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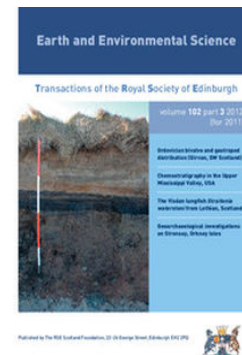
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VIII.—An Analytical Theory of the Equilibrium of an Isotropic Elastic Plate

John Dougall

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VIII.—An Analytical Theory of the Equilibrium of an Isotropic Elastic Plate.

By John Dougall, M.A. *Communicated by* Professor G. A. GIBSON.

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The following paper contains a purely analytical discussion of the problem of the deformation of an isotropic elastic plate under given forces. The problem is an unusually interesting one. It was the first to be attacked (by LAMÉ and CLAPEYRON in 1828) after the establishment of the general equations by NAVIER. The solution of the problem of normal traction given by these authors, when reduced to its simplest form, involves double integrals of simple harmonic functions of the coordinates. The integrals are of complicated form, and practically impossible to interpret, a fact which, without doubt, has had much to do with the neglect of the problem in later times, and the almost complete absence of attempts to establish the approximate theory on the basis of an exact solution. An even more serious defect of LAMÉ and CLAPEYRON'S solution is that the integrals, as they stand, do not converge. A flaw of this sort has often been treated lightly by physical writers, the non-convergence of an integral being regarded as due to the inclusion of an infinite but unimportant constant. In the present case, however, the infinite terms are not constant, but functions of the coordinates, and the modifications necessary to secure convergence, so far from being unimportant, lead directly to the most significant terms of the solution.

The next writer to deal with the exact problem was Sir W. THOMSON, who, at the end of the memoir in which he solved the problem of a spherical shell, indicated the form which the solution would take in the limiting case of a plate. His method, if carried out, would lead to integrals of the same form as LAMÉ'S.

Solutions of special problems have been given by other writers. Prof. LAMB has worked out the solution for an infinite solid subjected to normal pressure proportional to $\cos kx$, and verified in this particular case some of the results of the approximate theory of thin plates (*Proc. Lond. Math. Soc.*, vol. xxi., 1889-90).

The history of the approximate theory is well known and easily accessible. It will be sufficient here to refer to—

- (i) TODHUNTER and PEARSON'S *History of the Elasticity and Strength of Materials*.
- (ii) CLEBSCH'S standard treatise, *Théorie de l'élasticité des corps solides*, as translated by SAINT VENANT; in particular, Part I. chap. iii., and SAINT VENANT'S brilliant note on § 73.
- (iii) Prof. LOVE'S *Treatise on the Theory of Elasticity*, 1892, — especially the historical introductions to both volumes.

The various forms of the approximate theory rest partly upon the general equations of equilibrium, partly upon auxiliary hypotheses or physical principles. These principles are recognised as contained in the general equations, but on account of the

analytical difficulties in the way of deducing them rigorously, they are either simply assumed, or else supported by reasoning plausible rather than demonstrative.

In the following pages the problem is treated as a purely mathematical one, and the approximate theory for a finite plate deduced from an exact solution for an infinite plate. The main features of the method are—

(i) The use of Bessel functions in place of the simple harmonic functions of previous writers. Only the symmetrical forms, or functions of order zero, are required.

(ii) Transformation of the definite integrals, in terms of which the solutions are in the first place obtained, into series, by means of Cauchy's theory of contour integration and residues. The series involve Bessel functions of the second kind with complex argument, and are so highly convergent that the principal features of the strain represented by the solution can be made out with the utmost ease. (The transformations belong to a class discussed systematically, apparently for the first time, in a paper "On the Determination of Green's Function by means of Cylindrical and Spherical Harmonics," *Proc. Edin. Math. Soc.*, vol. xviii.)

(iii) Detailed solutions of the problems of internal force with vanishing face traction. The usual method of dealing with a general problem in Elasticity is to find a particular solution for the bodily force, and then to treat the problem of surface tractions completely. This is theoretically sufficient, but leaves the result in a complicated form, which in the present case must be simplified before practical applications can be made.

(iv) Use of Betti's Theorem (LOVE, *Elasticity*, vol. i. § 140) to develop a method analogous to the method of Green's function in the Theory of the Potential, by which the properties of the solution for a finite plate can be deduced from the infinite plate solution. (Cf. *Proc. Edin. Math. Soc.*, vol. xvi., "On a general Method of Solving the Equations of Elasticity.")

The results of the ordinary theory are fully confirmed, and extended in various directions. The infinite solid solution gives, of course, an exact particular solution for internal force and traction on the plane faces of a finite plate. At the head of the solution appear the terms given by the approximate theory. In the case of flexure, the equation of Lagrange is obtained to a second approximation.

The problem of a finite plate under given edge tractions cannot be completely solved, but exact solutions are given of certain problems relating to a circular plate. For a thin plate, with edge of any shape, the conditions satisfied at the edge by the principal terms of the exact solution are found to a degree of approximation beyond the reach of any theory which rests merely on the "principle of the elastic equivalence of statically equipollent loads." For example, the celebrated boundary conditions given by KIRCHHOFF, in correction of POISSON, are verified, and extended by the inclusion of terms of higher order.

In conclusion, it may be mentioned that the methods given here are equally applicable to the problem of the vibrations of a plate, and to the problems of the equilibrium and vibration of a finite circular cylinder, or of an open spherical shell. Some account of these applications I hope to publish shortly.

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INTRODUCTORY ANALYSIS.

(a) The Bessel function J is defined by the series

$$J_m(z) = \frac{z^m}{2^m \Gamma(m)} \left(1 - \frac{z^2}{2 \cdot 2m+2} + \frac{z^4}{2 \cdot 4 \cdot 2m+2 \cdot 2m+4} - \dots \right)$$

For the function of the second kind we take as definition

$$G_m(z) = \frac{\pi}{2} \frac{J_{-m}(z) - e^{-im\pi} J_m(z)}{\sin m\pi}$$

This makes $G_m z$ an analytic function of m , the value of which, when m is a real integer, is

$$G_m(z) = \left(\log 2 - \gamma + \frac{\pi i}{2} \right) J_m(z) - Y_m(z)$$

where $Y_m(z)$ is Neumann's function.

In this case, therefore, $G_m z = -\log z J_m z +$ a uniform function of z .

In the following pages we are concerned chiefly with the function of order zero

$$G_0 z = \left(\log 2 - \gamma + \frac{\pi i}{2} \right) \left(1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 \cdot 4^2} - \dots \right) - \log z \left(1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 \cdot 4^2} - \dots \right) - \frac{z^2}{2^2} + \left(1 + \frac{1}{2} \right) \frac{z^4}{2^2 \cdot 4^2} - \dots$$

When $\text{mod } z$ is very large, while the phase (argument) of z lies between $-\pi/2$ and $3\pi/2$, then approximately

$$G_m z = e^{-\frac{m\pi i}{2}} e^{i\left(z + \frac{\pi}{4}\right)} \sqrt{\frac{\pi}{2z}}.$$

Similarly, when the phase of z is between 0 and π (excluding those values)

$$J_m z = e^{-\frac{m\pi i}{2}} e^{-i\left(z - \frac{\pi}{4}\right)} \sqrt{\frac{1}{2\pi z}}.$$

(b) If x, y, z and ρ, ω, z are the rectangular and cylindrical coordinates of a point in space, so that $x = \rho \cos \omega, y = \rho \sin \omega$, then the most important property of the Bessel Functions is that each of the eight functions

$$(e^{\kappa z} \text{ or } e^{-\kappa z}) (J_m \kappa \rho \text{ or } G_m \kappa \rho) (\cos m\omega \text{ or } \sin m\omega)$$

satisfies Laplace's equation, or in other words is a potential function.

Hence

$$(\nabla^2 + \kappa^2) \cdot (J_m \kappa \rho \text{ or } G_m \kappa \rho) (\cos m\omega \text{ or } \sin m\omega) = 0.$$

Further, if

$$\begin{aligned} R^2 &= (x - x')^2 + (y - y')^2 \\ &= \rho^2 + \rho'^2 - 2\rho\rho' \cos(\omega - \omega') \end{aligned}$$

then

$$(\nabla^2 + \kappa^2) \cdot (J_0 \kappa R \text{ or } G_0 \kappa R) = 0.$$

Let now $I = \iint G_0 \kappa R f(x', y') dx' dy'$, the integral being taken over a finite area A .

Then $(\nabla^2 + \kappa^2)I = 0$, if (x, y) is without,

but $(\nabla^2 + \kappa^2)I = -2\pi f(x, y)$, if (x, y) is within this area; as easily follows from the theorem

$$\nabla^2 \iint \log R/(x', y') dx' dy' = 2\pi f(x, y).$$

The differential equation satisfied by I , together with the conditions that I and its first derivatives dI/dx , dI/dy are continuous throughout, define the value of the integral completely, and in many cases make its evaluation easy.

(c) For example, take $f(x, y) = J_m \beta \rho \cos m\omega$, with m an integer, so that

$$I = \iint G_m \kappa R J_m \beta \rho' \cos m\omega' \rho' d\rho' d\omega'$$

and suppose the area of integration to be a circle of radius a , with centre at the origin. For convenience in the proof, let the imaginary part of κ be positive.

$$\begin{aligned} \text{Then } I &= \frac{2\pi}{\beta^2 - \kappa^2} J_m \beta \rho \cos m\omega + A J_m \kappa \rho \cos m\omega, \quad \text{when } \rho < a, \\ &= B G_m \kappa \rho \cos m\omega, \quad \text{when } \rho > a \end{aligned}$$

A, B are determined from the conditions that I and $dI/d\rho$ are continuous at $\rho = a$. Thus we find

$$\begin{aligned} I &= \frac{2\pi}{\beta^2 - \kappa^2} J_m \beta \rho \cos m\omega + \frac{2\pi}{\beta^2 - \kappa^2} J_m \kappa \rho \cos m\omega (\kappa a G_m' \kappa a J_m \beta a - G_m \kappa a \beta a J_m' \beta a); \quad (\rho < a) \\ &= \frac{2\pi}{\beta^2 - \kappa^2} G_m \kappa \rho \cos m\omega (\kappa a J_m' \kappa a J_m \beta a - J_m \kappa a \beta a J_m' \beta a); \quad (\rho > a) \end{aligned}$$

By the principle of continuation in the Theory of Functions, the result is true whatever be the phase of κ . But when the phase of κ is diminished by 2π ,

$$G_m(\kappa e) \text{ is increased by } 2\pi i J_m(\kappa e)$$

and

$$G_m'(\kappa e) \text{ by } 2\pi i J_m'(\kappa e);$$

hence, equating the corresponding changes in I and its value, we obtain

$$\begin{aligned} &\int_0^a \int_0^{2\pi} J_0 \kappa R J_m \beta \rho' \cos m\omega' \rho' d\rho' d\omega' \\ &= \frac{2\pi}{\beta^2 - \kappa^2} J_m \kappa \rho \cos m\omega (\kappa a J_m' \kappa a J_m \beta a - J_m \kappa a \beta a J_m' \beta a). \end{aligned}$$

From this again it easily follows that in I and its value we may replace the G functions by the Y functions.

(d) We have

$$\begin{aligned} Y_0 \kappa \rho &= \log \kappa \rho J_0 \kappa \rho + \frac{1}{4} \kappa^2 \rho^2 - \dots \\ &= \log \kappa J_0 \kappa \rho + \log \rho (1 - \frac{1}{4} \kappa^2 \rho^2 \dots) + \frac{1}{4} \kappa^2 \rho^2 \dots \end{aligned}$$

Thus $\log \kappa J_0 \kappa \rho - Y_0 \kappa \rho$ is an integral function of κ , in which

$$\begin{aligned} \text{coefficient of } \kappa^0 &\text{ is } -\log \rho, \\ \text{and coefficient of } \kappa^2 &\text{ is } \frac{1}{4} \rho^2 \log \rho - \frac{1}{4} \rho^2. \end{aligned}$$

The functions $\log \rho$ and $\frac{1}{4}\rho^2 \log \rho - \frac{1}{4}\rho^2$ are thus in a sense degenerate forms of the Bessel Functions, and any theorem relating to the G or Y functions will yield a corresponding theorem in these.

