niceties by a peculiar tenderness of tone. Our regret that the author has not betrayed a perception of some theoretical refinements may well appear equally pedantic. F. Y. EDGEWORTH

Encyklopädie der mathematischen Wissenschaften. Band I, Heft 6. *Anwendungen der Wahrscheinlichkeitsrechnung auf Statistik.* Von LADISLAUS V. BORTKIEWIEZ. Band I, Heft 7. *Anwendungen der Mathematik auf National-ökonomie.* Von V. PARETO.

The German Encyclopædia of Mathematical Sciences has very properly included among its constituent treatises two relating to those branches of the human or moral sciences which require the use of mathematics, namely, the more abstract portions of political economy and statistics.

The scope and function of the mathematical method in political economy are indicated by Professor Pareto with that persuasive lucidity which charactérises his introduction to the subject in his articles in the *Giornale degli Economisti* and subsequent writings, among which we take the opportunity of calling attention to the *résumé* of the course of lectures given at the École des Hautes Études Sociales in Paris (1901–2). The fundamental equations of exchange are derived by Professor Pareto from the conception of utility, or, as he prefers to say, *ophelimity*, which in the mechanics of trade plays nearly the same *rôle* as the concept of force in mathematical physics. "With the equations given for exchange we must combine the equations of production and capitalisation, and thus we obtain the system of equations which completely determine the economic cycle for the case of free competition." The *simultaneity* of these equations was first clearly conceived by Professor Walras, the distinguished predecessor of Professor Pareto in the Chair of Political Economy at Lausanne. Without this conception, as Professor Pareto says, "useful detailed studies" may be attained, "but no insight into the whole system." When the prices, say p_1, p_2, &c., considered as unknown quantities, depend upon constants entering into several simultaneous equations, it is impossible to specify the particular constant which "determines," p_1 for example, and "it is an unfruitful strife when one party maintains that this constant is a_2, another party that it is a_1." The unmathematical economist vainly seeks some single attribute as the condition of price or ground of interest.

Among Professor Pareto's original contributions to the subject we may notice his study on the quantitative data with which the mathematical economist has to deal. As we understand Professor Pareto, these data do not comprise measurements of utility: psychical quantities, unlike physical, cannot be expressed as the sum of so many units. The exercise of *choice*, the preference of the economic man for one combination of goods to another, results in a system of *indifference-curves* which are comparable with the isobars or isotherms of physical science in that each successive curve denotes a greater intensity of the attribute under consideration, but differ in that the economic, unlike the physical, curves cannot be each labelled with a number. There is also to be noticed our author's claim to have proved by mathematical reasoning "what is by no means evident *à priori*," that the methods of production, the values of the "production coefficients" would be the same in an intelligent *régime* of socialism, as it is in the system of *laissez faire*. The dynamics of the economic system offer an attractive subject to one who is penetrated with the thought that "the analogies between mathematical economics and pure mechanics are numerous and far-reaching (*tiefgriefend*)." Elsewhere Professor Pareto has made reconnaissances in the almost unexplored region of economic *dynamics*, understanding that much-abused term in a genuine mathematical, not a vague biological signification. Here the leader contents himself with a Pisgah prospect, while he sagely thus concludes: "It is wiser not to anticipate (*vorzugreifen*) the future. For the present it is only the *static* of political economy that has been scientifically constructed and has produced useful results."

The leading part which is played by the theory of exchange or law of demand-and-supply in abstract economics is matched by the position of the law of error in the higher statistics. Professor Bortkiewiez shows how that law is applied to the solution of certain problems which we will illustrate by an English example, that which is afforded by Jevons' examination of nearly 100,000 sovereigns circulating in different parts of England. Sorted according to their date, these samples presented the following percentages:—

1817–19.	1820–29.	1830–39.	1840–49.	1850–59.	1860–67.
·2	7·4	7·0	16·9	28·6	38·3.[1]

To what degree of accuracy, within what limit of error can we be reasonably sure that if all the sovereigns in England had been counted, the proportions between the numbers of sovereigns bearing dates respectively 1817–19, 1820–29, and so on, would have corresponded to the proportions presented by the sample 100,000? For instance, is it practically certain that the actual number in all England of sovereigns dated 1820–29 was greater than the number of those bearing date 1830–39? How many samples are required to afford this certainty? If the simple treatment of such problems with which Laplace and Poisson were content is to be adopted, it must be granted that the samples were such as would have been obtained if we supposed all the

[1] *Investigations in Currency and Finance*, p. 274. *Cp.* p. 292. The sum of the percentages above quoted are less than 100 by a matter of 1·6 pertaining to Australian sovereigns.

