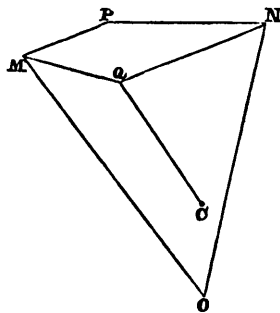


*Extension of Peaucellier's Theorem. By W. H. LAVERY.*

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The general problem is—"Given O and C fixed, and the lengths of OM, ON, CQ, QM, QN, MP, and NP; to determine the locus of P;" but a great simplification is obviously introduced by taking  $MP = MQ$  and  $NP = NQ$ ; for then part of the locus of P coincides with the locus of Q, and a factor divides out of the equation to the locus of P.



Take O the origin of rectangular coordinates, OC the axis of  $y$ , and let the coordinates of C, M, N, Q, P be respectively  $0, \beta; x_1, y_1; x_2, y_2; x', y'; x, y$ ; then the problem is this—"To eliminate  $x'y'x_1y_1x_2y_2$  between the seven equations,

$$x^2 + (y' - \beta)^2 = a^2 \dots\dots\dots (1),$$

$$x_1^2 + y_1^2 = r_1^2 \dots\dots\dots (2),$$

$$x_2^2 + y_2^2 = r_2^2 \dots\dots\dots (3),$$

$$(x' - x_1)^2 + (y' - y_1)^2 = \rho_1^2 \dots\dots\dots (4),$$

$$(x - x_1)^2 + (y - y_1)^2 = \rho_1^2 \dots\dots\dots (5),$$

$$(x' - x_2)^2 + (y' - y_2)^2 = \rho_2^2 \dots\dots\dots (6),$$

$$(x - x_2)^2 + (y - y_2)^2 = \rho_2^2 \dots\dots\dots (7);$$

$a, r_1, r_2, \rho_1$ , and  $\rho_2$  being respectively CQ, OM, ON, MQ (= MP), and NQ (= NP).

The following abbreviations will be found convenient:—

$$\delta_1^2 = \rho_1^2 - r_1^2 + \beta^2 - a^2,$$

$$\delta_2^2 = \rho_2^2 - r_2^2 + \beta^2 - a^2,$$

$$\lambda^2 = \delta_1^2 - \delta_2^2,$$

$$M_1^2 = x^2 + y^2 + r_1^2 - \rho_1^2,$$

$$M_2^2 = x^2 + y^2 + r_2^2 - \rho_2^2;$$

therefore

$$M_2^2 - M_1^2 = \lambda^2,$$

$$\left. \begin{aligned} K^2 &= x^2 + (y - \beta)^2 - a^2 \\ &= M_1^2 + \delta_1^2 - 2\beta y \\ &= M_2^2 + \delta_2^2 - 2\beta y \end{aligned} \right\} \dots\dots\dots (8).$$

[ $K^2$ , obviously, is the factor which is to divide out of the equation.]

$$R_1^2 = (x^2 + y^2) - (r_1 - \rho_1)^2,$$

$$R_2^2 = (x^2 + y^2) - (r_2 - \rho_2)^2,$$

$$S_1^2 = (r_1 + \rho_1)^2 - (x^2 + y^2),$$

$$S_2^2 = (r_2 + \rho_2)^2 - (x^2 + y^2),$$

$$A^2 = R_1 S_1 - R_2 S_2,$$

$$B^4 = \delta_2^2 R_1 S_1 - \delta_1^2 R_2 S_2,$$

$$D^4 = B^4 - 2\lambda^2 \beta x.$$

Now, from equations (2) and (5) find  $x_1$  and  $y_1$ , and from equations (3) and (7),  $x_2$  and  $y_2$ , in terms of  $x$  and  $y$ . Then from equations (4) and (6) determine  $x'$  and  $y'$  in terms of  $x_1$ ,  $y_1$ ,  $x_2$ , and  $y_2$ ; this, by substitution, gives the values of  $x'$  and  $y'$  in terms of  $x$  and  $y$ , substituting which values in equation (1), and substituting for  $M_1^2$  and  $M_2^2$  from equation (8), we have, as the equation to the locus of  $P$ ,

$$a^2 [K^2 A^2 - D^4]^2 = [y K^2 \lambda^2 + x D^4]^2 + [K^2 (x \lambda^2 - \beta A^2) - (y - \beta) D^4]^2.$$

This equation is to be divided through by  $K^2$ ; and all the terms are separately divisible by  $K^2$ , except the various coefficients of  $D^4$ ; the sum of these, however, being

$$-a^2 + x^2 + (y - \beta)^2,$$

is itself  $K^2$ ; therefore, dividing by  $K^2$ , the locus becomes finally

$$a^2 A^2 (K^2 A^2 - 2D^4) = y \lambda^2 (K^2 y \lambda^2 + 2x D^4) \\ + (x \lambda^2 - \beta A^2) \{ K^2 (x \lambda^2 - \beta A^2) - 2(y - \beta) D^4 \} + D^8;$$

or, more conveniently,

$$A^2 \{ (K^2 A^2 - 2D^4) (a^2 - \beta^2) - 2\beta y D^4 \} - D^8 \\ = \lambda^2 (A^2 K^2 + D^4) 2\beta x + \lambda^4 (x^2 + y^2) K^2.$$

As a particular case take  $\delta_1^2 = \delta_2^2 = \delta^2$ ;

then

$$D^4 = A^2 \delta^2,$$

for

$$\lambda^2 = \delta_1^2 - \delta_2^2 = 0;$$

and the general equation reduces to

$$(K^2 - 2\delta^2) (a^2 - \beta^2) - 2\beta y \delta^2 = \delta^4,$$

which is the equation to a circle with its centre on the line joining the fixed points; the circle degenerating into a straight line perpendicular to the fixed line when  $a^2 = \beta^2$ ; i.e., when  $CQ = CO$ .

The result arrived at is, therefore, an extension of Peaucellier's Theorem. This last proves that, when  $OM = ON$  and  $MQ = MP = NQ = NP$ , then the locus of  $P$  is a circle degenerating into a straight line when  $CQ = CO$ . By the present paper a similar result is shewn to be true when the links are bound by the more general conditions

$$MP = MQ, \quad NP = NQ, \quad \text{and} \quad OM^2 - MQ^2 = ON^2 - NQ^2.$$