ON A FORMULA FOR AN AREA

By W. H. YOUNG.

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PART I.—*Introduction.*

1. In the treatment of the area of a curve, it has been usual to suppose the curve to be a simple Jordan curve, in other words, a curve defined by a continuous (1, 1) correspondence with a straight line or circle. Such a curve, when closed, divides the plane into two distinct parts, the "inside" and "outside" of the curve, the former of which is a simply connected region. Such a curve can be approximated to by polygons, without loops, internal and external to the curve, and the area is defined as the common unique limit, whenever this exists (as it will if the curve is rectifiable), of the area of these polygons; and this area is the same as the content of the set of points inside the curve, together with, or omitting, the points of the curve itself, as we please.*

The idea, however, that an area, when given by a formula, is not always to be conceived of in this sense, is a familiar one. Thus, for example, the formula

$$A = \tfrac{1}{2}\int \{x\,dy - y\,dx\},$$

which in the case of a simple closed Jordan curve gives its area, when it exists, leads, when the integral is taken round a curve in which there are loops, only to the sum of the areas of those loops, when the area is regarded as a directed quantity (*directed area*), got by attaching an appropriate sign to the content of the inside of the loop, this sign being + when a point, describing the curve in the manner prescribed by the equations defining it, has the inside of the loop on its left (anticlockwise motion), and − when the inside of the loop is on the right (clockwise

* All these results are discussed in Chapters X and XIII of the *Theory of Sets of Points*, and references given.

motion). Thus in the case of the lemniscate, we are led to the value zero.

2. In the theory of the transformation of the variables in a multiple integral, based though it frequently is on geometrical representation, no application, as far as I know, of the idea of *directed area* is to be found. This is no doubt connected with the fact that it has been usual to regard the independent variables as always preserving their own sign, and either steadily increasing, or steadily decreasing. The moment, however, we introduce integration with respect to a function of bounded variation, this peculiarity in the process of integration disappears. The function of bounded variation will not, in general, be monotone, and may change sign. This fact may be of importance, even in one dimension. Thus in the method of integration by substitution, it is worth while to state the fundamental theorem in the case of a single variable in such a manner as to permit of the old variable x being a not necessarily monotone function of the new one t. This corresponds in the theory of lengths to the mode of estimation which consists in regarding journeys over the same ground in opposite directions as cancelling one another. The formula which lies at the basis of the theory is

$$\int dx = x = \int \frac{\partial x}{\partial t}\, dt,$$

and, as is well known, is true if x is any (absolutely convergent) integral with respect to t, and $\partial x/\partial t$ denote any one of its derivates.*

When we come to two dimensions, the question becomes at once more difficult and more interesting. Under what conditions, we ask, may we, in a multiple integral taken with respect to x and y, substitute for x and y functions of two new variables u and v?

Just as in one dimension the problem we have to solve is in its simplest form the determination of a formula for a directed length as the integral (or sum) of directed lengths, so, in our case, it is the problem of determining a formula for a directed area as the integral, *i.e.* the limit of the algebraic sum, of directed areas.

The problem is a more difficult one, not only owing to the comparative unfamiliarity of the concept of "directed area", but also because the analysis required involves considerations of a more subtle character.

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* H. Lebesgue, *Leçons sur l'intégration*. See, for instance, my paper on "Functions of Bounded Variation," *Quarterly Journal*, Vol. 42 (1910), p. 81.

The directed area formula itself, on the other hand, is well-known: it is

$$A = \iint \frac{\partial(x, y)}{\partial(u, v)} du\, dv,$$

where $\frac{\partial(x, y)}{\partial(u, v)}$ denotes the Jacobian

$$\begin{vmatrix} \frac{\partial x}{\partial u}, & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v}, & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Here $\frac{\partial x}{\partial u}$, $\frac{\partial y}{\partial u}$, $\frac{\partial x}{\partial v}$ and $\frac{\partial y}{\partial v}$ are in the first instance, and in the cases usually considered, the actual partial differential coefficients of the functions x and y with respect to u and v. But they may evidently without any inconvenience be regarded as any partial derivates, provided only they exist as differential coefficients, except at a set of plane content zero of values (u, v). What is required then is to establish this formula under conditions which have no reference to the sign of the Jacobian, or even to the signs of the partial derivates, a suitable definition having been previously devised for *area*.

3. At this point, however, it may be well to deal with the question which may already have suggested itself to the reader, as to the value of such considerations. It might be sufficient to reply generally by adducing merely the advantage of possessing a more general form for so fundamental a formula as that which constitutes the expression of the rule for the transformation of the variables in a multiple integral. The fewer the restrictions imposed on the functions with which it is concerned, the greater the potential usefulness of the formula. It is a great advantage not to have to prove *a priori* that such conditions are satisfied, even when in point of fact, *a posteriori*, this turns out to be the case. Moreover the greater generality of the statement of a theorem will often involve the necessity for devising a new method for proving it, and this may be actually not only more powerful, but more fruitful than that which presents itself more naturally in the simpler case. This occurs in the present instance; for the method to which I have been led enables us, even in the simpler case, to dispense with a number of useless restrictions, which

are seen to have their origin merely in the imperfections of the older method of proving the formula.

The mere fact that in one dimension these considerations are fundamental, is of itself sufficient reason for attempting to generalise them. I may say, however, that it is in the progress of researches into the area of curved surfaces that the necessity for modifying the usual definition of the area of a plane curve has been forced on my mind. It has been the modification which I have made in that concept, and the introduction of the analogous concept for *the area of a skew curve*, that have enabled me to overcome the difficulties in this theory which seem to have baffled all previous investigators. It has thus been rendered possible to evolve a simple and harmonious treatment not only of the concept of the area of a curved surface, but also of the analogous concepts in the case of curved manifolds of any number of dimensions in space of any number of dimensions.

4. To return to the generalisation of the concept of the *area of a closed curve*, it is clear that this should be such that it reduces in the particular case when the curve is a simple Jordan curve to the usual definition. As a particularly simple formula for the area of such a Jordan curve is known to us, it is open to us to attempt either to generalise the definition itself, or to devise a new definition to which this simple formula is still applicable. The latter will naturally suggest itself, if practicable; for it is desirable that in any case this formula should continue to hold good.

We are thus at once led to the following definition, which, it will be seen, is applicable to every closed curve for which the formula in question has a meaning, and accordingly certainly whenever the curve is rectifiable, *i.e.* has a length in Jordan's sense, or more generally if, the curve being defined by the equations

$$x = x(u), \quad y = y(u), \quad (a \leqslant u \leqslant b),$$

at least one of the continuous functions $x(u)$ and $y(u)$ has bounded variation.*

* The necessary and sufficient condition that the curve should be rectifiable is, as Jordan showed, that both $x(u)$ and $y(u)$ should be, not only continuous, but of bounded variation. If one of these functions at least has bounded variation, both being continuous, the plane content of the curve is easily shown to be zero.

Definition.

Inscribe in the curve, which we shall now suppose closed, any polygon in the usual way, so as, when the curve is rectifiable, to form an approximation to the length of the curve. Imagine vectors to be represented in respect of magnitude, line of action, and sense, by the sides of this polygon, supposed described in the sense in which u increases. Take the moments of these vectors about any point O in the plane. Then it may be shewn that—

(i) *The algebraic sum of these moments is independent of the position of the point O;* it is, of course, the moment of the couple equivalent to the imagined vectors, regarded as forces, and is equal to twice the area of the polygon in the usual sense, when the polygonal line does not cut itself.

(ii) *The number expressing this sum has a unique limit $2A$, as the number of the sides of the polygon is indefinitely increased and their lengths indefinitely decreased,* so that the perimeter of the polygon has the length of the curve, if rectifiable, as unique limit.

(iii) *The formula holds,*

$$A = \tfrac{1}{2}\int_a^b \{x(u)\,dy(u) - y(u)\,dx(u)\}.$$

This quantity A is defined to be the area of the curve.

5. It is not necessary for the purposes of this paper to give any other definition, but it will be easily seen that the area so defined is equivalent to the following, which may be regarded as the generalisation of the usual definition of the area of a simple Jordan curve.

The polygonal line inscribed in our closed curve may be thought of as forming a number of loops, some described in a clockwise, and others in an anti-clockwise manner. The sum of the areas of these loops, taken with proper sign, is identical with half the sum of the moments used in our definition, and will accordingly lead to the same unique limit A. And it should be noted that, though there is a certain degree of arbitrariness in the choice of these loops, the figure taken as a whole is perfectly definite in each case, and divides the whole plane up into connected sets of points, which, from the point of view of area, have a definite multiplicity. Thus certain points will be counted so many times positively, others negatively, while the rest will be regarded as external to the figure. As we proceed to the limit, certain of these sets will in essentials not change, just as in the gradual generation of the black intervals of a perfect set by means of increasing sets of intervals finite in number.