Thus by equating coefficients of κ^2 in the equation

$$(\nabla^2 + \kappa^2)(\log \kappa J_0 \kappa \rho - Y_0 \kappa \rho) = 0$$

we obtain

$$\begin{aligned}\nabla^2 \log \rho &= 0 \\ \nabla^2 (\tfrac{1}{4}\rho^2 \log \rho - \tfrac{1}{4}\rho^2) &= \log \rho\end{aligned}$$

and therefore

$$\nabla^4 (\tfrac{1}{4}\rho^2 \log \rho - \tfrac{1}{4}\rho^2) = 0.$$

We deduce at once

$$\nabla^2 \iint (\tfrac{1}{4}R^2 \log R - \tfrac{1}{4}R^2) f(x', y') dx' dy' = \iint \log R f(x', y') dx' dy'$$

and

$$\begin{aligned}\nabla^4 \iint (\tfrac{1}{4}R^2 \log R - \tfrac{1}{4}R^2) f(x', y') dx' dy' &= \nabla^2 \iint \log R f(x', y') dx' dy' \\ &= 2\pi f(x, y).\end{aligned}$$

(e) Again, from the addition theorem

$$Y_0 \kappa R = Y_0 \kappa \rho J_0 \kappa \rho' + 2 \sum_{m=1} Y_m \kappa \rho J_m \kappa \rho' \cos m(\omega - \omega'); \quad (\rho > \rho')$$

we deduce

$$\log R = \log \rho - \sum_{m=1} \frac{1}{m} \left(\frac{\rho'}{\rho}\right)^m \cos m(\omega - \omega'); \quad \rho > \rho'$$

and

$$\begin{aligned}\tfrac{1}{4}R^2 \log R - \tfrac{1}{4}R^2 &= (\tfrac{1}{4}\rho^2 \log \rho - \tfrac{1}{4}\rho^2) + \tfrac{1}{4}\rho'^2 \log \rho \\ &+ \left\{ (\rho - 2\rho \log \rho) \frac{\rho'}{4} - \frac{\rho'^3}{8\rho} \right\} \cos(\omega - \omega') \\ &+ \sum_{m=2} \frac{1}{4m} \left(\frac{\rho'}{\rho}\right)^m \left(\frac{\rho^2}{m-1} - \frac{\rho'^2}{m+1} \right) \cos m(\omega - \omega'); \quad (\rho > \rho').\end{aligned}$$

(f) In the same way, from the results of (c), we may deduce the value of the integral

$$I_0 = \int_0^a \int_0^{2\pi} (\tfrac{1}{4}R^2 \log R - \tfrac{1}{4}R^2) J_m \beta \rho' \cos m\omega' \rho' d\rho' d\omega'.$$

The form of the result varies in the cases $m=0$, $m=1$, $m>1$.

$m=0$:

$$\begin{aligned}I_0 &= \frac{2\pi}{\beta^4} J_0 \beta \rho + \frac{2\pi}{\beta^2} \left\{ \left(\frac{1}{\beta^2} - \frac{\rho^2}{4} \right) \left(\log a \beta a J_0' \beta a - J_0 \beta a \right) \right. \\ &\quad \left. + \frac{a^2}{4} (1 - \log a) \beta a J_0' \beta a + \frac{a^2}{4} (2 \log a - 1) J_0 \beta a \right\}, \quad \rho < a \\ &= \frac{2\pi}{\beta^2} \left\{ \left(\frac{1}{4} \rho^2 \log \rho - \frac{1}{4} \rho^2 \right) \left(-\beta a J_0' \beta a \right) + \log \rho \left(\frac{1}{\beta^2} - \frac{a^2}{4} \beta a J_0' \beta a + \frac{a^2}{2} J_0 \beta a \right) \right\}, \quad \rho > a.\end{aligned}$$

$m=1$:

$$\begin{aligned}I_0 &= \frac{2\pi}{\beta^4} J_1 \beta \rho \cos \omega + \frac{2\pi \rho \cos \omega}{\beta^2} \left\{ \left(\frac{1}{\beta^2} - \frac{\rho^2}{8} \right) \left(J_1 \beta a + \beta a J_1' \beta a \right) \right. \\ &\quad \left. + \frac{a^2}{4} (1 + 2 \log a) J_1 \beta a + \frac{a^2}{4} (1 - 2 \log a) \beta a J_1' \beta a \right\}, \quad \rho < a \\ &= \frac{2\pi a \cos \omega}{\beta^2} \left\{ \left(\frac{1}{\beta^2} + \frac{1}{4} \rho^2 - \frac{1}{2} \rho^2 \log \rho \right) \left(J_1 \beta a - \beta a J_1' \beta a \right) + \frac{a^2}{8} \left(\beta a J_1' \beta a - 3 J_1 \beta a \right) \right\}, \quad \rho > a.\end{aligned}$$

$m > 1$:

$$\begin{aligned} I_0 &= \frac{2\pi}{\beta^4} J_m \beta \rho \cos m\omega - \frac{2\pi}{\beta^2} \frac{\rho^m \cos m\omega}{2m\alpha^{m-1}} \left\{ \left(\frac{1}{\beta^2} - \frac{\rho^2}{4m+4} \right) (\beta \alpha J_m' \beta \alpha + m J_m \beta \alpha) \right. \\ &\quad \left. + \frac{\alpha^2}{4m-4} (\beta \alpha J_m' \beta \alpha + m-2 J_m \beta \alpha) \right\}, \quad \rho < \alpha \\ &= \frac{2\pi}{\beta^2} \frac{\alpha^m \cos m\omega}{2m\rho^{m-1}} \left\{ \left(\frac{1}{\beta^2} + \frac{\rho^2}{4m-4} \right) (m J_m \beta \alpha - \beta \alpha J_m' \beta \alpha) + \frac{\alpha^2}{4m+4} (\beta \alpha J_m' \beta \alpha - m + 2 J_m \beta \alpha) \right\}, \quad \rho > \alpha. \end{aligned}$$

The corresponding integrals with $\log R$ in place of $\frac{1}{4}R^2 \log R - \frac{1}{4}R^2$ may be obtained at once by taking ∇^2 of the above.

Also, all through we may write $\log(R/c)$, $\log(\rho/c)$, $\log(\alpha/c)$, instead of $\log R$, $\log \rho$, $\log \alpha$, this amounting merely to a change in the unit of length.

(g) By equating coefficients of like powers of β in the results of (c), (f) we can obtain

$$\begin{aligned} &\int_0^a \int_0^{2\pi} G_0 \kappa R \rho'^{m+2n} \cos m\omega' d\rho' d\omega' \\ \text{and} \quad &\int_0^a \int_0^{2\pi} \left(\frac{1}{4}R^2 \log R - \frac{1}{4}R^2 \right) \rho'^{m+2n} \cos m\omega' d\rho' d\omega'. \end{aligned}$$

In the case when $m = n = 0$

$$\begin{aligned} \int_0^a \int_0^{2\pi} G_0 \kappa R \rho' d\rho' d\omega' &= -\frac{2\pi}{\kappa^2} - \frac{2\pi}{\kappa^2} J_0 \kappa \rho \kappa \alpha G_0' \kappa \alpha, \quad \rho < \alpha \\ &= -\frac{2}{\kappa^2} G_0 \kappa \rho \kappa \alpha J_0' \kappa \alpha, \quad \rho > \alpha. \\ \int_0^a \int_0^{2\pi} \left(\frac{1}{4}R^2 \log R - \frac{1}{4}R^2 \right) \rho' d\rho' d\omega' &= \frac{\pi}{32} \left\{ \rho^4 + 4\rho^2 \alpha^2 (2 \log \alpha - 1) + \alpha^4 (4 \log \alpha - 5) \right\}, \quad \rho < \alpha \\ &= \left(\frac{1}{4} \rho^2 \log \rho - \frac{1}{4} \rho^2 \right) \pi \alpha^2 + \log \rho \frac{\pi \alpha^4}{8}, \quad \rho > \alpha. \end{aligned}$$

These results and those of (f) may easily be verified, or obtained, from the values of ∇^4 of the integral, with the conditions that I , $\frac{dI}{d\rho}$, $\nabla^2 I$, $\frac{d}{d\rho} \nabla^2 I$ are continuous at $\rho = \alpha$.

(h) In certain problems a class of potential functions occurs, which may be deduced from the fundamental potential $1/r$, where $r^2 = x^2 + y^2 + z^2$, by successive integration with respect to z .

Writing

$$\begin{aligned} u_1 &= \log(r+z) \\ u_2 &= z \log(r+z) - r \\ u_3 &= \frac{1}{2}(z^2 - \frac{1}{2}\rho^2) \log(r+z) - \frac{1}{4}rz + \frac{1}{4}\rho^2 \end{aligned}$$

we may easily verify that u_1 , u_2 , u_3 are potential functions, and that

$$\begin{aligned} \frac{du_2}{dz} &= u_1, \\ \frac{du_1}{dz} &= u_2, \\ \frac{du_1}{dz} &= \frac{1}{r} \end{aligned}$$

These z -integrals of $1/r$ may be expressed in the form of definite integrals involving the Bessel function J , analogous to the integral forms for r^{-1} and its z -derivatives,

$$r^{-1} = \int_0^\infty e^{-\kappa z} J_0 \kappa \rho d\kappa$$

$$\frac{d}{dz} r^{-1} = \int_0^\infty (-\kappa) e^{-\kappa z} J_0 \kappa \rho d\kappa, \text{ etc., where } z > 0.$$

We may notice that the value of $\int_0^\infty e^{-\kappa z} J_0 \kappa \rho d\kappa$ follows at once from the remark that it is a potential symmetrical about the axis of z , and taking on that axis the value $\int_0^\infty e^{-\kappa z} d\kappa = \frac{1}{z}$. We may use this idea to express u_1, u_2, u_3 in similar form.

For we have

$$\int_0^z (e^{-\kappa z} - e^{-\kappa c}) \frac{d\kappa}{\kappa} = -\log \frac{z}{c},$$

and

$$\int_0^\infty (e^{-\kappa z} - 1 + \kappa z e^{-\kappa c}) \frac{d\kappa}{\kappa^2} = z \log \frac{z}{c} - z$$

$$\int_0^z (e^{-\kappa z} - 1 + \kappa z - \frac{1}{2} \kappa^2 z^2 e^{-\kappa c}) \frac{d\kappa}{\kappa^3} = -\frac{1}{2} z^2 \log \frac{z}{c} + \frac{3}{4} z^2,$$

by integration with respect to z from 0 to z .

Hence

$$\int_0^\infty (e^{-\kappa z} J_0 \kappa \rho - e^{-\kappa c}) \frac{d\kappa}{\kappa} = -\log \frac{r+z}{2c}$$

$$\int_0^\infty (e^{-\kappa z} J_0 \kappa \rho - 1 + \kappa z e^{-\kappa c}) \frac{d\kappa}{\kappa^2} = z \log \frac{r+z}{2c} - r$$

$$\int_0^\infty (e^{-\kappa z} J_0 \kappa \rho - 1 + \kappa z - \frac{1}{2} \kappa^2 z^2 - \frac{1}{2} \rho^2 e^{-\kappa c}) \frac{d\kappa}{\kappa^3} = -\frac{1}{2} (z^2 - \frac{1}{2} \rho^2) \log \frac{r+z}{2c} + \frac{3}{4} zr - \frac{1}{4} \rho^2$$

because in each case the functions equated are symmetrical potentials, taking the same value on the axis of symmetry.

By putting $z=0$ in the first and third of these we obtain two integrals, of great importance in the following analysis,

$$\int_0^\infty (J_0 \kappa \rho - e^{-\kappa c}) \frac{d\kappa}{\kappa} = -\log \frac{\rho}{2c}$$

$$\int_0^\infty (J_0 \kappa \rho - 1 + \frac{1}{4} \kappa^2 \rho^2 e^{-\kappa c}) \frac{d\kappa}{\kappa^3} = \frac{1}{4} \rho^2 \log \frac{\rho}{2c} - \frac{1}{4} \rho^2.$$

There is no difficulty in generalising the above results, but those given are all that we shall require.

(i) With a view to indicating the broad lines of the treatment of the elastic problem given in the succeeding pages, a discussion on similar lines may be given here of a simple problem in potential, in which the attention is not distracted from the principles of the method by any complexity in the calculations.

The problem is to find the flow from a source situated between two parallel planes $z = \pm h$, under the condition that there is no flow across these planes.

We require a potential V , becoming infinite as $1/r$ at (x', y', z') , but with no other singularity at a finite distance, and such that $\frac{dV}{dz} = 0$ when $z = \pm h$.

$$\begin{aligned} \text{If } R &= \sqrt{(x' - x')^2 + (y' - y')^2}, \\ \text{then } r^{-1} &= \int_0^\infty e^{-\kappa(z \pm z')} J_0 \kappa R d\kappa, \quad \text{when } z > z' \\ &= \int_0^\infty e^{\kappa(z - z')} J_0 \kappa R d\kappa, \quad \text{when } z < z'. \\ \frac{d}{dz} r^{-1} &= \int_0^\infty (-\kappa) e^{-\kappa(z \pm z')} J_0 \kappa R d\kappa, \quad \text{when } z > z' \\ &= \int_0^\infty \kappa e^{\kappa(z - z')} J_0 \kappa R d\kappa, \quad \text{when } z < z'. \end{aligned}$$

We therefore begin by finding a potential

$$V_1 = (A \cosh \kappa z + B \sinh \kappa z) J_0 \kappa R$$

giving

$$\begin{aligned} \frac{dV_1}{dz} &= \kappa e^{-\kappa(z \pm z')} J_0 \kappa R \quad \text{on } z = h \\ &= -\kappa e^{\kappa(z \pm z')} J_0 \kappa R \quad \text{on } z = -h. \end{aligned}$$

We obtain

$$\begin{aligned} A \sinh \kappa h + B \cosh \kappa h &= e^{-\kappa(h - z')} \\ A \sinh \kappa h - B \cosh \kappa h &= e^{-\kappa(h + z')} \\ A &= e^{-\kappa h} \frac{\cosh \kappa z'}{\sinh \kappa h} \\ B &= e^{-\kappa h} \frac{\sinh \kappa z'}{\cosh \kappa h} \\ V_1 &= e^{-\kappa h} \left(\frac{\cosh \kappa z \cosh \kappa z'}{\sinh \kappa h} + \frac{\sinh \kappa z \sinh \kappa z'}{\cosh \kappa h} \right) J_0 \kappa R. \end{aligned}$$

If this could be integrated with respect to κ from 0 to ∞ , we should have a potential just balancing at the boundary the flow from the source.

