite out of itself and independently in single algebra, in the ordinary development of the subject (if at the same time somewhat of a prodigy and none too welcome) should nevertheless be capable of performing most useful work when the notation comes to be applied to what is, in appearance at least, an extraneous field; or, conversely, that a new subject-matter should be capable of receiving valuable, nay, indispensable aid from what, in its own native land is a sort of intellectual outcast, or, at best, a mere artifice. The view which I venture to advocate is the very opposite, *viz.*, that imaginary quantities have a real meaning in single algebra, and that, if a problem exists, it is to explain how this meaning finds its way into more concrete forms of inference and receives application in the material inferences of geometry.

I therefore propose to state:—

- (1) The logical theory of the imaginary.
- (2) To illustrate the application of the theory in some departments of mathematics.
- (3) To make a few remarks on the relation of the logical calculus of Boole to that of Grassmann's *Ausdehnungslehre* and to ordinary algebra.

The fundamental characteristics of algebra as contrasted with arithmetic is a certain indefiniteness attaching to its symbols. By this I do not mean that the letters employed may represent either known or unknown quantities, but the fact that the ultimate character of the quantity is left undetermined; and hence follows a surprising characteristic, that whereas in logic it is a fundamental principle that from truth only truth can follow, in certain operations of mathematics both true and false conclusions may equally follow from the data supplied. But inasmuch as the operations of mathematics are still, at bottom, conformable to logical laws, it will follow that a point will necessarily be reached when this indefiniteness will be removed. In logic, the indefiniteness which attaches to a disjunctive judgment, is necessarily got rid of, when that judgment is contradicted. In mathematics, the same point is reached, when we endeavour to extract the root of a negative quantity.

This gives us the clue to the logical theory of the Imaginary or the Imaginary of Logic as it may be termed. It was already recognised by De Morgan, and has since been pointed out and emphasised by Schroeder (*Operations-kreis des Logik Kalkuls* and *Algebra der Logik*) that the conjunctive "and" is the opposite of the disjunctive "or". In contradicting a disjunctive proposition, the contradictory is conjunctive.

We might therefore infer from this, that, inasmuch as $\sqrt{1}$ has two roots, one positive and the other negative, which are disjunctively related to each other as alternatives, so the $\sqrt{-1}$ will involve the same roots, no longer disjunctively but conjunctively related. Or, the expression $\sqrt{1}$ will mean, simply, that 1 is to be multiplied with another 1 similar in sign to itself; whereas, $\sqrt{-1}$ will mean a 1 which is to be multiplied by a 1 dissimilar in sign to itself. The whole mystery, therefore, underlying this symbol is that the identity or equivalence in the factors which is quantitatively implied, does not extend to the qualitative relation represented by the signs + and -. This is already recognised in all interpretations involving the concrete application of the imaginary symbol. If we substitute for $\sqrt{-1}$ a symbol, say, + (-), expressive of this logical analysis, we shall find it acquires different meanings in different systems of mathematical analysis. It is identical with the "law of duality" of Boole. If V be an independently interpretable logical function, $V(1-V)=0$. Boole terms this equation the condition of the interpretability of logical functions. It is quite clear that Boole regarded $V(1-V)$ as discharging an analogous function in logic to that performed by $\sqrt{-1}$ in algebra; though the work contains no hint of the substantial identity of the two, which is here maintained. In point of fact, the diverse application which the function receives in logic and in mathematics establishes points of contrast sufficient to obscure the identity. In logic, precisely that element is excluded which characterises the imaginary in its application to mathematics. I shall refer again to this point when concluding, here merely remarking that the only writer who has attempted to make a mathematical use of the purely logical form is Hegel, $V(1-V)$ is the Notion of Hegel.

If we turn from the logical calculus to trigonometry and substitute in De Moivre's theorem the symbol I have proposed for the usual $\sqrt{-1}$, we shall find that the results work out identically with the ordinary form of the theorem. We can apply also to this symbol the interpretation of rotation through a right angle, and as an immediate consequence we might arrive at that double interpretation of a line proposed many years ago by the late Prof. Sylvester in the *Messenger of Mathematics* and which subsequently formed the subject of controversy between Cayley and Sylvester in connexion with the Carnot-D'Alembert problem. The issue between these eminent mathematicians depended as I conceive on this, that Cayley and Mrs. Ladd-Franklin (who also

took part in the controversy) regarded the signs $+$ and $-$ as admitting of alternative interpretation in time or space, whereas Sylvester held the necessity of maintaining both interpretations at once.

Before leaving this subject of the application of the imaginary to geometry, I cannot forbear from touching on a question which has been agitated in the pages of *Nature* and elsewhere, *viz.*, the principle that the square of a vector should be negative. It has been claimed that to omit the $(-)$ is not only essential to the physicist but is more consistent with ordinary algebra. Here again, the principle of the interpretation of the imaginary, advocated above, gives the clue. The imaginary, as a conjunctive relation of $+$ and $-$, is on the one side identically related to a given direction, and in this relation would answer to the ordinary operations of algebra; but as non-identically or dissimilarly related (and in a directional calculus such as Hamilton's this is the dominant point of view) the $(-)$ sign must be retained. Hamilton was therefore justified in saying "every line in tridimensional space has its square equal to a negative number, which is one of the most novel but essential elements of the whole quaternion theory" (*Lectures*, p. 53).