6. We are now able to state the main result of the paper:—

Taking first the case of two dimensions, let

$$x = x(u, v), \qquad y = y(u, v),$$

be functions of (u, v) *possessing the property of having all their partial derivates with respect to* u *and* v *bounded for all values of* (u, v) *in the fundamental rectangle* $(a, c; b, d)$, *that is*

$$a \leqslant u \leqslant c, \qquad b \leqslant v \leqslant d,$$

and let A *be the area of the curve in the* (x, y)*-plane which is the image of the perimeter of this rectangle.*

Then $$A = \int_a^c du \int_b^d \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv,$$

where $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$ *represent any of the partial derivates of* x *and* y *with respect to* u *and* v.

More generally the same is true, if the partial derivates are not bounded, provided only the following conditions are satisfied:—

(i) $x(u, v)$ *is an integral with respect to* u, *and an integral with respect to* v;

(ii) $y(u, v)$ *is an integral with respect to* u, *and an integral with respect to* v;

(iii) $\frac{\partial y}{\partial v}$ *and* $\frac{\partial x}{\partial v}$ *are, except for a set of values of* v *of content zero, numerically less than summable functions of* v *alone,* $\mu(v)$ *and* $M(v)$ *say;*

(iv) *the same as* (iii) *with* u *and* v *interchanged, or more generally* $\iint \frac{\partial x}{\partial u}\mu(v)\, du\, dv$, $\iint \frac{\partial y}{\partial u} M(v)\, du\, dv$, *exist as absolutely convergent integrals.*

It will have been noticed that the conditions in the theorem just given are stated to be sufficient; there is no suggestion that they are necessary. But this is in accordance with what might be expected, as long, at least, as the conditions are expressed in terms of the quantities in the integrand. We know of no necessary and sufficient conditions in the case of the analogous problem in one dimension. Thus no knowledge of the nature of the derivate of a function constitutes a necessary as well as sufficient condition that its primitive function should be the integral of it. The analogy between the cases of one and two dimensions is not only of interest but it appears to belong to the nature of things: we are in presence, in fact, of a generalisation other than the usual one, of a function of a single variable, namely, of a functional relation between two planes which generalises for two dimensions the notion of a function of bounded variation of a single variable, and also that of integral. For the development of this idea I must refer my readers to a communication which I am making elsewhere.

7. I have not contented myself with proving these results in this general form, but I have begun by shewing how, by simpler methods, results, which, though less general than these, are none the less considerably more general than any hitherto formulated, may be obtained. I have hoped in this way to be of service to the worker in a double capacity, both as pioneer and guide, and this the more because I was actually led to the most general of my results by passing through the intermediate stages.

8. The expression for an n-dimensional volume which corresponds to that for an area in two dimensions, is formally an immediate generalisation of the latter. The actual proof of the more general formula, which involves for its proper comprehension a somewhat more developed analytical machinery, and a certain insight into the nature of the concept of surface, volume, and their generalisations, I have thought best to defer till after the publication of some of my results connected with the theory of these concepts. I have also not entered into the general question, referred to above, of the transformation of the variables in a multiple integral, connected as it is with the generalisation of the formula for, and of the concept of, a hyper-volume.

PART II.—*On the Notion of Area.*

10. As indicated in the introduction, we define the area of a curve as follows :—

Let the curve be defined by

$$x = x(u), \quad y = y(u),$$

where $x(u)$ and $y(u)$ are continuous one-valued functions of u, for values of u in the interval $(a \leqslant u \leqslant b)$, having the same values at a and b, viz.

$$x(a) = x(b), \quad y(a) = y(b),$$

so that the curve is *closed*, and may be called a *circuit*, or *contour*.

Divide the range of u up into any finite number of stretches, by means of dividing points $p_0 = a_1, p_1, p_2, \ldots, p_m = b$, each stretch being of length less than a certain norm e. Join the corresponding points

$$A, P_1, P_2, \ldots, P_{m-1}, A$$

on the curve to form an inscribed polygon.

Imagine forces, represented in magnitude, line of action, and sense, by the sides of this polygon in order, supposed determined in the sense of u increasing.

Denote by F the sum of the moments of these forces about any point in the plane, *i.e.* the moment of the resulting couple, counted positive when anti-clockwise in sense. Then, if, as the norm e tends to zero, so that $m \to \infty$, the number F has a unique limit $2A$, the limit A is called *the area of the curve.*

It is immediately evident that, with this definition, the area of the perimeter of a triangle is the area as defined in elementary geometry of the triangle, and that the area of an ordinary polygonal line, bounding a simply connected region is the area in the usual sense of that region.

THEOREM.—*If $x = x(u)$, $y = y(u)$ be any rectifiable* closed curve,† it necessarily possesses an area. Moreover this area is given by the formula*

$$A = \tfrac{1}{2}\int_a^b \{x(u)\,dy(u) - y(u)\,dx(u)\} \ \ldots,$$

* More generally, if *one* of the continuous functions $x(u)$ and $y(u)$ has bounded variation (see footnote to § 4), in which case also the expression (I) is perfectly defined.

† Jordan curve in the general sense, not necessarily *simple*.

the integrals being "Riemann" integrals, taken with respect to the functions of bounded variation $x(u)$ and $y(u)$.

Let x_0, y_0 be any fixed point in the plane of (x, y). Then the algebraic sum of the moments of the forces about (x_0, y_0) is equal to

$$F = \Sigma \begin{vmatrix} x_0, & x(u), & x(u+h) \\ y_0, & y(u), & y(u+h) \\ 1, & 1, & 1 \end{vmatrix} = \Sigma \begin{vmatrix} x(u)-x_0, & x(u+h)-x(u) \\ y(u)-y_0, & y(u+h)-y(u) \end{vmatrix},$$

where u is the coordinate appertaining to any one of the points of division $p_0, p_1, \ldots, p_{m-1}$, and $u+h$ that appertaining to the next point.

The coefficient of x_0 is

$$-\Sigma\{y(u+h)-y(u)\} = 0;$$

similarly that of y_0 is zero. Thus

$$F = \Sigma\{y(u+h)x(u)-y(u)x(u+h)\}$$
$$= \Sigma[\{y(u+h)-y(u)\}x(u)-\{x(u+h)-x(u)\}y(u)].$$

But, since the curve is rectifiable, the continuous functions $x(u)$ and $y(u)$ have bounded variation,* and therefore each has a "Riemann" integral with respect to the other. In other words,

$$\Sigma\{x(u+h)-x(u)\}y(u)$$

tends, when the norm e tends to zero, to a unique limit, and this is the Riemann integral

$$\int_a^b y(u)\,dx(u),$$

and a similar statement holds, with x and y interchanged. Therefore, as $e \to 0$,

$$F \to \int \{x(u)\,dy(u)-y(u)\,dx(u)\}.$$

This proves the theorem.

11. We now consider a correspondence between two planes, in which

* See footnote *supra*: it is only necessary that one of the functions should have bounded variation for the argument here.

the coordinates are respectively (x, y) and (u, v), defined by the equations

$$x = x(u, v), \qquad y = y(u, v).$$

Denote by the term *fundamental (u, v)-rectangle*, the rectangle $(a, c;\ b, d)$, that is $(a \leqslant u \leqslant c)$, $(b \leqslant v \leqslant d)$ in the (u, v)-plane, throughout which the correspondence is supposed to hold.

Corresponding to this rectangle we shall obtain a certain configuration of points in the (x, y)-plane, differing very essentially in character, in general, from a rectangle; we shall call this configuration *the rectangle-image*, and denote it by S.

We shall not suppose the correspondence (1, 1) in the two planes, but we shall at once make the hypothesis that *the image of every stretch on a straight line, parallel to a side of the (u, v)-rectangle, and terminated on its periphery is a rectifiable* curve.* This is equivalent to the condition that $x(u, v)$ and $y(u, v)$ should both be continuous functions of bounded variation of u for each fixed value of v, and also continuous functions of bounded variation of v, for each fixed value of u.

With this understanding it follows at once from the previous article, that the curve in the (x, y)-plane which is the image of the perimeter of any rectangle in the (u, v)-plane, whose sides are parallel to the axes of u and v, has an area.