sovereigns in circulation to be collected into one enormous box and a batch of 100,000 coins to be drawn therefrom at random. But things in the concrete are seldom so obligingly simple. Thus, in the case before us, there is reason to believe that the percentages of sovereigns bearing any assigned date were not the same in the different parts of the country where samples were obtained. It is as if the 100,000 samples taken at random were not all taken from one box, but some from one and some from another of several boxes in which the proportions between the number of coins bearing each date were not identical. When there exists this sort of heterogeneity in the sources from which the samples are derived—this abnormality as it has been called with reference to the simplest species of sortition—then the regulation method of eliminating chance, prescribed too generally by the older mathematicians, becomes, as Professor Bortkiewiez observes, "illusory and worthless."

In what cases then may this sort of abnormality be expected, on what conditions does it depend? The answer to this interesting question is given in the work before us, read in connection with the author's important treatise *On the Law of Small Numbers*.[1] We may partially illustrate his theory by our example. Supposing that several hundred thousand, instead of one hundred thousand, sample sovereigns had been taken, then the rule proper to the hypothesis of a simple sortition with as it were a single box, becomes less accurate; it is less accurate also when applied to the class of sovereigns dated 1850–9, or that dated 1840–9, than to the much rarer class of sovereigns dated 1817–9; other things being the same. But what those other things are, it is not easy to state with tolerable brevity in plain prose without the aid of symbols.

Considering how frequently the method of eliminating chance prescribed by Laplace proves illusory, we could wish that Professor Bortkiewiez had pronounced more decidedly upon the validity of a substitute which has been proposed. Suppose that a certain class of observed frequencies, such as birth rates for a series of years, do not behave like the proportions in samples taken from a single box, still may we not apply the law of error to this class of statistics for the solution of problems like those above examined, if we employ a coefficient of dispersion—a standard deviation or modulus—determined not on the "combinational" model, to use Professor Lexis' terminology, but according to the "physical" method, the data for which might have been obtained by observing the dispersion of birth rates at different times and places in previous experience.[2] Doubtless that empirical result would never rest on so large an inductive basis as the combinational coefficient. The procedure too would be at best provisional. We might always hope to break up the material into smaller categories characterised by the more satisfactory species of dispersion.

[1] *Das Gesetz der kleiner Zahlen.* Leipsig: Teubner, 1898.
[2] Cp. *Journal of the Statistical Society.* Dec. 1885.

We have not left ourselves space to consider Professor Bortkiewicz's application of his principles to tables of mortality and sickness. And we can only allude to the statement of general principles given by another authority, Professor Czuber, in the same volume. His concise treatise on the Calculus of Probabilities (Heft I) forms a good introduction to his well-known larger works. F. Y. EDGEWORTH

Abhandlungen zur Theorie der Bevölkerungs -und Moralstatistik. By W. LEXIS. (Jena: Gustav Fischer. 1903. Pp. 253.)

PROFESSOR LEXIS has brought together in this volume some of his earlier writings on statistical problems, and has added chapters designed to produce a consecutive treatment of the philosophy of statistical investigation. The parts, however, are rather disconnected. We have first an illuminating explanation of the author's graphic method of representing the life and death history of successive generations, together with an analytical discussion of the methods of obtaining the statistics which the demographer needs from those which the registrar furnishes, and other matter of purely technical interest; but as soon as we are thoroughly started as in an actuarial text-book, we are diverted to an essay on "The causes of the small fluctuations of statistical ratios," and this leads on to the chapters which deal with the normal law of error, the sex-ratio at birth and its relation to the normal law, and the theoretical examination of the stability of statistical series; while the book closes with a highly abstract discussion of the relation of statistical inquiries to the general methods of natural science.

In spite of the care with which these essays have been compiled, little is added by them to the debt which statisticians already owe the author; for the main lessons taught are the same as in his publications of more than 20 years ago, which have long been recognised as essential parts of modern statistical theory. We have long known that the observed sex-ratios at birth could be reproduced by a suitably arranged game of chance, that the dispersion of mortality about an age near three score and ten has a marked resemblance to the normal law of error, and we have had every opportunity of realising the essential difference between the fluctuations about a stable average in a series, and the variation of the average itself. Beyond these points Professor Lexis does not take us. He passes by the stormy controversy which has raged about the appropriateness of the normal law of error to represent actual statistical observations; he has no new examples to offer of series which do conform to the law, the sex-ratio is still the only one which fully satisfies his demands; he ignores absolutely the second approximation to the normal law, not indeed recognising that for most conceivable circumstances the normal law is essentially only approximate, and though he alludes to Professor Karl Pearson's work, it is only to reject his methods entirely in favour of pure *à priori*