But V_1 becomes infinite as $1/\kappa h$ at $\kappa = 0$, and the integration cannot be performed. We may, however, subtract from V_1 the (constant) potential $e^{-\kappa c}/\kappa h$, where c is an arbitrary positive quantity. This makes integration possible, without introducing any flow across the boundary.

A solution of the problem is then

$$V = \frac{1}{r} + \int_0^\infty \left\{ e^{-\kappa h} \left(\frac{\cosh \kappa z \cosh \kappa z'}{\sinh \kappa h} + \frac{\sinh \kappa z \sinh \kappa z'}{\cosh \kappa h} \right) J_0 \kappa R - \frac{e^{-\kappa c}}{\kappa h} \right\} d\kappa.$$

But this form of solution, while theoretically complete, is of little value because of the difficulty of interpretation. For example, it gives no indication of what on physical grounds we should expect to be the chief feature of the phenomenon, namely, the practically two-dimensional character of the flow at a moderate distance from the source.

The transformation to which we proceed brings this out as luminously as possible.

First, it is convenient to separate V_1 into its odd and even parts in κ , as is easily done by writing $\cosh \kappa h - \sinh \kappa h$ for $e^{-\kappa h}$.

Thus

$$V_1 = \left(-\cosh \overline{\kappa z - z'} + \frac{\cosh \kappa h}{\sinh \kappa h} \cosh \kappa z \cosh \kappa z' - \frac{\sinh \kappa h}{\cosh \kappa h} \sinh \kappa z \sinh \kappa z' \right) J_0 \kappa R.$$

Next we replace the term $1/r$ in V by the equivalent integral

$$\int_0^\infty e^{\mp \kappa(z-z')} J_0 \kappa R d\kappa.$$

Hence

$$V = \int_0^\infty \left\{ \left(\mp \sinh \overline{\kappa z - z'} + \frac{\cosh \kappa h}{\sinh \kappa h} \cosh \kappa z \cosh \kappa z' - \frac{\sinh \kappa h}{\cosh \kappa h} \sinh \kappa z \sinh \kappa z' \right) J_0 \kappa R - \frac{e^{-\kappa c}}{\kappa h} \right\} d\kappa,$$

the upper or lower sign being taken in the ambiguous term according as z is $>$ or $<$ z' .

When $R > 0$, this integral can be separated into the two

$$\begin{aligned} & \int_0^\infty J_0 \kappa R \left(\mp \sinh \overline{\kappa z - z'} + \frac{\cosh \kappa h}{\sinh \kappa h} \cosh \kappa z \cosh \kappa z' - \frac{\sinh \kappa h}{\cosh \kappa h} \sinh \kappa z \sinh \kappa z' - \frac{1}{\kappa h} \right) d\kappa \\ & + \int_0^\infty \left(J_0 \kappa R - e^{-\kappa c} \right) \frac{d\kappa}{\kappa h}. \end{aligned}$$

The value of the latter integral we have found to be $-\frac{1}{h} \log \frac{R}{2c}$.

The former integral is of the form $\int_0^\infty J_0 \kappa R F(\kappa) d\kappa$, where $F(\kappa)$ is an *odd* function of κ , vanishing for $\kappa = 0$. It may be expressed as a complex integral

$$\frac{1}{\pi i} \int G_0 \kappa R F(\kappa) d\kappa,$$

the path being from west to east along the whole of the real axis in the κ plane, for

$$G_0(\kappa R) - G_0(\kappa' \pi R) = \pi i J_0 \kappa R.$$

Now, from the original form of V_1 , and the integral forms of $1/r$, it is obvious that $F(\kappa)$ vanishes at infinity in the eastern half of the κ plane; being odd in κ it must vanish likewise in the western half.

Hence by Cauchy's Theorem, the integral $\frac{1}{\pi i} \int G_0 \kappa R F(\kappa) d\kappa$ is equal to twice the sum of the residues of the function $G_0 \kappa R F(\kappa)$ at its poles in the upper half of the κ plane, and

$$\begin{aligned} V = & -\frac{1}{h} \log \frac{R}{2c} \\ & + \frac{2}{h} \sum_{n=1}^\infty \cos \frac{n\pi z}{h} \cos \frac{n\pi z'}{h} G_0 \frac{in\pi R}{h} \\ & + \frac{2}{h} \sum_{n=0}^\infty \sin \left(n + \frac{1}{2} \right) \frac{\pi z}{h} \sin \left(n + \frac{1}{2} \right) \frac{\pi z'}{h} G_0 \left(n + \frac{1}{2} \right) \frac{i\pi R}{h}. \end{aligned}$$

(j) The solution indicates (i) a main current in two dimensions, defined by the potential $-\frac{1}{h} \log \frac{R}{2c}$, and (ii) an infinite series of local currents in three dimensions,

practically insensible when the distance from the source is a moderate multiple of the thickness of the plate. In the following pages we shall deduce analogous solutions for 'sources of strain' of the different types which may exist in an elastic solid, and develop these solutions in various directions. The corresponding development of the present solution is extremely easy, but would carry us too far. We merely mention that the 'main current' in the hydrodynamical problems corresponds to the 'principal modes of strain,' the determination of which is the object of the theory of thin plates. But there is one important distinction in the two cases. In the flow problems the exact conditions defining the 'main current' can always be found, and are indeed obvious; on the other hand, the analogous conditions in the strain problems can only be found by approximation.

(k) The following conventions seem to be very generally adopted, but to prevent any risk of ambiguity they may be stated explicitly here. Consider any continuous plane area A bounded externally by a closed curve C_0 , and internally by one or more closed curves C_1, C_2 , etc. At any point E of a bounding curve let Ex, Ey be drawn in the directions of the rectangular axes of coordinates. Let Ex, Ey be turned through an angle ϵ , which will be taken as positive when the rotation is counter-clockwise, until they coincide with $E\xi, E\eta$, the direction of $E\xi$ being that of the normal at E when drawn from within A towards the boundary. $E\xi, E\eta$ will be taken to be the positive directions of the normal and tangent at E , and if $f(x, y)$ be any function given within A , $\frac{df}{dn}$ and $\frac{df}{ds}$ will be used to denote the rates of variation of f per unit length in these positive directions.

The curvature at E is $\frac{d\epsilon}{ds}$ and is denoted by $1/\rho$. ρ is therefore positive when, in order to reach the centre of curvature from E, we have to proceed *into* the area A.

If we suppose the figure traced on level ground, a person proceeding along the boundary in the positive direction will have the area on his left, and the curvature will be positive when he is rotating about the vertical in the counter-clockwise sense.

The following formulæ relating to differentiation along the arc and normal will be much used in the later sections of the paper. Suppose the axes of x and y to coincide with the positive normal and tangent at a point O of the bounding curve. At a neighbouring point E (x, y) on this curve

[illegible]

By putting x, y, ϵ equal to zero, we have at ()

$$\frac{df}{dn} = \frac{df}{dx}; \quad \frac{df}{ds} = \frac{df}{dy} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \quad \quad \text{(ii)}$$

In terms of the displacements, the equations of equilibrium are therefore

$$\left. \begin{aligned} \mu \nabla^2 u + (\lambda + \mu) \frac{d\Delta}{dx} + X &= 0 \\ \mu \nabla^2 v + (\lambda + \mu) \frac{d\Delta}{dy} + Y &= 0 \\ \mu \nabla^2 w + (\lambda + \mu) \frac{d\Delta}{dz} + Z &= 0 \end{aligned} \right\} \quad . \quad . \quad . \quad (3)$$

When the bodily force is null, or $X = Y = Z = 0$, the following forms are easily shown to satisfy equations (3),

$$\begin{aligned} \text{(i)} \quad u &= 2 \frac{d\psi}{dy} \\ v &= -2 \frac{d\psi}{dx} \\ \text{(ii)} \quad u &= \frac{d\theta}{dx} \\ v &= \frac{d\theta}{dy} \\ w &= \frac{d\theta}{dz} \\ \text{(iii)} \quad u &= \frac{\lambda + 3\mu}{\lambda + \mu} \frac{d\phi}{dx} + 2z \frac{d^2\phi}{dz dx} \\ v &= \frac{\lambda + 3\mu}{\lambda + \mu} \frac{d\phi}{dy} + 2z \frac{d^2\phi}{dz dy} \\ w &= -\frac{\lambda + 3\mu}{\lambda + \mu} \frac{d\phi}{dz} + 2z \frac{d^2\phi}{dz^2} \end{aligned}$$

where ψ , θ , ϕ are potential functions, so that

$$\nabla^2\psi = 0, \quad \nabla^2\theta = 0, \quad \nabla^2\phi = 0.$$

These solutions have been used by BOUSSINESQ in his treatment of the problem of a solid bounded by a single plane $z = 0$. They are equally effective when the boundary consists of two parallel z -planes. Thus, as will explicitly appear in the sequel, and as might be proved at once, any solution of (3), with $X = Y = Z = 0$, in the space between the planes $z = \pm h$, can be expressed in the form

$$\left. \begin{aligned} u &= 2 \frac{d\psi}{dy} + \frac{d\theta}{dx} + \alpha \frac{d\phi}{dx} + 2z \frac{d^2\phi}{dz dx} \\ v &= -2 \frac{d\psi}{dx} + \frac{d\theta}{dy} + \alpha \frac{d\phi}{dy} + 2z \frac{d^2\phi}{dz dy} \\ w &= \frac{d\theta}{dz} - \alpha \frac{d\phi}{dz} + 2z \frac{d^2\phi}{dz^2} \end{aligned} \right\} \quad . \quad . \quad . \quad (4)$$

Here, and throughout the paper, the symbol α is used to denote the fraction $(\lambda + 3\mu)/(\lambda + \mu)$.

With these values of u , v , w the stresses across a z -plane, viz., \widehat{xx} , \widehat{xy} , \widehat{xz} , are given by

$$\left. \begin{aligned} \widehat{xx} &= \frac{d^2\psi}{dy dz} + \frac{d^2\theta}{dx dz} + \frac{d^2\phi}{dx dz} + 2z \frac{d^3\phi}{dx dz^2} \\ \widehat{xy} &= -\frac{d^2\psi}{dx dz} + \frac{d^2\theta}{dy dz} + \frac{d^2\phi}{dy dz} + 2z \frac{d^3\phi}{dy dz^2} \\ \widehat{xz} &= \frac{d^2\theta}{dz^2} - \frac{d^2\phi}{dz^2} + 2z \frac{d^3\phi}{dz^3} \end{aligned} \right\} \quad . \quad . \quad . \quad (5)$$

2. Force applied at a single point.

Returning to the solutions (i), (ii), (iii), we note that (i) and (ii) contribute nothing to the dilatation Δ , and (ii), (iii) nothing to the z -rotation $\omega_3 = \frac{1}{2} \left(\frac{dv}{dx} - \frac{du}{dy} \right)$.

These properties can be used to resolve any given displacement into its ψ , θ , ϕ components, the bodily force being null.

An example of fundamental importance is the displacement in an infinite solid due to a single force applied at a given point. Thus for a unit force applied at the origin in the direction of the axis of z we have

$$\left. \begin{aligned} u &= \frac{xz}{r^3} \\ v &= \frac{yz}{r^3} \\ w &= \frac{z^2}{r^3} + \frac{a}{r} \end{aligned} \right\} \text{each multiplied by } \frac{1}{8\pi\mu} \frac{\lambda + \mu}{\lambda + 2\mu} = \frac{1}{4\pi\mu(a+1)}$$

where $r^2 = x^2 + y^2 + z^2$; or say, for a Z force of $4\pi\mu(a+1)$ units applied at (x', y', z') we have

$$\left. \begin{aligned} u &= (z' - z) \frac{dr^{-1}}{dx} \\ v &= (z' - z) \frac{dr^{-1}}{dy} \\ w &= (z' - z) \frac{dr^{-1}}{dz} + ar^{-1} \end{aligned} \right\} \quad (6)$$

r^{-1} being written for $1/r$, where r is the distance from (x, y, z) to (x', y', z') . These give

$$\Delta = (a-1) \frac{dr^{-1}}{dz} ; \quad \omega_3 = 0.$$

But in (4)

$$\Delta = 2(1-a) \frac{d^2\phi}{dz^2} ; \quad \omega_3 = \frac{d^2\psi}{dz^2}.$$

Hence we take $\psi = 0$, and choose ϕ so that $\frac{d\phi}{dz} = -\frac{1}{2}r^{-1}$.