In the case of the perimeter of the fundamental (u, v)-rectangle, the area of the perimeter-image is moreover given by the formula

$$\begin{aligned} A = & \int_a^c \{x(u, b)\, dy(u, b) - y(u, b)\, dx(u, b)\} \\ & + \int_b^d \{x(c, v)\, dy(c, v) - y(c, v)\, dx(c, v)\} \\ & - \int_a^c \{x(u, d)\, dy(u, d) - y(u, d)\, dx(u, d)\} \\ & - \int_b^d \{x(a, v)\, dy(a, v) - y(a, v)\, dx(a, v)\}, \end{aligned} \tag{II}$$

which, if $x(u, v)$ and $y(u, v)$ are integrals with respect to v, when u is

* Or at least such a more general curve that the above formula holds, see previous footnotes.

constant, and integrals with respect to u, when v is constant, becomes*

$$A = \int_a^c \left\{ x(u, b)\frac{\partial y(u, b)}{\partial u} - x(u, d)\frac{\partial y(u, d)}{\partial u} - y(u, b)\frac{\partial x(u, b)}{\partial u} + y(u, d)\frac{\partial x(u, d)}{\partial u} \right\} du$$

$$+\int_b^d \left\{ x(c, v)\frac{\partial y(c, v)}{\partial v} - x(a, v)\frac{\partial y(a, v)}{\partial v} - y(c, v)\frac{\partial x(c, v)}{\partial v} + y(a, v)\frac{\partial x(a, v)}{\partial v} \right\} dv. \quad \text{(II}'\text{)}$$

12. Our object is to transform this expression for A into an integral of the form

$$A = \int_a^c du \int_b^d \frac{\partial(x, y)}{\partial(u, v)} du\, dv, \quad \text{(III)}$$

with the minimum of additional conditions which the circumstances permit.

If we make sufficient additional assumptions the transformation is almost immediate.

Thus, if we suppose that, not only $x(u, v)$ and $y(u, v)$, but also their partial derivates $\frac{\partial x}{\partial u}$, $\frac{\partial y}{\partial u}$, are integrals with respect to v, and $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial v}$ integrals with respect to u, so that also $x\frac{\partial y}{\partial u}$ and $y\frac{\partial x}{\partial u}$ are integrals with respect to v, and $x\frac{\partial y}{\partial v}$ and $y\frac{\partial x}{\partial v}$ are integrals with respect to u, we may directly carry out the process of transformation of (II′) into (III), by writing (II′) in the form

$$A = \int_a^c du \int_b^d \frac{\partial}{\partial v} \left\{ y(u, v)\frac{\partial x(u, v)}{\partial u} - x(u, v)\frac{\partial y(u, v)}{\partial u} \right\} dv$$

$$-\int_b^d dv \int_a^c \frac{\partial}{\partial u} \left\{ y(u, v)\frac{\partial x(u, v)}{\partial v} - x(u, v)\frac{\partial y(u, v)}{\partial v} \right\} du,$$

* Readers of my memoir on "Integration with respect to a Function of Bounded Variation" in these *Proceedings*, Ser. 2, Vol. 13 (1914), pp. 111–150, should notice that the formula

$$\int_a^b f(x)\, dg(x) = \int_a^b f(x)\, g'(x)\, dx,$$

when $g(x)$ is an integral, though not expressly given there, follows easily, by the methods there explained, from the particular case in which $f(x)$ is a simple l or u function, in which case the formula is manifestly true.

which, when we differentiate out and cancel the terms that appear in each of the two integrals, reduces to (III).

We notice that this direct process of transformation fails, when the mixed repeated differential coefficients $\frac{\partial^2 y}{\partial u \partial v}$, $\frac{\partial^2 x}{\partial u \partial v}$, do not exist at every point of some set of positive content: even when they exist *almost everywhere*, the process fails if $\frac{\partial x}{\partial u}$, $\frac{\partial y}{\partial u}$ are not integrals with respect to v, and $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial v}$ integrals with respect to u.

13. We have accordingly to employ an indirect method, if we are to obtain results of anything like a general character.

For this purpose it is convenient to use the fundamental property of a system of forces, namely that it is not altered by the introduction of any number of pairs of equal opposite forces.

Imagine the fundamental (u, v)-rectangle divided up in any manner into a finite number of sub-rectangles, by parallels to the axes of u and v. Then divide each sub-rectangle into a pair of triangles by the diagonal which intercepts positive lengths on both axes (*i.e.* sloping down from left to right).

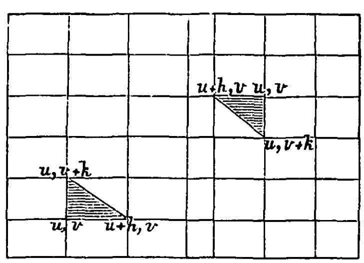

FIG. 1.

Denoting by (u, v) that vertex of such a triangle at which there is a right angle, and by $(u+h, v)$ and $(u, v+k)$ the other vertices, the area of the triangle is $\frac{1}{2}hk$. Adjoined to each such point (u, v) of division, there will then be two* pairs (h, k), in one of which h and k are both positive, and in the other both negative (Fig. 1).

We shall denote by $\bar{h}$ and $\bar{k}$ the greatest of the absolute values of all

* Exception of course made of the points (u, v) marked on the perimeter of the fundamental rectangle, which have each one pair (h, k). It will be readily seen that this does not affect our reasoning, and need not therefore be specially referred to in detail.

the h's and k's which occur in this division of the whole fundamental rectangle into sub-rectangles. The vertices of the triangles we shall for brevity refer to as the "marked points" at this stage.

14. Let the corresponding marked points in the (x, y)-plane in the same order as their correspondents,

$$(u, v), \quad (u+h, v), \quad (u, v+k)$$

be

$$(x, y), \quad (x+\Delta x, y+\Delta y), \quad (x+\Delta' x, y+\Delta' y).$$

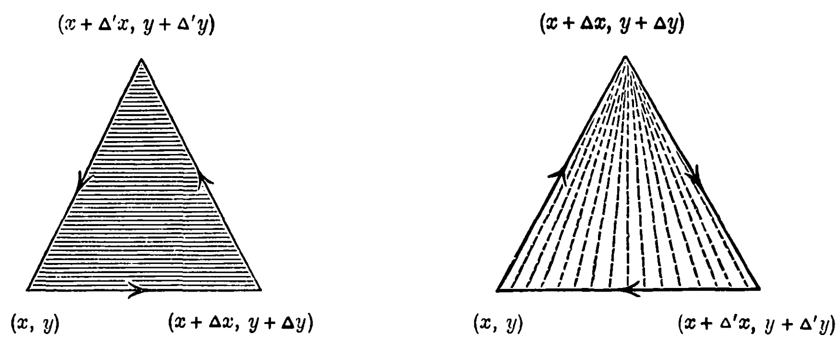

FIG. 2.

With the usual convention as to *sign*, the area being here a directed quantity, the area of the triangle in the (x, y)-plane having these vertices is

$$\frac{1}{2}\begin{vmatrix} x, & y, & 1 \\ x+\Delta x, & y+\Delta y, & 1 \\ x+\Delta' x, & y+\Delta' y, & 1 \end{vmatrix}$$

$$= \tfrac{1}{2}(\Delta x\, \Delta' y - \Delta' x\, \Delta y) = \tfrac{1}{2}\{x(u+h, v) - x(u, v)\}\{y(u, v+k) - y(u, v)\}$$
$$- \tfrac{1}{2}\{x(u, v+k) - x(u, v)\}\{y(u+h, v) - y(u, v)\},$$

and is positive or negative according as the order of the vertices so arranged is anticlockwise or clockwise. The area may even be zero, if the three points in the (x, y)-plane are collinear, or if two or more coincide.

15. It will be remarked that it is only the vertices (*marked points*) of the triangles in the two planes which "correspond", not the sides. It is convenient to speak of the two triangles as being "related". The "rela-

tion" is completely determined by, but not identical in general with, our "correspondence". It is only the *three-point*, so to speak, in the (x, y)-plane, and not *the three-side* which is the image of the "related" figure in the (u, v)-plane, images being determined by the correspondence

$$x = x(u, v), \quad y = y(u, v).$$

It is clear, however, that we can utilise the areas of these triangles for the purpose we have in view. In fact the sum of their areas, taken with proper sign, is precisely half the moment of the system of forces represented by the polygonal figure in the (x, y)-plane whose vertices correspond to the marked points in order on the perimeter of the fundamental (u, v)-rectangle. To see this we merely have to replace the directed area of each triangle, regarded as a couple, by forces along the sides, represented also in magnitude, line of action and sense by these sides, the sense being that given by the order of the vertices. Each side of such a triangle which is not "related" to a stretch on the perimeter of the fundamental (u, v)-rectangle, will occur as a common side of two triangles, as in the (u, v)-plane, and the two forces along it will necessarily be of opposite sense, and accordingly cut one another out.

We thus arrive at the conclusion that the area A for which we are seeking a second expression, this time as a double integral, is the limit of the area of the polygonal figure, when $\bar{h} \to 0$, $\bar{k} \to 0$.

16. Along the perimeter of the fundamental (u, v)-rectangle, x and y are functions of a single variable t, which for two of the sides of the rectangle is denoted by u, and the increment by h, and for the other two sides of the rectangle is denoted by v, and the increment by k.

It follows accordingly, by the theory of the "Riemann" integral, since A has been expressed by the sum of such integrals (Formula II), that it is immaterial whether we make the $\bar{h}$'s and $\bar{k}$'s approach zero simultaneously or consecutively. The sum of the areas of the triangles has accordingly necessarily a unique double limit, while it is open to us to calculate this double limit by any process by which $\bar{h}$ and $\bar{k}$ approach zero.