The analysis we have given admits of other illustrations as in Determinants. I now pass on to the more general question of its effect on our conception of the relation of the logical calculus to other branches of analysis. The idea of a symbolical calculus which should be perfectly general and applicable to all kinds of investigations is one which has frequently presented itself to both logicians and mathematicians. The idea occurs in the *Discours de la Methode* of Descartes. Such a calculus, Leibnitz seems to have had before his mind under the name of *Characteristica universalis* and Comte also in a passage in the *Synthèse Subjective* seems to have contemplated the same idea. It might also be said that Newton's definition of algebra as *Arithmetica universalis* implies the conception as an ultimate consequence. Boole maintained such a view in an article in the *Philosophical Magazine*. The idea of the isolation of the specious, universal or formal element of arithmetic or any other science, seems to lead to the conception of a theory of forms which should be perfectly pure and admit of general application, varied only by the conditions of the peculiar matter to which such a calculus is applied. Thus, the "principle of the permanence of equivalent forms" is regarded by Peacock as expressing the law of transition from an algebra arithmetically conditioned to a more universal, a symbolical algebra. The

only question seems to be, at what point shall this transition process cease, and *a priori* there seems no reason why it should cease, before it has brought the processes and inferences of every science within its scope. This is the real difficulty, in attempting to generalise from particular operations a general calculus of functions or operations. Shall it include under it, for example, the symbolism of chemistry? Such a universal science seems to become as empty of all real content as the old Aristotelian logic. The attempt to evolve in a symbolic calculus certain laws and methods common to a variety of symbols of operation, is apt to leave as a residuum simply the general notions of similarity and difference, connexion and separation. The result is formal logic, and formal logic not brought into any organic connexion with the material from which it is evolved.

The same problem thus presents itself to the logician and mathematician only viewed from opposite sides. The mathematician rises from the conception of particular laws and operations to the conception of the most general laws governing all operations. Logicians begin with the latter, but have not been successful in throwing light on the former. They have either dismissed the forms of mathematical inference as material consequences, or added them on empirically to inductive logic. A tendency has of late arisen to bring some of the forms of mathematical inference under what has been called the logic of relatives, but no satisfactory theory of the relation of such forms of inference to the ordinary logic has been put forward. Jevons seems to regard them as disguised cases of formal inference. De Morgan represents the opposite tendency and rather looks at formal inference itself as a refined residuum of material inference. In reality I believe there exists the closest connexion between all the forms of logical inference and of material inference, but the relation is not one of generalisation. In the theory of the imaginary which we have been discussing, we have only one instance out of many of such connexion. The only writer who has in general attempted to conceive the various categories of objective logic, not only as standing in systematic connexion with one another, but also as organically connected with the forms of subjective logic, is Hegel. His theory of the organic growth of the one from the other is opposed to the view which we are about to indicate, but has in common with it that it does not present the relation as one of mere degree of generality. The connexion of these two systems of forms is too vast a subject to be treated within the limits of this paper. It is only possible to briefly point out the general

distinction which exists between the processes of formal and material thought and their relations to concrete sciences, such as mathematics. The view we shall present is an expansion of the following remark of Grassmann's in the *Einleitung* to the *Ausdehnungslehre* of 1844 :—

“Die formalen Wissenschaften betrachten entweder die allgemeinen Gesetze des Denkens, oder sie betrachten das Besondere durch das Denken gesetzte, ersteres die Dialektik (die Logik) letzteres die reine Mathematik”.

This passage contrasts logical inference based on universal laws of thought with mathematical resting on the particular. In principle, the calculus of Boole's *Laws of Thought* is identical with the ordinary logic. No inference can be drawn in the former which cannot also be drawn in the latter. It has all the weakness of formal logic in dealing with material consequence. It may be regarded as the limiting case of material inference. If we compare such a calculus of logic with the calculus of the *Ausdehnungslehre*, we shall find that the leading characteristic of the former is that it treats the terms with which it deals as self-identical units, which may coincide or not but between which no other relation can exist. The equations by which it is distinguished are :—

$$\begin{aligned}x^2 &= x. \\x(1-x) &= 0.\end{aligned}$$

On the other hand the *Ausdehnungslehre* presents equations the opposite of this :—

$$\begin{aligned}a^2 &= 0. \\a b &= - b a.\end{aligned}$$

It is evident that the literal symbols in the first set of equations have their value residing in themselves; those of the second set have their value in relation to each other, and in the character of that relation. The one calculus views the units with which it deals, as identical, self-related, coinciding or not coinciding but otherwise unrelated to each other. The other calculus regards its objects as existing only in relation, as constituted by relation to something different and out of that relation becoming zero. The connexion between these two algebras is not external or contingent. They are united by reason of the necessary synthesis of thought with objects of experience. Ordinary mathematics employs both processes of inference. In imaginary expressions the absolute disconnexion which the abstract use of the negative in ordinary logic involves is overcome by means of the opposite principle of relativity and necessary synthesis.