Now the functions $\log(r+z-z')$ and $-\log(r-z-z')$ are both potentials having r^{-1} for z -derivative; the former is without singular point in the region $z > z'$, the latter in the region $z < z'$. We may without confusion use a single symbol to denote either function indifferently, and define

$$\frac{d^{-1}r^{-1}}{dz^{-1}} = \begin{cases} \log(r+z-z') & \text{when } z > z' \\ -\log(r-z-z') & \text{when } z < z' \end{cases} \quad (7)$$

We may therefore take

$$\phi = -\frac{1}{2} \frac{d^{-1}r^{-1}}{dz^{-1}}.$$

Comparison of the displacement w in (4) and (6) gives now

$$\frac{d\theta}{dz} = z' \frac{dr^{-1}}{dz} + \frac{a}{2} r^{-1}.$$

For a Z force of $4\pi\mu(a+1)$ units at (x', y', z') we have therefore

$$\left. \begin{aligned} \psi &= 0 \\ \theta &= \frac{1}{2} r^{-1} + \frac{a}{2} r^{-1} r^{-1} \\ \phi &= -\frac{1}{2} r^{-1} r^{-1} \end{aligned} \right\} \quad (8)$$

It is easy to verify that these values of ψ , θ , ϕ substituted in (4) do actually reproduce equations (6).

Similarly for an X force of $4\pi\mu(u+1)$ units at (x', y', z') we find

$$\left. \begin{aligned} \psi &= -\frac{\alpha+1}{2} \frac{d}{dy} \frac{d^{-2}r^{-1}}{dz^{-2}} \\ \theta &= z \frac{d}{dx} \frac{d^{-1}r^{-1}}{dz^{-1}} - \frac{\alpha}{2} \frac{d}{dx} \frac{d^{-2}r^{-1}}{dz^{-2}} \\ \phi &= -\frac{1}{2} \frac{d}{dx} \frac{d^{-2}r^{-1}}{dz^{-2}} \end{aligned} \right\} \quad (9)$$

Here $\ell^{(-z)r^{-1}}_{(z^{-z})}$ denotes a potential function having $\ell^{(-1)r^{-1}}_{(z^{-1})}$ for z -derivative, and is defined by the equations

$$\begin{aligned} \frac{d^{-2}r^{-1}}{dz^{-2}} &= (z-z') \log(r+z-z') - r && \text{when } z > z' \\ &= -(z-z') \log(r-z+z') - r && \text{when } z < z' \end{aligned} \quad (10)$$

It may be observed that the necessity for dealing separately with the two regions $z > z'$ and $z < z'$ in these cases is not inconsistent with the theorem of (4), which refers only to a displacement free from singularity in the space considered.

3. Solution of the problem of normal traction.

Coming now to the problems relating to a solid bounded by the two parallel planes $z=h$ and $z=-h$, we begin with the simplest of these, and seek a solution of the equations of equilibrium giving

$X=Y=Z=0$ throughout the body ;

the normal stress

$$\begin{aligned} \hat{u} &= f(x, y) & \text{on } z = h, \\ &= 0 & \text{on } z = -h; \end{aligned}$$

the tangential stresses $\widehat{ex}, \widehat{ey} = 0$ on both faces $z = \pm h$. The arbitrary function $f(x, y)$, which we shall suppose to vanish at all points without a given finite area A , is expressed in a form amenable to analytical treatment in the familiar theorem

$$\lim_{\epsilon \rightarrow 0} \iint \frac{\epsilon f(x', y') dx' dy'}{\{(x-x')^2 + (y-y')^2 + \epsilon^2\}^{\frac{3}{2}}} = 2\pi f(x, y) \quad (11)$$

the integral being taken over the area A.

(If we imagine the plane $z = 0$ to be covered with attracting matter of surface density $f(x, y)$, then the theorem expresses the well-known relation between the density at

These terms of negative degree are potentials contributing nothing to the stresses on $z = \pm h$, as we see from (12), since $\kappa J_0 \kappa R$ contains no terms of negative degree. They might therefore be subtracted from the expressions (13) without affecting the satisfaction of the conditions in (12). This simple subtraction would, however, introduce terms not integrable right up to the upper limit, at least after c is put equal to zero, as eventually it will be. The difficulty is met by subtracting from $4\mu\phi$, not $H/\kappa^3 + K/\kappa$, but $H/\kappa^3 + Ke^{-\kappa h}/\kappa$; and from $4\mu\theta$, not $L/\kappa^3 + M/\kappa$, but $L/\kappa^3 + Me^{-\kappa h}/\kappa$.

(There are, of course, any number of equally suitable modifications; instead of $e^{-\kappa h}$ we might take $e^{-\kappa z}$ or $1/(1+\kappa^2)$, for instance.) A solution of the preliminary problem of normal traction equal to $\epsilon/(R^2+\epsilon^2)^{\frac{3}{2}}$ on the face $z=h$ is thus obtained in the form

$$\begin{aligned} 4\mu\phi = & - \int_0^\infty e^{-\kappa z} \left\{ \frac{\cosh \kappa h}{\kappa (\sinh 2\kappa h - 2\kappa h)} J_0 \kappa R \sinh \kappa z \right. \\ & \left. + \frac{\sinh \kappa h}{\kappa (\sinh 2\kappa h + 2\kappa h)} J_0 \kappa R \cosh \kappa z - \frac{H}{\kappa^3} - \frac{K e^{-\kappa h}}{\kappa} \right\} d\kappa \\ 4\mu\theta = & \int_0^\infty e^{-\kappa z} \left\{ \frac{\cosh \kappa h + 2\kappa h \sinh \kappa h}{\kappa (\sinh 2\kappa h - 2\kappa h)} J_0 \kappa R \sinh \kappa z \right. \\ & \left. + \frac{\sinh \kappa h + 2\kappa h \cosh \kappa h}{\kappa (\sinh 2\kappa h + 2\kappa h)} J_0 \kappa R \cosh \kappa z - \frac{L}{\kappa^3} - \frac{M e^{-\kappa h}}{\kappa} \right\} d\kappa. \end{aligned} \quad (14)$$

The solution of the original general problem is found by multiplying by $f(x', y')/2\pi$, integrating with respect to x', y' over the area A, and finally taking the limit for $\epsilon=0$. But a glance at the forms near $\kappa=\infty$ of the functions in (14) shows that the triple integrals are absolutely convergent, it being supposed that $-h < z < h$. Hence we may integrate with respect to x', y' first, and by a well-known theorem the limits for $\epsilon=0$ may then be found by simply putting $\epsilon=0$ in the integrands, provided the resulting integrals are convergent, as they manifestly are.

This gives the value of ϕ , for example, in the form

$$\int_0^\infty d\kappa \iint f(x', y') F(\kappa) dx' dy',$$

but, always provided $-h < z < h$, we may change this if we please into

$$\iint f(x', y') dx' dy' \int_0^\infty F(\kappa) d\kappa.$$

Finally, we may with great advantage confine our study in the first place to what is usually spoken of as a *unit element* of normal traction at (x', y', h) . The area A enclosing this point is diminished, and the intensity of traction increased without limit, so that $\iint f(x', y') dx' dy'$ remains equal to unity. The resulting solution is simply that of (14), but with ϵ put equal to zero within the integral signs.

As we have just seen, the solution for the general case can at any time be found from this elementary solution (15) by multiplying by $f(x', y')/2\pi$ and integrating over the area A.

4. *Flexural and extensional components of the strain. Disadvantages of the solution in definite integrals.*

In the elementary solution each of the potentials ϕ, θ may with advantage be decomposed into an odd and an even part in z . Thus, for an element of normal

traction of $8\pi\mu$ units at (x', y', h) , a solution is given by

$$\phi = \phi_0 + \phi; \quad \theta = \theta_0 + \theta.$$

where

$$\left. \begin{aligned} \phi_0 &= \int_0^\infty \left\{ -\frac{\cosh \kappa h}{\kappa (\sinh 2\kappa h - 2\kappa h)} J_0 \kappa R \sinh \kappa z + \frac{3z}{4\kappa^3 h^3} + \frac{e^{-\kappa h}}{\kappa} \left(\frac{2z^3 - 3R^2 z}{16h^3} + \frac{9}{40} \frac{z}{h} \right) \right\} d\kappa \\ \theta_0 &= \int_0^\infty \left\{ \frac{\cosh \kappa h + 2\kappa h \sinh \kappa h}{\kappa (\sinh 2\kappa h - 2\kappa h)} J_0 \kappa R \sinh \kappa z - \frac{3z}{4\kappa^3 h^3} - \frac{e^{-\kappa h}}{\kappa} \left(\frac{2z^3 - 3R^2 z}{16h^3} + \frac{9}{40} \frac{z}{h} + \frac{3}{2} \frac{z}{h} \right) \right\} d\kappa \end{aligned} \right\} \quad (16)$$

$$\left. \begin{aligned} \phi_c &= \int_0^\infty \left\{ -\frac{\sinh \kappa h}{\kappa (\sinh 2\kappa h + 2\kappa h)} J_0 \kappa R \cosh \kappa z + \frac{e^{-\kappa h}}{4\kappa} \right\} d\kappa \\ \theta_c &= \int_0^\infty \left\{ \frac{\sinh \kappa h + 2\kappa h \cosh \kappa h}{\kappa (\sinh 2\kappa h + 2\kappa h)} J_0 \kappa R \cosh \kappa z - \frac{3e^{-\kappa h}}{4\kappa} \right\} d\kappa \end{aligned} \right\} \quad (17)$$

The conditions satisfied at the faces by the partial solutions (16) and (17) are easily made out. For when ϕ, θ are both odd functions of z , then $\widehat{xx}, \widehat{zy}$ are even and \widehat{zz} odd; but when ϕ, θ are even functions of z , then $\widehat{xx}, \widehat{zy}$ are odd, \widehat{zz} even, as is obvious from (5). Hence (16) gives equal values of opposite sign for \widehat{zz} at corresponding points on $z = \pm h$; (17) gives equal values of the same sign.

It follows that (16) is the solution for elements of normal force of $4\pi\mu$ units at each of the points $(x', y', h), (x', y', -h)$, the force being in the positive direction of Oz in each case, and therefore a traction on $z = h$, but a pressure on $z = -h$; in (17) the only difference is that the force is a traction on both planes.

Hence, also, (16) subtracted from (17) will give the solution for traction on $z = -h$ alone.

Each of the integrals in (16), (17) defines a potential function without singularity at a finite distance in the space between the planes $z = \pm h$, and all the successive derivatives with respect to x, y, z of any of these functions may be calculated by differentiation within the sign of integration, provided we are dealing with a point actually within the solid, so that $-h < z < h$.

The solutions defined by these integrals are therefore formally satisfactory. It is, however, a serious objection to them that they do not lend themselves readily to interpretation, and it is not easy to make out from them any of the simple laws which the ordinary approximate theory leads us to anticipate.

In particular, the solutions in their present form throw no light on the question of the behaviour of the functions and their derivatives at points the distance of which from the sources of strain is great in comparison with the thickness of the plate, a question of great importance for the application to the thin-plate theory.

The analytical transformations to which we now proceed reduce the solutions to a form entirely free from these objections. Each of the integrals is shown to be composed of two parts of very different character. The first part represents a function the value of which diminishes with great rapidity as the distance from the source increases, while the remaining part is a function of very simple form. Each solution is thus resolved into a permanent or persistent element and a local, transitory, or decaying element, the latter being insignificant beyond the immediate vicinity of the source.

5. *Transformation of the definite integrals into series by means of Cauchy's Theorem.*

The integral ϕ_0 of (16) can be written as the sum of three integrals, namely—

$$\phi_0 = \int_0^\infty G_0 \kappa R \left\{ -\frac{\cosh \kappa h \sinh \kappa z}{\kappa (\sinh 2\kappa h - 2\kappa h)} + \frac{3z}{4\kappa^3 h^3} + \frac{1}{\kappa} \left(\frac{z^3}{8h^3} + \frac{9}{40} \frac{z}{h} \right) \right\} d\kappa \\ - \frac{3z}{4h^3} \int_0^\infty (G_0 \kappa R - 1 + \frac{1}{4} \kappa^2 R^2 e^{-\kappa h}) \frac{d\kappa}{\kappa^3} \\ - \left(\frac{z^3}{8h^3} + \frac{9}{40} \frac{z}{h} \right) \int_0^\infty (G_0 \kappa R - e^{-\kappa h}) \frac{d\kappa}{\kappa} \quad (18)$$

It should be observed that the first and third of these integrals cease to converge when $R = 0$. Hence the transformation does not apply to points on the line $x = x'$, $y = y'$, the normal to the plate through the sources.