We shall find it convenient to make first one of these quantities tend to zero, and afterwards the other. We are thus able to deal with the separate terms which occur in the expression for the area of each triangle.

It is not indeed evident, and probably not true, that, on the hypotheses which we propose to make, the double limit is unique, unless all the terms due to all the triangles are taken into account at one and the same time.

17. We have thus to consider the summation of pairs of expressions, one member of each pair being of the type

$$\tfrac{1}{2}\{x(u+h, v)-x(u, v)\}\{y(u, v+k)-y(u, v)\}$$

$$-\tfrac{1}{2}\{x(u, v+k)-x(u, v)\}\{y(u+h, v)-y(u, v)\},$$

where $h > 0$, $k > 0$, and the other of a similar type, except that then $h < 0$, $k < 0$.

We proceed to discuss the summation formed from one member of each pair separately, and write, with an obvious notation

$$F_1 = \tfrac{1}{2}\sum_a^c \sum_b^d \{x(u+h, v)-x(u, v)\}\{y(u, v+k)-y(u, v)\},$$

$$(h > 0,\ k > 0) \text{ or } (h < 0,\ k < 0).$$

It follows by the theory of the "Riemann" integral, as already utilised, that, equally whether h and k are always positive in the summation, or always negative, this expression tends, if we make $\bar{k} \to 0$, to

$$\tfrac{1}{2}\int_b^d \sum_a^c \{x(u+h, v)-x(u, v)\}\, dy(u, v).$$

Accordingly, treating the other terms in the same way, we get

$$A = \underset{h\to 0}{\mathrm{Lt}}\ \tfrac{1}{2}\int_b^d \Big[\sum_a^c \{x(u+h, v)-x(u, v)\}\, dy(u, v)$$

$$-\sum_a^c \{y(u+h, v)-y(u, v)\}\, dx(u, v)\Big], \quad \text{(IV)}$$

where each (u, v) appears in two terms, one with $h > 0$, and one with $h < 0$.

18. We have now to make such assumptions as will enable us, in the result just obtained, actually to proceed to the limit with $\bar{h}$.

There is, however, one case so simple that it is best treated by means of the double summation.

THEOREM.—*If $x(u, v)$ and $y(u, v)$ are such functions that the partial differential coefficients $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial u}$, and $\frac{\partial y}{\partial v}$ everywhere exist and are continuous functions of (u, v), then the area A of the rectangle-image is given by the expression*

$$A = \int_a^c du \int_b^d \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv.$$

By the Theorem of the Mean for Differential Coefficients, we may write

$$F_1 = \tfrac{1}{2} \sum_a^c \sum_b^d \frac{\partial x(u+\theta_1 h, v)}{\partial u} \frac{\partial y(u, v+\theta_2 k)}{\partial v} hk,$$

where $$|\theta_1| \leqslant 1, \quad |\theta_2| \leqslant 1.$$

Now we may suppose the division of the fundamental (u, v)-rectangle such that the oscillation of each of the partial differential coefficients $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$ is less than ϵ [since these differential coefficients are continuous functions of (u, v)], in each of the sub-rectangles into which the fundamental (u, v)-rectangle has been divided.

Thus the expression may be written

$$F_1 = \tfrac{1}{2} \sum_a^c \sum_b^d \left\{ \frac{\partial x}{\partial u} + \theta_3 \epsilon \right\} \left\{ \frac{\partial y}{\partial v} + \theta_4 \epsilon \right\} hk.$$

Accordingly the whole area may be obtained by considering a summation of the form

$$\sum_a^c \sum_b^d \left\{ \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right\} hk + \epsilon M,$$

where M is a bounded function of the various quantities concerned, and making all the h's and k's diminish without limit.

By the theory of the double integral, we then necessarily get a quantity which differs from the required double integral by a quantity of the type $\theta \epsilon B$, where $|\theta| \leqslant 1$, and B is a finite constant. As ϵ is as small as we please, this proves the required result.

19. We shall now, as in (II′), suppose that $x(u, v)$ *and* $y(u, v)$ *are each integrals with respect to* v *for each fixed value of* u, *and integrals with respect to* u *for each fixed value of* v.

Then our formula (IV) takes the form*

$$A = \operatorname*{Lt}_{h \to 0} \tfrac{1}{2} \int_b^d \sum_a^c \Big[\{x(u+h, V) - x(u, V)\} \frac{\partial y(u, V)}{\partial V} - \{y(u+h, V) - y(u, V)\} \frac{\partial x(u, V)}{\partial V} \Big] \partial V. \quad \text{(V)}$$

* In formulæ like the present involving (1) a variable with respect to which we sum, assuming a finite number of values at each stage, and therefore in all a countably infinite set of values, dense everywhere, and (2) a continuous variable with respect to which we integrate. I use small letters u, v for the former and capitals U, V for the latter. It may be remarked that, as regards the fixed values, the supposition made so far only applies to the countably infinite sets u, v.

In our method of procedure, and therefore in this formula, we may change u into U, and V into v, if at the same time we interchange h and k, and therefore write $\bar{k}$ in place of $\bar{h}$.

20. What we have first to consider is the circumstances under which we can proceed to the limit in the formula (V) under the sign of integration. We shall employ the known pair of sufficient conditions*

(1) *the integrands have a unique limit, when* $\bar{h} \to 0$, *except for a set of values of* v *of content zero ; and*

(2) *the integrands form a bounded succession, or, more generally, are all numerically less than one and the same summable function.*

It is, of course, not necessary for the existence of the unique limit in (V), that the integrand should have a unique limit. In point of fact in §§ 24–31 we shall utilise this circumstance to obtain more general conditions. It would seem, however, that even the results we are about to obtain in the paragraphs which immediately follow, constitute a considerable advance on any hitherto formulated.

21. Making use of the conditions (1) and (2) just stated, we are able to prove the following theorem :—

THEOREM 2.—*If* $x(u, v)$, $y(u, v)$ *are functions satisfying the following conditions :—*

(i) *The four partial derivates* $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}$ *are bounded functions of* (u, v) ; *and*

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* W. H. Young, "On Semi-integrals and Oscillating Successions of Functions," § 27. More general conditions are given in § 23. I take this opportunity of pointing out that, in § 24 of that paper, the reasoning employed does not prove the theorem stated, but only that obtained from it by adding the restriction that the succession should be bounded below (above). The following corrections are therefore necessary :—

P. 309, between lines 3 and 4, *read* "bounded below (above)," and line 5, *delete* "no."
Line 6, *delete* "positive (negative)," and *insert* "zero for unique double limit."
After line 6, *insert* "If the succession is bounded below, we may suppose the functions all positive, so that it is sufficient to show that there is no positive double limit."
Line 7, *after* "obvious" *insert* "then."
P. 310, § 25. In the enunciation *insert after* "functions" the words "bounded below (above)," and *delete* in the last line "above (below)."

The corresponding corrections in the proof and in § 26 can be easily supplied by the reader.

(ii) $\dfrac{\partial x}{\partial u}$ *and* $\dfrac{\partial y}{\partial u}$ *are continuous with respect to v, except at the points (u, v) of a plane set of content zero ; or*

$\dfrac{\partial x}{\partial v}$ *and* $\dfrac{\partial y}{\partial v}$ *are continuous with respect to u, except at the points (u, v) of a plane set of content zero.**

Suppose first for definiteness that $\dfrac{\partial y}{\partial v}$ and $\dfrac{\partial x}{\partial v}$ have the continuity properties specified in the statement of the theorem, so that, in particular, $\dfrac{\partial y}{\partial v}$ is for each fixed value of v, which does not belong to a certain set of values of v of zero content, continuous with respect to u, except for a set of values of u of zero content.

Then, except for a set of values of v of zero content,

$$\sum_{a}^{c} \{x(u+h,\ V)-x(u,\ V)\}\,\frac{\partial y(u,\ V)}{\partial V},$$

which is the first term in the integrand of V, certainly has a unique limit

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* It is this theorem which corresponds most closely with that obtained by E. W. Hobson in these *Proceedings*, Ser. 2, Vol. 8 (1909), pp. 22–39, "On some Fundamental Properties of Lebesgue Integrals in a Two-Dimensional Domain." More especially pp. 32–39.

Though only a special case of the general theorem obtained below, it appears to differ essentially only from that due to Hobson in the omission of some of Hobson's restrictions. These latter are seen in fact to be unnecessary in the light of the methods employed in the present paper. Among these unnecessary restrictions, it may be particularly noted that in our statement the set of zero content need not be closed, moreover it is not necessary to hypothecate that the corresponding (x, y)-set should have zero content.

The most striking simplification is that which corresponds to the omission of all those conditions entailed by the supposition, necessarily made by Hobson in his method of treatment, that the correspondence between the two planes should be (1, 1), and that further it should be what he calls a "normal" correspondence, this latter involving quite a series of conditions given on p. 37 *loc. cit.*, relating to an intermediate transformation, obtained by eliminating v, between the equations giving x and y in terms of u and v.

It may fairly be contended that this series of conditions is only introduced *ad hoc*. They constitute a part of the scaffolding, and *a priori* reasons would accordingly lead to the conclusion that their removal must be possible. If I lay stress on this point it is because nothing could better illustrate the necessity of a treatment of area such as that employed in the present paper.

It may be added that, while in theory the limitations there introduced are serious, in practice the determination of whether the correspondence is "normal" or not, even if the elimination could be carried out, might well present difficulties.

when $h \to 0$. In fact, as $\frac{\partial x}{\partial u}$ is bounded, $x(U, V)$ is an integral with respect to U, and therefore $\frac{\partial y}{\partial V}$, regarded as a function of U, has an integral with respect to $x(U, V)$ for each fixed value of V not belonging to a set of zero content, namely,

$$\int_a^c \frac{\partial y(U, V)}{\partial V}\, dx(U, V) = \int_a^c \frac{\partial x}{\partial U}\,\frac{\partial y}{\partial V}\,\partial U.$$

Indeed the condition for the existence of this "Riemann" integral is that, V being fixed, the variation of $x(U, V)$ over the set of points U where $\frac{\partial y}{\partial V}$ is discontinuous with respect to u, should be zero: this will be the case provided this set of points u has zero content, since $x(u, v)$ is an integral with respect to u. Hence by the definition of a Riemann integral with respect to a function of bounded variation, the first term of the integrand in (V) has this "Riemann" integral for limit, when $\bar{h} \to 0$.

Precisely the same reasoning applies of course to the second term of the integrand in (V), interchanging x and y. Thus this second term tends to

$$\int_a^c \frac{\partial y}{\partial U}\,\frac{\partial x}{\partial V}\,dU,$$

as $h \to 0$.

On the other hand, the first case of condition (2) of § 20 is clearly satisfied, since the integrand in (V) consists only of bounded functions, $x(u, v)$ and $y(u, v)$ being now necessarily continuous functions of (u, v), owing to the boundedness of all their derivates.

If secondly it be $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$ which have the continuity properties, the argument is the same, except that the parts played by u and v are to be interchanged, the corresponding alteration being also effected in the formula (V). In this case $\bar{h}$ tends first to zero and then $\bar{k}$.

Thus our theorem is proved.

22. It should be remarked that our reasoning does not permit us to deduce the same result from the possession of the continuity property by $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial v}$, for example. This will be clear if we reflect that, though A is known to be the unique double limit of our summation, it is not known

that the two constituent summations, whose difference is the whole summation, have each a unique double limit. Our method shews that, with the hypothesis of Theorem 2, they both have repeated limits in the particular order determined by whichever of the alternative hypotheses (ii) we take. The difference of these two repeated limits may be therefore used to calculate A as a repeated limit in the same order. But we should not be justified in asserting that A was the difference of the repeated limit of one constituent summation in one order and the repeated limit in the other order of the second summation.

23. It will be noticed that the reasoning employed in proving Theorem 2 really gives us more than is stated in that theorem. But it does not seem worth while to delay passing on to the general result of the paper. As, however, in Theorem 2, the derivates are all bounded, it is perhaps worth while to give the following simple enunciation, which avoids this assumption, though in some respects its demands are greater than those of Theorem 1.

THEOREM 3.—*If $x(u, v)$ and $y(u, v)$ are functions of bounded variation of (u, v), and are integrals with respect to u for each fixed value of v, and integrals with respect to v for each fixed value of u, then*

$$A = \int_a^c \int_b^d \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv.$$

We may again employ the formula (V).

That the integrand in this formula has a unique limit follows from a fundamental property of a function of bounded variation in two variables,* namely, that the mixed differential coefficients of the second order necessarily exist and are finite, except for a plane set of content zero. Hence it follows that the continuity property in the enunciation of Theorem 2 holds for the partial derivates of the first order, and therefore, as before, each of the two terms in the integrand in (V) leads to a unique limit when $\bar{h} \to 0$.

Further, that we may introduce the limit under the integral sign, is easily seen as follows.

* W. H. Young, "Sur la dérivation des fonctions à variation bornée", *Comptes Rendus*, Vol. 164 (1917).

We may write

$$x(u, v) = x_1(u, v) - x_2(u, v),$$

$$y(u, v) = y_1(u, v) - y_2(u, v),$$

where each of the functions x_1, x_2, y_1 and y_2 is a $(+, +)$-monotone increasing function of (u, v), that is a function whose increment with respect to each variable is a positive ($\geqslant 0$) monotone increasing function of the other variable: the same is therefore true when we replace the increment by the upper or lower derivate, or differential coefficient.

Hence
$$0 \leqslant \frac{\partial y_i(u, v)}{\partial v} \leqslant \frac{\partial y_i(c, v)}{\partial v}, \quad (i = 1 \text{ or } 2),$$

and
$$\frac{\partial y_i(c, v)}{\partial v} = \mu_i(v),$$

being one of the derivates of a monotone function of v, is a summable function of v.

Therefore, for $i = 1$ or 2, $j = 1$ or 2,

$$0 \leqslant \sum_a^c \{x_j(u+h, v) - x_j(u, v)\} \frac{\partial y_i}{\partial v} \leqslant \mu_i(v) \sum_a^c \{x_j(u+h, v) - x_j(u, v)\}$$

$$\leqslant \mu_i(v) \{x_j(c, v) - x_j(a, v)\} \leqslant \mu_i(v) \{x_j(c, d) - x_j(a, b)\}.$$

Thus the first term in the integrand in (V) becomes the sum of four terms with proper signs, each of which is $\geqslant 0$ and not greater than a summable function of v. The same applies to the second term in the integrand of (V).

For each of these terms, therefore, the more general criterion (2) of § 20 is satisfied, while a unique limit when $\bar{h} \to 0$ exists, since x_i and y_i, like x and y, are functions of (u, v) of bounded variation. Thus we may, for each of these eight terms separately, introduce the limit under the integral sign, and therefore we may proceed to the limit under the integral sign, when the integrand is that given in (V). This proves the theorem.

24. Hitherto we have required that the integrand in (V) should, except at a set of content zero, itself have a unique limit when $\bar{h} \to 0$; in other words, we have based our reasoning on the theory of integrable successions of functions. The moment we let this hypothesis drop, we require the methods of the much more difficult theory of the integrability of oscillating successions of functions.

If we suppose the integrand in (V) to be bounded, we are at once sure that the limits of the two terms whose difference is the summation tending to A as limit, must lie between the integrals of the upper and lower limits of the corresponding part of the integrand, and accordingly between the integrals with respect to v of the "Darboux" upper and lower integrals of $\frac{\partial y}{\partial v}$ taken with respect to $x(u, v)$, for constant v, and of $\frac{\partial x}{\partial v}$ taken with respect to $y(u, v)$, for constant v. As the "Lebesgue" integral always lies between the "Darboux" upper and lower integrals, it is natural to attempt to formulate theorems in which the "Lebesgue" integral, though not approached by the integrand is, when integrated with respect to v, the required limit.

25. For our present purpose it is convenient to write the formula (V) as follows:—

$$A = \underset{h\rightarrow}{\text{Lt}}\ \tfrac{1}{2}\int_b^d dV \sum_a^c \left[\int_u^{u+h} \left\{ \frac{\partial x(U, V)}{\partial U}\,\frac{\partial y(u, V)}{\partial V} - \frac{\partial y(U, V)}{\partial U}\,\frac{\partial x(u, V)}{\partial V} \right\} dU. \right. \tag{VI}$$

We shall now transform the two terms under the limit, writing, for instance, the first of these in the form

$$\int_b^d dV \int_a^c \frac{\partial x(U, V)}{\partial U}\,\frac{\partial \phi_r(U, V)}{\partial V}\, dU, \tag{α}$$

where we have introduced an auxiliary function ϕ_r at the r-th stage in our passage to the limit, when $\bar{h}$ assumes the value $\bar{h}_r$, of a certain sequence

$$\bar{h}_1 > \bar{h}_2 > \dots > \bar{h}_r > \dots \rightarrow 0.$$

This function $\phi_r(U, V)$ is defined as equal to $y(u, V)$ in the half-open interval $(u \leqslant U < u+h)$, for each of these intervals in the division made at the r-th stage. *Accordingly $\phi_r(U, V)$ is, for constant V, constant in stretches.*

Taking any point U, such that

$$u < U < u+h,$$

we then have

$$\phi_r(U, V) = y(u, V), \tag{β}$$

so that, *for constant* U, $\phi_r(U, V)$ *is an integral*, and

$$\frac{\partial\phi_r(U, V)}{\partial V} = \frac{\partial y(u, V)}{\partial V}, \qquad (\gamma)$$

which gives the expression (a) as equivalent to the first term under the limit sign in (VI).