Consider now the first integral in (18). The function of κ multiplying $G_0 \kappa R$ within the integral sign is an odd function of κ vanishing for $\kappa = 0$. Hence, as in (i), the integral is equivalent to the complex integral

$$\int \frac{1}{\pi i} G_0 \kappa R \left\{ -\frac{\cosh \kappa h \sinh \kappa z}{\kappa (\sinh 2\kappa h - 2\kappa h)} + \frac{3z}{4\kappa^3 h^3} + \frac{1}{\kappa} \left(\frac{z^3}{8h^3} + \frac{9}{40} \frac{z}{h} \right) \right\} d\kappa,$$

the path of integration running from west to east along the whole of the real axis, and just avoiding the origin, which is a singular point of $G_0 \kappa R$, on the north or upper side.

On this path take points E, W at distances $n\pi/2h$ to the right and left of the origin, and on EW as side describe a square E W A B in the upper part of the plane. The integral over each of the sides W A, A B, B E is easily proved to have zero for limit when n tends to infinity through positive integral values.

Hence, by Cauchy's fundamental theorem, the integral over the path W E is equal to the sum of the residues of the integrand at its poles in the upper half of the κ plane, multiplied by $2\pi i$, that is, to the series

$$\sum_{\kappa} G_0 \kappa R \left(-\frac{\cosh \kappa h \sinh \kappa z}{\kappa h (\cosh 2\kappa h - 1)} \right),$$

the summation extending over the zeroes of the function $\sinh 2\kappa h - 2\kappa h$ in the upper half of the κ plane, in the order of their moduli.

If ζ_n is a zero of the function $\sinh \zeta - \zeta$, the corresponding zero of $\sinh 2\kappa h - 2\kappa h$ is $\kappa_n = \zeta_n/2h$, and

$$G_0 \kappa_n R = G_0 \frac{\zeta_n R}{2h} = \sqrt{\frac{\pi h}{\zeta_n R}} e^{\frac{i\pi}{4}} e^{\frac{i\zeta_n R}{2h}} \text{ approximately,}$$

and we see that this part of ϕ_0 , with its successive derivatives, is practically insensible when R is a very moderate multiple of $2h$. (Cf. § 7, *infra*.)

As to the other two integrals which occur in (18), we have proved in (h) that

$$\int_0^\infty (J_0 \kappa R - 1 + \frac{1}{4} \kappa^2 R^2 e^{-\kappa h}) \frac{d\kappa}{\kappa^3} = \frac{1}{4} R^2 \log \frac{R}{2h} - \frac{1}{4} R^2,$$

and

$$\int_0^\infty (J_0 \kappa R - e^{-\kappa h}) \frac{d\kappa}{\kappa} = -\log \frac{R}{2h}.$$

These functions will occur so often that it will be convenient to reserve an invariable symbol for the former of them, say

$$\chi(R), \text{ or simply } \chi, = \left. \begin{aligned} &\frac{1}{4} R^2 \log \frac{R}{2h} - \frac{1}{4} R^2 \\ &\nabla^2 \chi = \log \frac{R}{2h} \end{aligned} \right\} \quad (19)$$

and then

The persistent part of ϕ_0 is therefore

$$-\frac{3z}{4h^3} \chi + \left(\frac{z^3}{8h^3} + \frac{9}{40} \frac{z}{h} \right) \nabla^2 \chi,$$

which is the sum of two potential functions,

$$-\frac{3}{4h^3} \left(z\chi - \frac{1}{6} z^3 \nabla^2 \chi \right) \quad \text{and} \quad \frac{9}{40h} z \nabla^2 \chi.$$

6. Types of the particular solutions composing the general solution.

A glance at the relation between the results just obtained and the form of ϕ_0 in (16) enables us to write down at once the corresponding transformations of θ_0 , ϕ_e , θ_e .

Collecting the results, we find

$$\left. \begin{aligned} \phi_0 &= \sum_{\kappa} G_0 \kappa R \left(- \frac{\cosh \kappa h \sinh \kappa z}{\kappa h (\cosh 2\kappa h - 1)} - \frac{3}{4h^3} \left(z\chi - \frac{1}{6} z^3 \nabla^2 \chi \right) + \frac{9}{40h} z \nabla^2 \chi \right) \\ \theta_0 &= \sum_{\kappa} G_0 \kappa R \left(\frac{\cosh \kappa h + 2\kappa h \sinh \kappa h}{\kappa h (\cosh 2\kappa h - 1)} \sinh \kappa z + \frac{3}{4h^3} \left(z\chi - \frac{1}{6} z^3 \nabla^2 \chi - 2h^2 z \nabla^2 \chi \right) - \frac{9}{40h} z \nabla^2 \chi \right) \end{aligned} \right\} \quad (20)$$

where κ is a zero of $\sinh 2\kappa h - 2\kappa h$, with positive imaginary part.

$$\left. \begin{aligned} \phi_e &= \sum_{\kappa} G_0 \kappa R \left(- \frac{\sinh \kappa h \cosh \kappa z}{\kappa h (\cosh 2\kappa h + 1)} + \frac{1}{4} \nabla^2 \chi \right) \\ \theta_e &= \sum_{\kappa} G_0 \kappa R \left(\frac{\sinh \kappa h + 2\kappa h \cosh \kappa h}{\kappa h (\cosh 2\kappa h + 1)} \cosh \kappa z - \frac{3}{4} \nabla^2 \chi \right) \end{aligned} \right\} \quad (21)$$

where κ is a zero of $\sinh 2\kappa h + 2\kappa h$, with positive imaginary part.

The solution must give \widehat{xx} , \widehat{xy} , \widehat{zz} all equal to zero at the two plane faces of the plate, except when $R=0$, and we are thus prepared to find that the strain defined by the terms corresponding to any one root κ gives zero stress across $z = \pm h$. Thus in (20) \sum_{κ} contains a series of particular solutions of the type

$$(i) \quad \left. \begin{aligned} \phi &= -\cosh \kappa h \sinh \kappa z F(x, y) \\ \theta &= (\cosh \kappa h + 2\kappa h \sinh \kappa h) \sinh \kappa z F(x, y) \end{aligned} \right\} \begin{aligned} &\text{where } (\nabla^2 + \kappa^2)F = 0 \\ &\sinh 2\kappa h - 2\kappa h = 0 \end{aligned} \quad (22)$$

Calculating the stresses by means of (5), we find

$$\begin{aligned}\frac{d\theta}{dz} + \frac{d\phi}{dz} + 2z \frac{d^2\phi}{dz^2} &= (2\kappa^2 h \sinh \kappa h \cosh \kappa z - 2\kappa^2 z \cosh \kappa h \sinh \kappa z) F \\ \frac{d^2\theta}{dz^2} - \frac{d^2\phi}{dz^2} + 2z \frac{d^3\phi}{dz^3} &= 2\kappa^2 (\cosh \kappa h + \kappa h \sinh \kappa h) \sinh \kappa z F - 2\kappa^2 z \cosh \kappa h \cosh \kappa z F\end{aligned}$$

both of which vanish when $z = \pm h$.

We have further in (20) a solution of the type

$$(ii) \quad \left. \begin{aligned} \phi &= -zF + \frac{1}{6}z^3 \nabla^2 F \\ \theta &= -zF - \frac{1}{6}z^3 \nabla^2 F - 2h^2 z \nabla^2 F \end{aligned} \right\} F \text{ a function of } x, y \text{ with } \nabla^4 F = 0.$$

From this, by (4), (5),

$$\begin{aligned} u &= \frac{d}{dx} \left\{ -(\alpha+1) \left(zF - \frac{1}{6}z^3 \nabla^2 F \right) + 2 \left(\frac{1}{3}z^3 - h^2 z \right) \nabla^2 F \right\} \\ v &= \frac{d}{dy} \left\{ -(\alpha+1) \left(zF - \frac{1}{6}z^3 \nabla^2 F \right) + 2 \left(\frac{1}{3}z^3 - h^2 z \right) \nabla^2 F \right\} \\ w &= (\alpha+1) \left(F - \frac{1}{2}z^2 \nabla^2 F \right) + 2(z^2 - h^2) \nabla^2 F \\ \widehat{zx} &= 4\mu(z^2 - h^2) \frac{d}{dx} \nabla^2 F \\ \widehat{zy} &= 4\mu(z^2 - h^2) \frac{d}{dy} \nabla^2 F \\ \widehat{zz} &= 0 \end{aligned} \quad (23)$$

The solution for unit normal traction on $z = h$ contains a strain of this type, with $F = \chi(R) \cdot 3/32\pi\mu h^3$.

Lastly, in (20) there is a solution of the type

$$(iii) \quad \left. \begin{aligned} \phi &= -zF \\ \theta &= -zF \end{aligned} \right\} F \text{ a function of } x, y \text{ with } \nabla^2 F = 0,$$

giving

$$\begin{aligned} u &= -(\alpha+1)z \frac{dF}{dx} \\ v &= -(\alpha+1)z \frac{dF}{dy} \\ w &= (\alpha+1)F \end{aligned} \quad \widehat{zx} = \widehat{zy} = \widehat{zz} = 0 \quad (24)$$

Obviously this is merely a degenerate case of (ii).

Again in (21) we have a series of solutions of the type

$$(iv) \quad \left. \begin{aligned} \phi &= -\sinh \kappa h \cosh \kappa z F(x, y) \\ \theta &= (\sinh \kappa h + 2\kappa h \cosh \kappa h) \cosh \kappa z F(x, y) \end{aligned} \right\} \begin{aligned} &\text{where } (\nabla^2 + \kappa^2)F = 0 \\ &\sinh 2\kappa h + 2\kappa h = 0 \end{aligned} \quad (25)$$

In this, as in (i), $\widehat{zx} = \widehat{zy} = \widehat{zz} = 0$ on $z = \pm h$.

(v) The persistent part of (21) is of the type

$$\begin{aligned} \phi &= F(x, y); \quad \theta = -3F(x, y), \text{ where } \nabla^2 F = 0 \\ u &= (\alpha-3) \frac{dF}{dx}, \quad v = (\alpha-3) \frac{dF}{dy}; \quad \widehat{zx} = \widehat{zy} = \widehat{zz} = 0 \end{aligned} \quad (26)$$

giving

and $(2n-1)\pi + \pi/2$, and it may be shown, as in the previous case, that there is actually one root with η between these limits for all positive integral values of n .

Also

$$\eta = (2n-1)\pi + \pi/2 - \epsilon; \quad \cosh \xi = 2n\pi - \pi/2$$

$$\epsilon = 2 \log \sqrt{4n-1} \pi / (4n-1) \pi.$$

To a first approximation

$$\zeta_n = \log \sqrt{4n-1} \pi + (2n - \frac{1}{2})\pi i \quad (28)$$

In addition to the roots of (27), (28) we have of course a corresponding series in the second quadrant, the images of these in the axis of imaginaries.

8. *Approximate forms of the n^{th} terms of the infinite series, when n is large.*

It may be useful to give in terms of n approximate forms for the general terms of (20), (21) corresponding to the n^{th} roots in the first quadrant.

(i) ϕ_0 and θ_0 .

$$\kappa h = \frac{1}{2} \log 4n\pi + (n + \frac{1}{4})\pi i$$

$$\sinh \kappa h = \frac{1}{2} e^{\kappa h} = \frac{1}{2} (4n\pi)^{\frac{1}{2}} e^{(n+\frac{1}{4})\pi i}$$

$$\frac{\cosh \kappa h}{\kappa h (\cosh 2\kappa h - 1)} = \frac{1}{\sinh \kappa h \cdot 2 \sinh^2 \kappa h} = \frac{4}{(4n\pi)^{\frac{3}{2}}} e^{-(n+\frac{1}{4})\pi i} (-i)$$

$$\sinh \kappa z = \frac{1}{2} (e^{\kappa z})_h - \frac{1}{2} (e^{\kappa h})^{-\frac{z}{h}}$$

$$= \frac{1}{2} \left\{ (4n\pi)^{\frac{z}{2h}} e^{(n+\frac{1}{4})\frac{\pi i z}{h}} - (4n\pi)^{-\frac{z}{2h}} e^{-(n+\frac{1}{4})\frac{\pi i z}{h}} \right\}$$

Hence in ϕ_0 the general term

$$= \frac{i}{2n\pi} G_0 \kappa R \left\{ (4n\pi)^{\frac{z-h}{2h}} e^{(n+\frac{1}{4})\frac{\pi i(z-h)}{h}} - (4n\pi)^{-\frac{z+h}{2h}} e^{-(n+\frac{1}{4})\frac{\pi i(z+h)}{h}} \right\};$$

in θ_0 , the same as this, with the factor $i/2n\pi$ omitted.