26. As we proceed, stage by stage, the point U being fixed, the point u will either coincide with U, or will move up to u as limit. In either case, since $y(U, V)$ is a continuous function of U, $y(u, V)$ tends to $y(U, V)$ as unique limit, when $u \to U$. Thus

$$\operatorname*{Lt}_{r\to\infty} \phi_r(U, V) = y(U, V).$$

Since therefore $y(U, V)$ and $\phi_r(U, V)$ are both integrals with respect to V, we may write

$$\operatorname*{Lt}_{r\to\infty} \int_b^V \frac{\partial\phi_r(U, V)}{\partial V}\, dV = \int_b^V \frac{\partial y(U, V)}{\partial V}\, dV. \qquad (\delta)$$

27. We consider first, for simplicity, the case when the partial derivates of x and y with respect to u and v are all bounded. It will be seen that we require no further condition, if this be assumed. We have, in fact, the following theorem :—

THEOREM 4.—*If the partial derivates* $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial u}$, $\frac{\partial y}{\partial v}$ *are all bounded, then*

$$A = \int_a^c \int_b^d \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv.$$

For, if the derivates all lie between $\pm M$, the functions $\frac{\partial\phi_r}{\partial V}$ also lie between $\pm M$, for all values of r, U and V, by reason of (γ): they form therefore, as $r \to \infty$, a bounded succession, whose integrals converge to an integral, by the relation (δ). By a well-known theorem,* we may then insert the bounded function $\frac{\partial u(U, V)}{\partial U}$ under the integral sign on both sides of the relation (δ). Thus

$$\operatorname*{Lt}_{r\to\infty} \int_b^d \frac{\partial x(U, V)}{\partial U} \frac{\partial\phi_r(U, V)}{\partial V}\, dV = \int_b^d \frac{\partial x(U, V)}{\partial U} \frac{\partial y(U, V)}{\partial V}\, dV.$$

* W. H. Young, "Successions of Integrals and Fourier Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 11 (1912), p. 62, see also below § 32.

Since these integrands, and therefore the integrals, are bounded functions of U, we may now integrate term-by-term with respect to U, and get

$$\operatorname*{Lt}_{r\to\infty}\int_a^c dU\int_b^d \frac{\partial x}{\partial U}\frac{\partial \phi_r}{\partial V}dV = \int_a^c dU\int_b^d \frac{\partial x}{\partial U}\frac{\partial y}{\partial V}dV.$$

Changing the order of integration, we see by (a) that the first term under the limit in (VI) leads, when $r\to\infty$, that is $\bar{h}\to 0$, to the integral on the right in the last equation. Interchanging x and y, as we may, since the hypotheses are symmetrical, we get the second term in (VI). Thus, subtracting, we get the required result.

28. We have stated and proved Theorem 4 separately, because of its great simplicity of treatment, and the relative ease with which it is proved, oscillating successions being particularly easy to deal with when they are bounded. This facility, however, is really largely due to the fact that we are then able to assert with certainty that the integral $\int_E f_n(x)\,dx$ of the typical function $f_n(x)$ of the succession over a set E, has, as the content E tends to zero, and simultaneously $n\to\infty$, the unique double limit zero. Now the convergence of this integral to zero is also secured when $|f_n(x)|$, without being bounded, is less than or equal to a summable function, independent of n, and we are naturally led to enquire whether the condition of boundedness of the partial derivates may not be replaced by the corresponding more general condition. In the next theorem it will be proved that this is the case. We must, however, it will be noticed, specifically introduce into the conditions the fact that x and y are integrals with respect to each variable separately, a condition which fulfils itself when the derivates are bounded.

THEOREM 5.—*If*

(i) *$x(u, v)$ and $y(u, v)$ are integrals with respect to u, for each fixed value of v, and integrals with respect to v, for each fixed value of u;*

(ii) *each of the partial derivates of x and y with respect to v is, except at most for a set of values of v of content zero, numerically less than, or equal to, a summable function of v independent of u;**

* We might only postulate this condition for a countably infinite set of values of u, dense everywhere. We should, however, only formally gain in generality, since when the condition is satisfied in this form, it is satisfied in the form given in the text.

In fact, if for a countably infinite set of values of u, dense everywhere, we have, except

(iii) *the same as* (ii) *with u and v interchanged*;

then
$$A = \int_a^c \int_b^d \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv.$$

29. In the proof of Theorem 5 we shall make use of the following theorem on the integration of oscillating successions of functions, the proof will be given in a subsequent article:

If the integrals of a succession of summable functions $f_1(x), f_2(x), \ldots$ converge to an integral,

$$\underset{n \to \infty}{\mathrm{Lt}} \int f_n(x)\, dx = \int q(x)\, dx,$$

then we may insert under the integral signs any the same function $g(x)$, provided $\int_E f_n(x)dx$ and $\int_E f_n(x)\, g(x)\, dx$, both have the unique double limit zero, when $E \to 0$, $n \to \infty$, and the various integrals exist.

30. To prove Theorem 5, we start once more with the formula (VI),

for a set of values of V of content zero,

$$-\mu(V) \leqslant \frac{\partial}{\partial v} y(u, V) \leqslant \mu(V), \tag{1}$$

we see, integrating with respect to V, that

$$y(u, V) + \int \mu(V)\, dV \quad \text{and} \quad \int \mu(V)\, dV - y(u, V)$$

are monotone increasing functions of V. But, if we write

$$Z_1(U, V) = y(U, V) + \int \mu(V)\, dV,$$

$Z_1(U, V)$ will be, like $y(U, V)$, a continuous function of U, and therefore a monotone increasing function of V, since it is the limit of $Z_1(u, V)$, when u tends to U as limit by means of the points of our countable set. Similarly,

$$Z_2(U, V) = \int \mu(V)\, dV - y(U, V)$$

is a monotone increasing function of V. Therefore, except for a set of values of V of content zero,

$$-\mu(V) \leqslant \frac{\partial}{\partial v} y(U, V) \leqslant \mu(V), \tag{1a}$$

for *all* values of U. This proves the above statement.—[*Note added May* 11*th*, 1919.]

and consider the expression (a) of § 25, viz.

$$\int_b^d dV \int_a^c \frac{\partial x(U, V)}{\partial U} \frac{\partial \phi_r(U, V)}{\partial V} dU. \tag{a}$$

Supposing $\lambda(U)$ and $\mu(V)$ to be the functions hypothecated greater than $\left|\frac{\partial x(U, V)}{\partial U}\right|$ and $\left|\frac{\partial y(U, V)}{\partial V}\right|$ respectively, we have, by (γ),

$$\left|\frac{\partial x(U, V)}{\partial U} \frac{\partial \phi_r(U, V)}{\partial V}\right| = \left|\frac{\partial x(U, V)}{\partial U} \frac{\partial y(u, V)}{\partial V}\right| < \lambda(U)\mu(V).$$

Therefore, as $E \to 0$ and $r \to \infty$, we have, integrating over a set E,

$$\left|\int_E \frac{\partial \phi_r(U, V)}{\partial V} \partial V\right| \leqslant \int_E \mu(V) dV \to 0, \tag{1}$$

and
$$\left|\int_E \frac{\partial x(U, V)}{\partial U} \frac{\partial \phi_r(U, V)}{\partial V} \partial V\right| \leqslant \lambda(U) \int_E \mu(V) dV \to 0. \tag{2}$$

Since then the integrals of the succession of functions $\frac{\partial \phi_r(U, V)}{\partial V}$, for constant U, converge to an integral, namely, as shewn in § 25,

$$\underset{r \to \infty}{\mathrm{Lt}} \int \frac{\partial \phi_r(U, V)}{\partial V} \partial V = \int \frac{\partial y(U, V)}{\partial V} \partial V, \tag{δ}$$

it follows from the theorem on oscillating successions just quoted, that, for every fixed value of U,

$$\underset{r \to \infty}{\mathrm{Lt}} \int_b^d \frac{\partial x(U, V)}{\partial U} \frac{\partial \phi_r(U, V)}{\partial V} \partial V = \int_b^d \frac{\partial x(U, V)}{\partial U} \frac{\partial y(U, V)}{\partial V} \partial V,$$

since these integrals certainly exist.

Moreover, since each of these integrals is numerically

$$\leqslant \lambda(U) \int_b^d \mu(V) dV,$$

that is a summable function of U, we may integrate the equation last obtained term-by-term with respect to U, and write

$$\underset{r \to \infty}{\mathrm{Lt}} \int_a^c dU \int_b^d \frac{\partial x}{\partial U} \frac{\partial \phi_r}{\partial V} dV = \int_a^c dU \underset{r \to \infty}{\mathrm{Lt}} \int_b^d \frac{\partial x}{\partial U} \frac{\partial \phi_r}{\partial V} dV = \int_a^c dU \int_b^d \frac{\partial x}{\partial U} \frac{\partial y}{\partial V} dV,$$

whence, the hypotheses being symmetrical with respect to x and y, the required result follows, as before (§ 27), from (VI).

31. *We may here remark that the assumptions made in Theorem* 5 *secure the boundedness with respect to* (u, v) *of the variation of each of the dependent variables* x *and* y *with respect to one of the independent variables* u, v, *the other being regarded as a parameter.* Hence also x and y are themselves bounded functions of (u, v).

Indeed, by (i), the total variation of x is $\int \left| \frac{\partial x}{\partial u} \right| du$, v being regarded as a parameter, and

$$|x(u, v) - x(a, v)| = \left| \int_a^x \frac{\partial x}{\partial u} du \right| \leqslant \int_a^x \left| \frac{\partial x}{\partial u} \right| du \leqslant \int_a^c \lambda(u)\, du,$$

where $\lambda(u)$ is the summable function hypothecated as $\geqslant \left| \frac{\partial x}{\partial u} \right|$. This suggests that this property, which does not seem to entail necessarily the hypotheses of Theorem V, may be used in place of one of the conditions there imposed. It is easily seen, in fact, that our reasoning in § 30 was more general than is required for the mere proof of Theorem V. We have accordingly the following slightly more general, if less elegant theorem :—

THEOREM VI.—*If*

(i) $x(u, v)$ *and* $y(u, v)$ *are integrals with respect to* u *for each fixed value of* v, *and integrals with respect to* v *for each fixed value of* u ;

(ii) $$\left| \frac{\partial y(u, V)}{\partial V} \right| \leqslant \mu(V), \quad \left| \frac{\partial x(u, V)}{\partial V} \right| \leqslant M(V),$$

where $\mu(V)$ *and* $M(V)$ *are summable functions of* V *independent of* u ;*

(iii) *the total variations of* $x(U, V)$ *and* $y(U, V)$ *with respect to* U *for constant* V *are bounded functions of* V *and therefore of* (U, V) ; *or more generally possess absolutely convergent integrals with respect to the integrals of* $\mu(V)$ *and* $M(V)$ *respectively* ;

then $$A = \int_a^c \int_b^d \frac{\partial(x, y)}{\partial(u, v)} du\, dv.$$

For, if $X(U, V)$ denote the total variation of $x(U, V)$ for constant V,

* This condition need only hold for a countable everywhere dense set of values of u.

we have, by (i),

$$X(U, V) = \int \left| \frac{\partial x(U, V)}{\partial U} \right| dU,$$

and, by (iii), is either a bounded function of V, or is at any rate such that $\mu(V)\, X(U, V)$ is summable with respect to V,

$$\int \mu(V)\, X(U, V) dV = \int dV \int \mu(V) \left| \frac{\partial x(U, V)}{\partial U} \right| dU.$$

Therefore, since the integrand is positive, we may change the order of integration, and write

$$\int dU \int \mu(V) \left| \frac{\partial x(U, V)}{\partial U} \right| dV = \int dV \int \mu(V) \left| \frac{\partial x(U, V)}{\partial U} \right| dU.$$

Hence
$$u(V) \left| \frac{\partial x(U, V)}{\partial U} \right|$$

is a summable function of V, except possibly for a set of values of U of content zero. Therefore, excepting these values of U,

$$\left| \int_E \frac{\partial x(U, V)}{\partial U} \frac{\partial \phi_r(U, V)}{\partial V} dV \right| \leqslant \int_E \left| \frac{\partial x(U, V)}{\partial U} \right| \mu(V) dV \to 0,$$

as in (2) in the proof of Theorem 5. The relation (1) of course still holds, as well as the relation (δ) of § 26. Therefore except for a set of values of U of content zero,

$$\underset{r \to \infty}{\text{Lt}} \int_b^d \frac{\partial x}{\partial U} \frac{\partial \phi_r}{\partial V} dV = \int_b^d \frac{\partial x}{\partial U} \frac{\partial y}{\partial V} \partial V.$$

But these integrals, which certainly exist, are numerically

$$\leqslant \int_b^d \left| \frac{\partial x}{\partial U} \right| \mu(V)\, dV,$$

which is a summable function of U, its integral being, as we saw,

$$= \int_b^d \mu(V)\, X(U, V)\, dV.$$

Therefore we may integrate our equation term-by-term with respect to U, whence as before the theorem follows.

PART III.—*On Oscillating Successions of Functions.*

32. We now proceed to prove the theorem on successions, quoted in § 29, for any number of variables.

If x denote any ensemble of variables, finite in number [*e.g.* $x = (u, v)$, when we are working in a plane], and the integral sign be understood to imply multiple integration of the proper number of dimensions, a summable function $f(x)$ has the characteristic property that its integral over any set of points of content E tends to zero, as $E \to 0$. We write this

$$\int_E f(x)\, dx \to 0,$$

reserving the symbol $\int f(x) dx$ for indefinite integration over any "interval" (rectangle, block, &c., according to the number of dimensions of x). In particular, if $f(x)$ is independent of one or more of the variables implied in x, this property holds; or if $f(x) = f_1(u) f_2(v)$, is the product of a function of certain of the variables, denoted by u, and a function of the remaining variables, denoted by v.

We have already used [tacitly in (2) of § 19] this property of a summable function of a single variable, in connection with the theorem on successions of functions which states that the analogous condition

$$\underset{\binom{n \to \infty}{E \to 0}}{\text{Lt}} \int_E f_n(x)\, dx = 0,$$

where the limit is now a double one, is sufficient in order that the succession of summable functions $f_1(x)$, $f_2(x)$, ..., which converges except at a set of content zero to a summable function $f(x)$, should be ***absolutely integrable***, that is,

$$\underset{n \to \infty}{\text{Lt}} \int_S f_n(x)\, dx = \int_S f(x)\, dx,$$

integration being over any and every set of points S.

As already remarked, this same condition plays a part of some importance, when we merely know that the *integrals* of the succession converge to an integral. We have, in fact, the following general theorem :—

THEOREM.—*If the integrals of the succession of summable functions*

$$f_1(x),\ f_2(x),\ \ldots,\ f_n(x),\ \ldots$$

converge to an integral

$$\underset{n\to\infty}{\mathrm{Lt}} \int f_n(x)\,dx = \int q(x)\,dx, \tag{1}$$

and the usual condition is satisfied, namely,

$$\underset{\binom{n\to\infty}{E\to 0}}{\mathrm{Lt}} \int_E f_n(x)\,dx = 0, \tag{2}$$

then we may insert under the integral sign in (i) *any the same function* $g(x)$, *such that all the functions* $f_n(x)\,g(x)$, *as well as* $q(x)\,g(x)$, *are summable, and*

$$\underset{\binom{n\to\infty}{E\to 0}}{\mathrm{Lt}} \int_E f_n(x)\,g(x)\,dx = 0.$$

Moreover in this case integration may be over any set S.

COR. 1.—*The theorem holds if for all values of n,*

$$|f_n(x)| \leqslant \phi(x),$$

where $\phi(x)$ *is a summable function of the variables x, such that* $\phi(x)\,g(x)$ *is summable.*

COR. 2.—*If x denote the ensemble of certain variables, denoted by u, and certain other variables, denoted by v, it is sufficient in order that we may insert a function* $g(x)$ *under both integral signs in the relation (supposed to hold)*

$$\int f_n(x)\,dx \to \int q(x)\,dx, \tag{1}$$

if for all values of n,

$$|f_n(x)| < \phi(u), \qquad |q(x)| < \psi(v),$$

where $\phi(u)$ *and* $\psi(v)$ *are functions only of the variables implied in u and v respectively.*

33. The proof of these results in the explicit case of two variables $x = (u, v)$, is given here for reference; the reasoning is perfectly general for any number of variables, when we regard u and v as implying each a combination of other variables.

The proof of the main theorem requires certain Lemmas.

LEMMA 1.—*If*

$$\operatorname*{Lt}_{\binom{n\to\infty}{E\to 0}} \int_E f_n(x)\,dx = 0,$$

then also

$$\operatorname*{Lt}_{\binom{n\to\infty}{E\to 0}} \int_E |f_n(x)|\,dx = 0.$$

For let E be any set and divide it into two parts E'_n and E''_n, such that in E'_n the function $f_n(u, v)$ is $\geqslant 0$, and in E''_n it is < 0. Then, if $E \to 0$, $E'_n \to 0$, and $E''_n \to 0$. Therefore the first of the above relations is satisfied when we replace E by E'_n, or by E''_n; subtracting we therefore get the second of the above relations.

LEMMA 2.—*If, integrating over any rectangle,*

$$\int f_n(x)\,dx \to \int q(x)\,dx, \quad (n \to \infty), \tag{1}$$

then the same is true over any set of points, provided

$$\operatorname*{Lt}_{\binom{n\to\infty}{E\to 0}} \int_E f_n(x)\,dx = 0. \tag{2}$$

Indeed (1) being true over any rectangle, is true over any finite number of non-overlapping rectangles. Now, if S be any closed plane set, we can enclose it in a finite number of squares, having points of S as centres, and sides of length e parallel to any two given directions at right angles. These squares in general overlap; but producing their sides indefinitely, we divide them up into non-overlapping rectangles, finite in number, containing the whole set S inside them or on their perimeters.