In both

$$G_0 \kappa R = \sqrt{\frac{h}{2nR}} e^{-(n+\frac{1}{4})\frac{\pi R}{h}} e^{\frac{iR \log 4n\pi}{2h}}$$

(ii) ϕ_e and θ_e .

$$\kappa h = \frac{1}{2} \log 4n\pi + (n - \frac{1}{4})\pi i$$

$$\cosh \kappa h = \frac{1}{2} (4n\pi)^{\frac{1}{2}} e^{(n-\frac{1}{4})\pi i}$$

In ϕ_e the general term

$$\frac{i}{2n\pi} G_0 \kappa R \left\{ (4n\pi)^{\frac{z-h}{2h}} e^{(n-\frac{1}{4})\frac{\pi i(z-h)}{h}} + (4n\pi)^{-\frac{z+h}{2h}} e^{-(n-\frac{1}{4})\frac{\pi i(z+h)}{h}} \right\}$$

In θ_e the same, with the factor $i/2n\pi$ omitted.

$$G_0 \kappa R = \sqrt{\frac{h}{2nR}} e^{-(n-\frac{1}{4})\frac{\pi R}{h}} e^{\frac{iR \log 4n\pi}{2h}}.$$

9. *The solution for arbitrary normal traction. Questions for discussion.*

The solution of the general problem of given normal traction requires the multiplication of the functions in (20), (21) by $f(x', y')$ and integration with respect to x', y' over a finite area A. There is no difficulty in showing that these integrations can be performed term by term, and that the resulting series converge absolutely.

When the solution of the general problem has thus been obtained in terms of series of surface integrals, several questions present themselves for treatment, among which may be specially mentioned

(i) For how many times in succession may these series be differentiated term by term with respect to the coordinates x, y, z ?

(ii) When the thickness of the plate is infinitesimal, but $f(x, y)$ does not vary as h tends to zero, what are the orders of the various parts of the solution, and of the related physical quantities?

(iii) How are the answers to these two questions affected by discontinuity in the applied traction, or its x, y derivatives?

A perfectly general discussion of these questions would be tedious and difficult, and it will probably be more useful to consider the points suggested in the light of a special case, in which the integrations required can be performed, and the outstanding features of the solution can be grasped with comparative ease.

10. Detailed solution of a special case. Term by term differentiations.

The solution we propose to work out is to satisfy the following boundary conditions:—

$$\begin{aligned} \widehat{zz} &= \begin{aligned} &4\pi\mu J_m(\beta\rho) \cos m\omega, \text{ on } z = h \\ &-4\pi\mu J_m(\beta\rho) \cos m\omega, \text{ on } z = -h \end{aligned} \left. \vphantom{\begin{aligned} &4\pi\mu J_m(\beta\rho) \cos m\omega, \text{ on } z = h \\ &-4\pi\mu J_m(\beta\rho) \cos m\omega, \text{ on } z = -h \end{aligned}} \right\} \text{ when } \rho < a \\ &= 0 && \text{on } z = \pm h, \text{ when } \rho > a \\ \widehat{zx} = \widehat{zy} &= 0, && \text{on } z = \pm h. \end{aligned}$$

ρ, ω, z are the cylindrical coordinates of the point (x, y, z) , so that $x = \rho \cos \omega$, $y = \rho \sin \omega$.

β is any constant, and m is an integer.

The solution, obtained from (20) by integration, is

$$\begin{aligned} \phi &= \phi_1 + \phi_2; \quad \theta = \theta_1 + \theta_2, \quad \text{where} \\ \phi_1 &= -\frac{3}{4h^3} \left(zF - \frac{1}{6} z^3 \nabla^2 F \right) + \frac{9}{40h} z \nabla^2 F \\ \theta_1 &= \frac{3}{4h^3} \left(zF - \frac{1}{6} z^3 \nabla^2 F - 2hz z \nabla^2 F \right) - \frac{9}{40h} z \nabla^2 F \end{aligned} \quad (29)$$

$$\text{with } F = \iint \chi(R) J_m(\beta\rho') \cos m\omega' \rho' d\rho' d\omega'$$

$$\begin{aligned} \phi_2 &= \sum_{\kappa} \left(- \frac{\cosh \kappa h \sinh \kappa z}{\kappa h (\cosh 2\kappa h - 1)} \right) \iint G_0 \kappa R J_m \beta \rho' \cos m\omega' \rho' d\rho' d\omega' \\ \theta_2 &= \sum_{\kappa} \left(\frac{\cosh \kappa h + 2\kappa h \sinh \kappa h}{\kappa h (\cosh 2\kappa h - 1)} \right) \sinh \kappa z \quad (\text{same integral}) \end{aligned} \quad (30)$$

the integrals being taken over the circle of radius a .

Consider in the first place the part of the solution defined by ϕ_2, θ_2 .

The value of the surface integral in (30) takes different forms when $\rho >$ and $< a$.

As proved in (c)

$$\begin{aligned} \text{when } \rho < \alpha, \text{ the integral} &= \frac{2\pi}{\beta^2 - \kappa^2} J_m \beta \rho \cos m\omega + \frac{2\pi}{\beta^2 - \kappa^2} J_m \kappa \rho \cos m\omega (\kappa \alpha J_m' \kappa \alpha J_m \beta \alpha - G_m \kappa \alpha \beta \alpha J_m' \beta \alpha) \\ \text{,, } \rho > \alpha, \text{ ,,} &= \frac{2\pi}{\beta^2 - \kappa^2} J_m \kappa \rho \cos m\omega (\kappa \alpha J_m' \kappa \alpha J_m \beta \alpha - J_m \kappa \alpha \beta \alpha J_m' \beta \alpha). \end{aligned}$$

Now when $\kappa\rho, \kappa\alpha$ are both large,

$$J_m \kappa \alpha J_m \kappa \rho = \frac{i}{2\kappa \sqrt{\alpha\rho}} e^{i\kappa(\rho-\alpha)},$$

which, with its derivatives, is very small when $(\rho - \alpha)/2h$ is even moderately large. Thus, in the space without the cylinder $\rho = \alpha$, the part of the strain given by ϕ_2, θ_2 is, when h is small, insensible except in the immediate neighbourhood of that cylinder. The same remark applies to the strain within the cylinder, so far as it is given by the parts of ϕ_2, θ_2 arising from the second term in the value of the surface integral.

We naturally inquire, how do these rapidly decaying parts of the solution behave, and what is the order of magnitude of the corresponding displacements and stresses, at points actually on the surface $\rho = \alpha$? Now, taking for example the value of ϕ_2 in the external region, namely,

$$\phi_2 = \sum_{\kappa} (-) \frac{\cosh \kappa h \sinh \kappa z}{\kappa h (\cosh 2\kappa h - 1)} \frac{2\pi}{\beta^2 - \kappa^2} G_m \kappa \rho \cos m\omega \left(\kappa \alpha J_m' \kappa \alpha J_m \beta \alpha - J_m \kappa \alpha \beta \alpha J_m' \beta \alpha \right) \quad (30')$$

we see from § 8 (i) that when $\rho = \alpha$, the general term has the approximate form

$$\frac{\Lambda}{n^3} \left\{ n^{\frac{z-h}{2h}} e^{(n+1)\frac{\pi i(z-h)}{h}} - n^{\frac{z+h}{2h}} e^{-(n+1)\frac{\pi i(z+h)}{h}} \right\},$$

Λ being independent of n . Moreover, each differentiation of ϕ_2 with respect to ρ or z will remove a factor $1/n$ from the general term. Hence *three* such differentiations, but no more, are permissible, if $-h < z < h$. But from (4) it is clear that none of the displacements requires more than two, and none of the stresses more than three of these differentiations for their calculation. As for θ_2 , the general term is of one order higher in n than the corresponding term in ϕ_2 , but in compensation for this only two differentiations are required to find the stresses. Hence, so far as the decaying part of the solution is concerned, displacements and stresses at $\rho = \alpha$ may be found by means of term by term differentiation, and subsequent substitution of α for ρ .

Again, considering the order of these various quantities in h , regarded as infinitesimal, and remembering that κh and κz are of order zero in h , we see that the expression for ϕ_2 in (30') and the corresponding expression for θ_2 are of order h^2 when $\rho = \alpha$, and each differentiation with respect to ρ or z diminishes the order by one. Hence the displacements at $\rho = \alpha$ are of order h and the stresses of order zero, so far as they arise from the decaying part of the solution ϕ_2, θ_2 .

11. *The same special problem. Summation of two infinite series.*

An important part of the strain given by ϕ_2, θ_2 remains to be considered, namely, that which arises from the term $2\pi/(\beta^2 - \kappa^2) \cdot J_m \beta \rho \cos m\omega$ in the value of the surface integral for the case when the point (ρ, ω) is within the cylinder $\rho = a$. Denoting these parts of ϕ_2, θ_2 by ϕ_3, θ_3 , we have

$$\begin{aligned}\phi_3 &= 2\pi J_m \beta \rho \cos m\omega \sum_{\kappa} (-) \frac{\cosh \kappa h \sinh \kappa z}{(\beta^2 - \kappa^2) \kappa h (\cosh 2\kappa h - 1)} \Bigg| \dots \dots \dots (31) \\ \theta_3 &= 2\pi J_m \beta \rho \cos m\omega \sum_{\kappa} \frac{(\cosh \kappa h + 2\kappa h \sinh \kappa h) \sinh \kappa z}{(\beta^2 - \kappa^2) \kappa h (\cosh 2\kappa h - 1)} \Bigg| \dots \dots \dots\end{aligned}$$

We note in the first place that ϕ_3 admits of three, and θ_3 of two term by term differentiations with respect to z when $-h < z < h$, while x, y differentiations can be performed without restriction. Combining this result with those already obtained, we see that in the complete solution in terms of surface integrals, all the differentiations necessary to give the displacements and strains or stresses at any point in the body of the plate can be performed on the series term by term.

When h is small, ϕ_3 and θ_3 are of order h^2 , and a z -differentiation lowers the order by one. This can be seen from the series, or otherwise, for, as we shall now show, the value of the series can be found in finite terms.

Consider the function of κ ,

$$\frac{\cosh \kappa h \sinh \kappa z}{(\beta^2 - \kappa^2) \kappa (\sinh 2\kappa h - 2\kappa h)}.$$

This function, multiplied by κ , vanishes at infinity at all points of the path E W A B described in § 5; hence the sum of its residues vanishes. The function being odd in κ , the residues at the poles $\kappa = \pm \kappa'$ are equal. Thus

$$\begin{aligned}& 2 \text{ (series of residues at zeroes of } \sinh 2\kappa h - 2\kappa h \text{ in upper part of plane)} \\ & + 2 \text{ (residue at } \kappa = \beta) + \text{(residue at } \kappa = 0) = 0.\end{aligned}$$

The residue at $\kappa = \beta$ is

$$(-) \frac{\cosh \beta h \sinh \beta z}{2\beta^2 (\sinh 2\beta h - 2\beta h)}.$$

Also if

$$\frac{\cosh \kappa h \sinh \kappa z}{\kappa (\sinh 2\kappa h - 2\kappa h)} = \frac{A}{\kappa^3} + \frac{B}{\kappa} + \dots, \text{ near } \kappa = 0,$$

then the residue at $\kappa = 0$ is

$$\frac{A}{\beta^4} + \frac{B}{\beta^2}.$$

Hence

$$\sum_{\kappa} (-) \frac{\cosh \kappa h \sinh \kappa z}{(\beta^2 - \kappa^2) \kappa h (\cosh 2\kappa h - 1)} = - \frac{\cosh \beta h \sinh \beta z}{\beta^2 (\sinh 2\beta h - 2\beta h)} + \frac{A}{\beta^4} + \frac{B}{\beta^2}.$$

It may be noted that $A/\beta^4 + B/\beta^2$ are simply the terms of negative degree in the expansion of

$$\frac{\cosh \beta h \sinh \beta z}{\beta^2 (\sinh 2\beta h - 2\beta h)}$$

in ascending powers of β .

Hence, putting in the values of A and B from (16), (18),

$$\begin{aligned} \phi_3 &= 2\pi J_m \beta \rho \cos m\omega \left\{ -\frac{\cosh \beta h \sinh \beta z}{\beta^2 (\sinh 2\beta h - 2\beta h)} + \frac{3z}{4\beta^4 h^3} + \frac{1}{\beta^2} \left(\frac{z^3}{8h^3} + \frac{9}{40} \frac{z}{h} \right) \right\} \\ \text{and similarly} \quad \theta_3 &= 2\pi J_m \beta \rho \cos m\omega \left\{ \frac{(\cosh \beta h + 2\beta h \sinh \beta h) \sinh \beta z}{\beta^2 (\sinh 2\beta h - 2\beta h)} - \frac{3z}{4\beta^4 h^3} - \frac{1}{\beta^2} \left(\frac{z^3}{8h^3} + \frac{69}{40} \frac{z}{h} \right) \right\} \end{aligned} \quad (32)$$

12. The same problem. Final form of the solution.

We come lastly to ϕ_1 , θ_1 of § 10.

The function F requires separate formulæ for its expression in the three cases $m=0$, $m=1$, $m>1$, but in all cases

$$\begin{aligned} F &= (2\pi/\beta^4) J_m \beta \rho \cos m\omega + F_1, & \text{when } \rho < a \\ F &= & F_2, & \text{when } \rho > a \end{aligned}$$

where F_1 and F_2 satisfy the equations $\nabla^4 F_1 = 0$, $\nabla^4 F_2 = 0$. The values of F_1 , F_2 for the various cases are given in (f).