Denoting these rectangles by D and the complementary set to S in these rectangles by E, we see that, as $e \to 0$, the content E also tends to zero. For otherwise there would be at least one point belonging to all the sets E, and this is not the case, since S is a closed set, and is therefore itself the complete inner limiting set of the set of squares, as $e \to 0$.

Thus $$\int_D f_n(x)\,dx = \int_S f_n(x)\,dx + \int_E f_n(x)\,dx,$$

where the latter integral tends, by our hypothesis, to zero when $e \to 0$, and therefore, as we have seen $E \to 0$.

Again, $$\int_D q(x)\,dx = \int_S q(x)\,dx + \int_E q(x)\,dx,$$

where the latter integral also tends to zero, by the fundamental property of a summable function of $x = (u, v)$.

Thus, since (1) has been shewn to hold for integration over D, so that

$$\int_S f_n(x)\,dx + \int_E f_n(x)\,dx \to \int_S q(x)\,dx + \int_E q(x)\,dx,$$

we have in consequence of what has been pointed out

$$\int_S f_n(x)\,dx \to \int_S q(x)\,dx.$$

This proves that (1) holds for integration over any closed set.

But if S is any measurable plane set, it is the sum of a closed plane closed set Σ and a set E whose content may be taken to be as small as we please. Thus we express our double integral over S as the sum of one over Σ and another over E, of which the last tends to zero, if we take a succession of such sets (Σ, E) with $E \to 0$. Since the equation (1) holds for integration over the closed set Σ, we have

$$\int_S f_n(x)\,dx - \int_E f_n(x)\,dx \to \int_S q(x)\,dx - \int_E q(x)\,dx,$$

which leads again to

$$\int_S f_n(x)\,dx \to \int_S q(x)\,dx,$$

since the other two integrals tend to zero when $E \to 0$. This proves the equation (1) for integration over any measurable set S.

LEMMA 3.—*If in addition to the conditions* (1) *and* (2) *of Lemma* 2, *we have*

$$|f_n(u, v)| \leqslant \phi(u), \tag{3}$$

we may take $q(u, v)$ *in* (1) *to be such that*

$$|q(u, v)| \leqslant \phi(u). \tag{4}$$

For, if not, for some point (u, v),

$$\phi(u) < |q(u, v)|,$$

and therefore $$\phi(u) + e < |q(u, v)|,$$

for some value of e.

Let us take together all the points (u, v) at which this last relation holds good for one and the same e, and $q(u) \geqslant 0$, and denote this set by S. We then have, by Lemma 2, for a sufficiently large n,

$$\iint_S [\phi(u)+e]\, du\, dv \leqslant \iint_S q(u, v)\, du\, dv \leqslant \iint_S f_n(u, v)\, du\, dv + \tfrac{1}{2}eS,$$

which is obviously only possible if S is a set of zero content. Since this is true for all positive values of e, it follows that, except at a set of content zero, we have, when $q(u, v)$ is not negative,

$$q(u, v) \leqslant \phi(u).$$

Similarly, except at a set of content zero, when $q(u, v)$ is negative,

$$-\phi(u) \leqslant q(u, v).$$

LEMMA 4.—*If* $$\iint_E f_n(u, v)\, du\, dv \to 0,$$

the succession of integrals

$$\iint_S f_n(u, v)\, du\, dv$$

is bounded and oscillates uniformly and homogeneously.

To prove that the succession of integrals is bounded, let us determine E_0 and u_0, so that, for $E < E_0$ and $u > u_0$,

$$-e < \iint_E f_n(u, v)\, du\, dv < e.$$

Next let us determine an integer k so that

$$kE_0 > (d-b)(c-a).$$

Then, if S be any set in the rectangle $(a, b;\, c, d)$, and we divide the whole rectangle into k equal parts, each of these parts, and therefore each of the sub-sets of S in them, has content less than E_0. Hence, by the above, for $n \geqslant n_0$,

$$-ke < \iint_S f_n(u, v)\, du\, dv < ke.$$

Thus the whole set of integrals, from $u = u_0$ onwards, is bounded, and therefore the whole set is bounded.

To prove the oscillation uniform and homogeneous, let us add to and subtract from the set S any sets E_1, E_2, where $E_1 \to 0$, $E_2 \to 0$. Then, denoting the new set by T,

$$\iint_T f_n(u, v)\,du\,dv = \iint_S f_n(u,v)\,du\,dv + \iint_{E_1} f_n(u, v)\,du\,dv - \iint_{E_2} f_n(u, v)\,du\,dv.$$

By hypothesis the second and third integrals on the right tend to zero. Therefore the lowest and highest double limits on the left are respectively the lowest and highest limits of $\iint_S f_n(u, v)\,du\,dv$. In other words, the succession of integrals oscillates uniformly. Moreover, this is true whether we let n describe all integers or only a sub-set of integers tending to infinity; thus the oscillation is uniform and homogeneous.

This proves the theorem.

COR. 1.—*The results of the theorem are still true if we substitute* $|f_n(x)|$ *for* $f_n(x)$.

COR. 2.—*There is in every succession of the integrals a subsuccession which converges uniformly to an integral.*

COR. 3.—*If the integrals converge, they converge uniformly to an integral.*

34. We can now prove the theorem given in § 32.

THEOREM.—*If* $$\int_a^c \int_b^d f_n(u, v)\,du\,dv \to \int_a^c \int_b^d q(u, v)\,du\,dv, \tag{1}$$

and, when $E \to 0$, $n \to \infty$,

$$\iint_E f_n(u, v)\,du\,dv \to 0, \tag{2}$$

then we may insert in (1) *under each integral sign any function* $g(u, v)$ *such that* $f_n(u, v)\,g(u, v)$ *for all values of* n *and* $g(u, v)\,q(u, v)$ *are summable, and when* $E \to 0$, $n \to \infty$,

$$\iint_E f_n(u, v)\,q(u, v)\,du\,dv \to 0. \tag{3}$$

For, by (2), using Lemma 2,

$$\iint_S f_n(u, v)\,du\,dv \to \iint_S q(u, v)\,du\,dv, \tag{4}$$

where S is any set.

This remains true when we insert any constant c under the integral sign on both sides, and therefore, also, when we add together any finite number of such relations. That is, *denoting by* $g(u, v)$ *a function which assumes only a finite number of values :*

$$\iint_S f_n(u, v)\, g(u, v)\,du\,dv \to \iint_S q(u, v)\, g(u, v)\; du\,dv. \tag{5}$$

But any bounded function $g(u, v)$ being given, we can find a function of the type just considered, say $g_r(u, v)$ differing from $g(u, v)$ at every point by less than $1/r$. Hence, as $n \to 0$,

$$\int_a^c \int_b^d f_n(u, v)\, g(u, v)\, du\,dv - \int_a^c \int_b^d f_n(u, v)\, \{g(u, v) - g_r(u, v)\}\, du\,dv$$

$$\to \int_a^c \int_b^d q(u, v) g(u, v)\, du\,dv - \int_a^c \int_b^d q\,(u, v)\, \{(u, v)\, g_r(u, v)\}\; du\,dv,$$

or, which is the same thing,

$$\int_a^c \int_b^d f_n(u, v) g\,(u, v)\, du\,dv - \theta \int_a^c \int_a^d |f_n(u, v)|\, du\,dv/r$$

$$\to \int_a^c \int_b^d q(u, v) g\,(u, v)\, du\,dv - \theta' \int_a^c \int_b^d |q\,(u, v)|\, du\,dv/r,$$

where $|\theta| \leqslant 1$, $|\theta'| \leqslant 1$.

But, by Lemma 4, the second terms on each side are numerically less than a finite constant divided by r, and therefore are as small as we please, r being large enough. Thus we obtain again the relation (3), $g(u, v)$ being any bounded function.

Finally, when $g(u, v)$ is merely summable, we divide it into the difference of two functions, each of which is $\geqslant 0$ and summable. It is therefore only necessary to consider one of these.

Let then $g(u, v) \geqslant 0$, and define an auxiliary bounded function $g_r(u, v)$ equal to $g(u, v)$ except at the points of the set E where $g(u, v) > r$, since $g(u, v)$ is summable, $E \to 0$ when $r \to \infty$.

Since, by what has been shewn, the theorem is true for $g_r(u, v)$, we have, since $g-g_r=0$, except on E when $g-g_r=g$,

$$\int_a^c\int_b^d f_n(u, v)\, g(u, v)\, du\, dv - \iint_E f_n(u, v) g(u, v)\, du\, dv$$

$$\rightarrow \int_a^c\int_b^d q(u, v)\, g(u, v)\, du\, dv - \iint_E q(u, v) g(u, v)\, du\, dv.$$

By (3), the relation (5) again results, since by hypothesis $g(u, v)\, q(u, v)$ is summable. This proves the relation (5) for any summable function $g(u, v)$ satisfying the conditions of the enunciation. Hence the theorem is true. The proof of the corollaries may be left to the reader.