When $\rho < a$, the term $(2\pi/\beta^4) J_m \beta \rho \cos m\omega$ of F, taken by itself in ϕ_1 and θ_1 , would give

$$\begin{aligned} \phi_1 &= -\frac{3}{4h^3} \frac{z}{\beta^4} - \frac{1}{\beta^2} \left(\frac{z^3}{8h^3} + \frac{9}{40} \frac{z}{h} \right) \\ \theta_1 &= \frac{3}{4h^3} \frac{z}{\beta^4} + \frac{1}{\beta^2} \left(\frac{z^3}{8h^3} + \frac{69}{40} \frac{z}{h} \right) \end{aligned} \left\{ 2\pi J_m \beta \rho \cos m\omega \right\}$$

These are precisely the terms of negative degree (both in β and in h) with signs changed, in the expressions for ϕ_3 and θ_3 given at the end of § 11.

If, then, we take this part of ϕ_1 , θ_1 along with ϕ_3 , θ_3 we have the complete solution in the form

$$\begin{aligned} \phi &= -2\pi J_m \beta \rho \cos m\omega \frac{\cosh \beta h \sinh \beta z}{\beta^2 (\sinh 2\beta h - 2\beta h)} \\ &+ 2\pi \sum_{\kappa} (-) \frac{\cosh \kappa h \sinh \kappa z}{(\beta^2 - \kappa^2) \kappa h (\cosh 2\kappa h - 1)} J_m \kappa \rho \cos m\omega (\kappa a G_m' \kappa a J_m \beta a - G_m \kappa a \beta a J_m' \beta a) \\ &\quad - \frac{3}{4h^3} \left(z F_1 - \frac{1}{6} z^3 \nabla^2 F_1 \right) + \frac{9}{40h} z \nabla^2 F_1 \\ \theta &= 2\pi J_m \beta \rho \cos m\omega \frac{(\cosh \beta h + 2\beta h \sinh \beta h) \sinh \beta z}{\beta^2 (\sinh 2\beta h - 2\beta h)} \\ &+ 2\pi \sum_{\kappa} \frac{(\cosh \kappa h + 2\kappa h \sinh \kappa h) \sinh \kappa z}{(\beta^2 - \kappa^2) \kappa h (\cosh 2\kappa h - 1)} J_m \kappa \rho \cos m\omega (\kappa a G_m' \kappa a J_m \beta a - G_m \kappa a \beta a J_m' \beta a) \\ &\quad + \frac{3}{4h^3} \left(z F_1 - \frac{1}{6} z^3 \nabla^2 F_1 - 2h^2 z \nabla^2 F_1 \right) - \frac{9}{40h} z \nabla^2 F_1 \\ &\quad \text{when } \rho < a. \end{aligned} \quad (33)$$

and

$$\begin{aligned} \phi &= 2\pi \sum_{\kappa} (-) \frac{\cosh \kappa h \sinh \kappa z}{(\beta^2 - \kappa^2) \kappa h (\cosh 2\kappa h - 1)} J_m \kappa \rho \cos m\omega (\kappa a J_m' \kappa a J_m \beta a - J_m \kappa a \beta a J_m' \beta a) \\ &\quad - \frac{3}{4h^3} \left(z F_2 - \frac{1}{6} z^3 \nabla^2 F_2 \right) + \frac{9}{40h} z \nabla^2 F_2 \\ \theta &= 2\pi \sum_{\kappa} \frac{(\cosh \kappa h + 2\kappa h \sinh \kappa h) \sinh \kappa z}{(\beta^2 - \kappa^2) \kappa h (\cosh 2\kappa h - 1)} G_m \kappa \rho \cos m\omega (\kappa a J_m' \kappa a J_m \beta a - J_m \kappa a \beta a J_m' \beta a) \\ &\quad + \frac{3}{4h^3} \left(z F_2 - \frac{1}{6} z^3 \nabla^2 F_2 - 2h^2 z \nabla^2 F_2 \right) - \frac{9}{40h} z \nabla^2 F_2, \\ &\quad \text{when } \rho > a. \end{aligned} \quad (34)$$

In (33) and (34) each line represents a potential function; in (33) the first lines define a particular solution giving the proper values of the tractions at the surface, as may be seen from (13); the partial solutions given by the κ series give zero surface tractions, and represent a strain insensible except in the vicinity of $\rho = \alpha$; and the solutions defined by the last lines, being of the form (23), give zero surface tractions.

From these remarks it follows immediately that the solution (33), (34) satisfies all the conditions of the problem in the two regions $\rho < \alpha$, $\rho > \alpha$, taken separately. To verify the solution completely, it would be necessary to show in addition that certain conditions are fulfilled at the surface $\rho = \alpha$, namely,

- (i) that the displacements and strains are continuous at this surface, and
- (ii) that the integral value of the stresses \widehat{xx} , \widehat{xy} , \widehat{yz} over any small area lying partly within and partly without the cylinder $\rho = \alpha$, on either of the plane faces of the plate, tends to zero when the area is indefinitely diminished.

The condition (i) ensures the 'synexis' of the solution across the surface $\rho = \alpha$, and can be proved by showing, as may easily be done by means of summations similar to those of § 11, that ϕ , θ , $\frac{d\phi}{d\rho}$ and $\frac{d\theta}{d\rho}$ are continuous at that surface. For by the Theory of the Potential this carries with it the continuity of all the derivatives of ϕ and θ , and therefore of the displacements and stresses, as well as of all their derivatives, under the proviso, of course, that $-h < z < h$.

The condition (ii), or some equivalent, is required in order to exclude the possibility of stresses with finite resultant passing into the solid through the lines $z = \pm h$, $\rho = \alpha$; or, in other words, in order to ensure that the solution is not partly due to linear elements of traction at these lines.

13. *Order of the various parts of the solution, when h is small.*

The final form of the solution, as exhibited in (33), (34) was obtained by combining parts of ϕ_1 , θ_1 with ϕ_2 , θ_2 , and until this was done, it was not immediately evident that ϕ and θ were potentials. Thus the part of the solution arising from the imaginary values of κ , or from any one of them, is not, within the region of applied traction, a potential by itself, and the same is true of the ϕ_1 , θ_1 part, which may be considered as coming from the zero values of κ . This has sometimes to be taken into account in calculating the stresses; the formula for \widehat{zz} , for example, in (5) requires additional terms if, while u , v , w are still given by (4), ϕ and θ are not potentials.

On the other hand, the separation of the solution into the two parts (29), (30) has this very marked advantage that, when h is very small, the first part gives the terms of the two lowest orders in h of ϕ , θ , namely those of orders h^{-2} and h^0 , while the second part, as we have already seen, contains no terms of lower order than h^2 . When, however, we come to calculate displacements and stresses, the separation is less simple, mainly in consequence of the fact that x , y differentiations do not change the order of ϕ_1 , θ_1 , but diminish the order of ϕ_2 , θ_2 by one for each differentiation.

The following table, which may be deduced immediately from the results of §§ 10, 11, shows the order in h of displacements and stresses arising from ϕ_1, θ_1 and ϕ_2, θ_2 respectively.

	ϕ_1, θ_1	ϕ_2, θ_2	
		$\rho \neq a$	$\rho = a$
u, v	-2, 0	2	1
w	-3, -1	1	1
$\widehat{xx}, \widehat{xy}, \widehat{yy}$	-2, 0	0	0
$\widehat{zx}, \widehat{zy}$	-1	1	0
\widehat{zz}	0	0	0

It thus appears that the first part of the solution gives all the displacements to a second approximation, and all the stresses but \widehat{zz} to a first approximation. With regard to \widehat{zz} , it should be observed that the solutions depending on F_1, F_2 contribute nothing to it, so that, within $\rho = a$, its value comes altogether from the particular solution, and without $\rho = a$, its value is zero beyond the immediate vicinity of that surface.

14. *Methods and results of the special case extended to the general problem of arbitrary normal traction.*

One feature of the solution expressed by equations (33), (34) we have already found useful, especially in the important case when h is small, in such a way that βh and h/a are small fractions. We refer to the explicit separation in the solution of a purely local element, entirely negligible except within a certain strip of breadth comparable with the thickness of the plate, from an element of a persistent or permanent character, with an area of influence not affected by the indefinite diminution of h .

Another advantage of the form of solution in (33), (34) is that the particular solution for the space within which the traction is applied is found in such a form that it can be readily expanded in powers of h , so as to give the terms of positive order in the infinitesimal h , as well as those of negative order which were already separated in (29). Thus in the particular solution, or first line of ϕ in (33), the factor

$$\frac{\cosh \beta h \sinh \beta z}{\beta^2 (\sinh 2\beta h - 2\beta h)}$$

can be expanded in ascending powers of β , the series converging if $|2\beta h| < |\zeta_1|$, where ζ_1 is the complex root of $\sinh \zeta - \zeta = 0$ with smallest modulus. Since z is of the same order as h , and we are supposing β independent of h , it is clear that the terms of the series will be of ascending order in h .

We shall now show how the solution for the general case when the given normal traction is a function of x, y of unspecified form may be transformed, under certain restrictions, so as to yield the advantages to which we have been referring as pertaining to the solution (33), (34).

The problem we suppose to be the same as that stated at the beginning of § 10,

but with $f(x, y)$ instead of $J_m \beta \rho \cos m\omega$, and with any continuous area A instead of the circle within $\rho = a$.

The solution will then be of the form defined in (29), (30), but the integral of (29) will now be

$$F = \iint \chi(R) f(x', y') dx' dy'$$

and the integral of (30)

$$1 = \iint G_0(\kappa R) f(x', y') dx' dy'.$$

Since $\nabla^2 \chi(R) = \log(R/2h)$ we have

$$\left. \begin{aligned} \nabla^4 F &= \nabla^2(\nabla^2 F) = 2\pi f(x, y) \\ (\nabla^2 + \kappa^2)I &= -2\pi f(x, y) \end{aligned} \right\} \quad (35)$$

also

If $f(x, y)$ and its derivatives of the first two orders are finite within A, we may transform I by Green's Theorem. Thus, excluding from the area A an infinitesimal circle about (x, y) as centre, and writing ∇'^2 for $\frac{d^2}{dx'^2} + \frac{d^2}{dy'^2}$,

$$I = -\frac{1}{\kappa^2} \iint f(x', y') \nabla'^2 G_0 \kappa R dx' dy'$$

or

$$\left. \begin{aligned} I &= -\frac{1}{\kappa^2} \iint G_0 \kappa R \nabla'^2 f(x', y') dx' dy' \\ &\quad - \frac{1}{\kappa^2} \int \left\{ f(x', y') \frac{d}{dn} G_0 \kappa R - G_0 \kappa R \frac{d}{dn} f(x', y') \right\} ds \\ &\quad - \frac{2\pi}{\kappa^2} f(x, y) \end{aligned} \right\} \quad (36)$$

the line integral being taken round the boundary of A.

If this three-termed equivalent of the integral I be substituted in the series for ϕ_2 and θ_2 , each of these series may be subdivided into three, ϕ_2 for instance into

$$(i) \quad \sum_{\kappa} \frac{1}{\kappa^2} \frac{\cosh \kappa h \sinh \kappa z}{\kappa h (\cosh 2\kappa h - 1)} \iint G_0 \kappa R \nabla'^2 f(x', y') dx' dy',$$

a series of the same general form as the original series, but at once more convergent, and of two orders higher in h ;

$$(ii) \quad \sum_{\kappa} \frac{1}{\kappa^2} \frac{\cosh \kappa h \sinh \kappa z}{\kappa h (\cosh 2\kappa h - 1)} \int \left\{ f(x', y') \frac{d}{dn} G_0 \kappa R - G_0 \kappa R \frac{d}{dn} f(x', y') \right\} ds,$$

which corresponds to a strain local to the boundary of A;

$$(iii) \quad 2\pi f(x, y) \sum_{\kappa} \frac{1}{\kappa^2} \frac{\cosh \kappa h \sinh \kappa z}{\kappa h (\cosh 2\kappa h - 1)}, \text{ a series which can be summed in the same way}$$

as (31), being in fact simply the first series of (31) with $\beta = 0$. The sum is therefore

$$2\pi f(x, y) \cdot \text{coefficient of } \kappa^0 \text{ in } \frac{-\cosh \kappa h \sinh \kappa z}{\kappa^2 (\sinh 2\kappa h - 2\kappa h)}.$$

We may now, by repetition of the same transformation, obtain a similar threefold

equivalent for the series (i), and continue the process as far as the continuity of the derivatives of $f(x, y)$ will permit. We should thus obtain, in place of ϕ_2 ,

$$(i) \quad (-)^n \sum_{\kappa} (-)^{\frac{1}{\kappa^{2n}}} \frac{\cosh \kappa h \sinh \kappa z}{\kappa h (\cosh 2\kappa h - 1)} \iint A_0 \kappa R \nabla^{2n} f(x', y') dx' dy',$$

a function of order h^{2n+2} .

(ii) A series of line-integrals which we need not write down, corresponding to a local perturbation at the edge of A, and giving the edge values of the relative part of ϕ up to terms in h^{2n} .

(iii) A series of n terms

$$2\pi(c_0 f - c_2 \nabla^2 f + c_4 \nabla^4 f - \dots + (-)^{n-1} c_{2n-2} \nabla^{2n-2} f),$$

where c_{2r} is the coefficient of κ^{2r} in the ascending power expansion of $(-)^{\frac{\cosh \kappa h \sinh \kappa z}{\kappa^2 (\sinh 2\kappa h - 2\kappa h)}}$, and is obviously a rational integral function of z and h of degree $2r+2$. If the function $f(x, y)$ has its derivatives of every finite order continuous throughout the area A, the process can be carried to as high a value of n as we please, and we can thus obtain the values of ϕ_2, θ_2 to any required order in h . It should be noticed, however, that the series (iii) is not necessarily convergent when continued to infinity, as we may see by taking as an example $f = \cos ax$, when the series would become $2\pi \cos ax (c_0 + c_2 a^2 + c_4 a^4 + \dots)$, which is the expansion (without the terms of negative degree) of

$$-2\pi \cos ax \frac{\cosh ah \sinh az}{a^2 (\sinh 2ah - 2ah)},$$

and is therefore divergent if $|2ah| > |\zeta_1|$, ζ_1 being the complex root of $\sinh \zeta - \zeta = 0$ with smallest modulus. The form of the condition suggests that in ordinary cases the series will be convergent if h is small enough; and when this is so, this part of ϕ_2, θ_2 taken along with ϕ_1, θ_1 will define an exact particular solution within A, giving the proper values of the surface tractions, and arranged in terms of ascending order in h .

As a special case, the series will terminate if, for some finite value of n , $\nabla^{2n} f = 0$, and in particular if f be a rational integral function of x, y . (It may be noted here that the solution for $f = \rho^{m+2p} \cos m\omega$ might be obtained from the solution for $f = J_m \beta \rho \cos m\omega$ by expanding in powers of β , and equating coefficients of β^{m+2p} in conditions and solution.)

Looking back now to the ϕ_1, θ_1 part of the solution, and having regard to (18), (29), (35), we see that we may write symbolically

$$\begin{aligned} F &= 2\pi \nabla^{-4} f; \quad \nabla^2 F = 2\pi \nabla^{-2} f, \quad \text{and} \\ \phi_1 &= 2\pi (c_{-4} \nabla^{-4} f - c_{-2} \nabla^{-2} f). \end{aligned}$$

The particular solution to any order in h is then given by

$$\phi = 2\pi (c_{-4} \nabla^{-4} f - c_{-2} \nabla^{-2} f + c_0 f - c_2 \nabla^2 f + \dots)$$

where c_{2r} is the coefficient of κ^{2r} in the expansion of $(-)^{\frac{\cosh \kappa h \sinh \kappa z}{\kappa^2 (\sinh 2\kappa h - 2\kappa h)}}$, for negative as well as positive values of r ; or, as we may put it, this particular part of ϕ is given by expanding $(-2\pi)^{\frac{\cosh \kappa h \sinh \kappa z}{\kappa^2 (\sinh 2\kappa h - 2\kappa h)}}$, writing $-\nabla^2$ for κ^2 , and operating on $f(x, y)$.

15. *Independent symbolical solution of the general problem.*

The form of the last result suggests a method of dealing with the problem from the beginning, which, though not easy to develop independently with thorough rigour, has the advantage of conciseness, and will therefore be useful in giving a rapid account both of the foregoing solution and of those to be obtained in the following pages.

We begin by observing that

$$\begin{aligned} \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} \right) \sinh \kappa z f(x, y) &= \sinh \kappa z \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \kappa^2 \right) f(x, y) \\ &= \sinh \kappa z (\nabla^2 + \kappa^2) f(x, y). \end{aligned}$$

Hence $\sinh \kappa z f(x, y)$ is a potential function, provided we regard κ as an operator such that $\kappa^2 = -\nabla^2$. We may, if we please, take $\kappa = i\nabla$, but it will not be necessary to interpret odd powers of the operator ∇ .

On this understanding, it is obvious from (12), (13) that we obtain a solution giving

$$\begin{aligned} \widehat{v} = \widehat{y} &= 0 && \text{on } z = \pm h \\ \widehat{z} &= 0 && \text{on } z = -h \\ \widehat{z} &= f(x, y) && \text{on } z = h, \text{ within the area } A, \end{aligned}$$

by taking

$$\left. \begin{aligned} 4\mu\phi &= -\frac{\cosh \kappa h}{\kappa^2(\sinh 2\kappa h - 2\kappa h)} \sinh \kappa z f - \frac{\sinh \kappa h}{\kappa^2(\sinh 2\kappa h + 2\kappa h)} \cosh \kappa z f \\ 4\mu\theta &= \frac{\cosh \kappa h + 2\kappa h \sinh \kappa h}{\kappa^2(\sinh 2\kappa h - 2\kappa h)} \sinh \kappa z f + \frac{\sinh \kappa h + 2\kappa h \cosh \kappa h}{\kappa^2(\sinh 2\kappa h + 2\kappa h)} \cosh \kappa z f \end{aligned} \right\} \quad (37)$$

Now, taking as a specimen the first term of $4\mu\phi$, we observe that the function of κ

$$(-) \frac{\cosh \kappa h}{\kappa^2(\sinh 2\kappa h - 2\kappa h)} \sinh \kappa z$$

vanishes at infinity round the path W A B E of § 5.

Hence the function is represented by the sum of its polar elements. (If κ_1 be a simple pole of the function, and if in the vicinity of this pole the function $= A_1/(\kappa - \kappa_1) + \text{finite}$, then $A_1/(\kappa - \kappa_1)$ is the polar element at this pole, and A_1 is the residue there. The point $\kappa = 0$ is a multiple pole, and the polar element there has the form $A/\kappa^4 + B/\kappa^2$, these being the terms of negative degree in the expansion of the function near $\kappa = 0$).

Taking the elements belonging to $\pm \kappa_1$ together we obtain

$$\begin{aligned} (-) \frac{\cosh \kappa h \sinh \kappa z}{\kappa^2(\sinh 2\kappa h - 2\kappa h)} &= \frac{A}{\kappa^4} + \frac{B}{\kappa^2} + A_1 \left(\frac{1}{\kappa - \kappa_1} - \frac{1}{\kappa + \kappa_1} \right) + \dots \\ &= \frac{A}{\kappa^4} + \frac{B}{\kappa^2} + \frac{2A_1\kappa_1}{\kappa^2 - \kappa_1^2} + \dots \\ &= \frac{A}{\kappa^4} + \frac{B}{\kappa^2} + (-) \frac{\cosh \kappa_1 h \sinh \kappa_1 z}{\kappa_1 h (\cosh 2\kappa_1 h - 1)} \frac{1}{\kappa^2 - \kappa_1^2} + \dots \end{aligned}$$

the series extending over the poles with positive imaginary part. When we put $\kappa^2 = -\nabla^2$, this part of $4\mu\phi$ becomes

$$A\nabla^{-4}f - B\nabla^{-2}f + \sum_{\kappa} \frac{\cosh \kappa h \sinh \kappa z}{\kappa h (\cosh 2\kappa h - 1)} \frac{1}{\nabla^2 + \kappa^2} f,$$

where κ is no longer an operator, but simply a root of $\sinh 2\kappa h - 2\kappa h = 0$.

Now since

$$(\nabla^2 + \kappa^2) \iint G_0 \kappa R f(x', y') dx' dy' = -2\pi f(x, y),$$

one value of

$$\frac{1}{\nabla^2 + \kappa^2} f \quad \text{is} \quad -\frac{1}{2\pi} \iint G_0 \kappa R f(x', y') dx' dy'$$

Similarly one value of

$$\nabla^{-2} f \quad \text{is} \quad \frac{1}{2\pi} \iint \log(R/2h) f(x', y') dx' dy'$$

and of

$$\nabla^{-4} f \quad \frac{1}{2\pi} \iint \left(\frac{1}{4} R^2 \log \frac{R}{2h} - \frac{1}{4} R^2 \right) f(x', y') dx' dy'$$

Hence for the first part of $4\mu\phi$ we obtain

$$\begin{aligned} & -\frac{1}{2\pi} \frac{3z}{4h^3} \iint \chi(R) f(x', y') dx' dy' + \frac{1}{2\pi} \left(\frac{z^3}{8h^3} + \frac{9}{40} \frac{z}{h} \right) \iint \nabla^2 \chi(R) f(x', y') dx' dy' \\ & + \frac{1}{2\pi} \sum_{\kappa} (-) \frac{\cosh \kappa h \sinh \kappa z}{\kappa h (\cosh 2\kappa h - 1)} \iint G_0 \kappa R f(x', y') dx' dy' \end{aligned} \quad (38)$$

which agrees with our previous solution, the element of which is given in (20).

Further, the results obtained at the end of § 14 clearly agree with what we should get by expanding the function of κ in (37) in ascending power series and interpreting.

As an example of this use of equations (37) we may find to a first approximation the value of \widehat{w} at points not very close to the edge of A.

From (5)

$$\begin{aligned} 2(\widehat{w}) &= \frac{d^2}{dz^2} (4\mu\theta) - \frac{d^2}{dz^2} (4\mu\phi) + 2z \frac{d^3}{dz^3} (4\mu\phi) \\ &= \frac{(2 \cosh \kappa h + 2\kappa h \sinh \kappa h) \sinh \kappa z - 2\kappa z \cosh \kappa h \cosh \kappa z}{\sinh 2\kappa h - 2\kappa h} \cdot f \\ &\quad + \frac{(2 \sinh \kappa h + 2\kappa h \cosh \kappa h) \cosh \kappa z - 2\kappa z \sinh \kappa h \sinh \kappa z}{\sinh 2\kappa h + 2\kappa h} \cdot f \\ &= (3h^2 - z^2)zf/2h^3 + f. \end{aligned}$$

Thus

$$\widehat{w} = \{ (3h^2 - z^2)z/4h^3 + 1/2 \} f(x, y), \quad (40)$$

and we verify at a glance that this gives the proper values at the faces.

16. *The problem of tangential face traction. Solution for an element of traction.*

We will now pass to the problem in which the given surface traction is tangential. Taking the direction of the traction parallel to the axis of x , we may take for conditions

$$\begin{aligned} \widehat{w}_x &= f(x, y) & \text{on } z = +h \\ \widehat{w}_x &= 0 & \text{on } z = -h \\ \widehat{w}_y &= 0 & \text{on } z = \pm h \end{aligned} \quad (41)$$

According to the method explained in § 3, we begin with the function $\kappa J_0(\kappa R)$ in place of $f(x, y)$, and determine potentials ψ, θ, ϕ giving

$$\begin{aligned} & \left. \begin{aligned} \frac{d^2\psi}{dydz} + \frac{d^2\theta}{dx dz} + \frac{d^2\phi}{dx dz} + 2z \frac{d^3\phi}{dz^3 dx} &= \frac{1}{2\mu} \kappa J_0 \kappa R & \text{on } z = h \\ &= 0 & \text{on } z = -h \\ -\frac{d^2\psi}{dx dz} + \frac{d^2\theta}{dy dz} + \frac{d^2\phi}{dy dz} + 2z \frac{d^3\phi}{dz^3 dy} &= 0 & \text{on } z = \pm h \\ \frac{d^2\theta}{dz^2} - \frac{d^2\phi}{dz^2} + 2z \frac{d^3\phi}{dz^3} &= 0 & \text{on } z = \pm h \end{aligned} \right\} \quad (42) \end{aligned}$$

Since $J_0 \kappa R = -\frac{1}{\kappa^2} \frac{d^2}{dx^2} J_0 \kappa R - \frac{1}{\kappa^2} \frac{d^2}{dy^2} J_0 \kappa R$, it is clear that these equations will all be satisfied if we take

$$\psi = \frac{d\psi'}{dz}, \theta = \frac{d\theta'}{dz}, \phi = \frac{d\phi'}{dz} \quad (43)$$

where

$$\left. \begin{aligned} \frac{d\psi'}{dz} &= -\frac{1}{2\mu} \frac{1}{\kappa} J_0 \kappa R \quad \text{on } z=h \\ &= 0 \quad \text{on } z=-h \end{aligned} \right\} \quad (44)$$

$$\left. \begin{aligned} \frac{d\theta'}{dz} + \frac{d\phi'}{dz} + 2z \frac{d^2\phi'}{dz^2} &= -\frac{1}{2\mu} \frac{1}{\kappa} J_0 \kappa R \quad \text{on } z=h \\ &= 0 \quad \text{on } z=-h \end{aligned} \right\} \quad (45)$$

$$\frac{d^2\theta'}{dz^2} - \frac{d^2\phi'}{dz^2} + 2z \frac{d^3\phi'}{dz^3} = 0 \quad \text{on } z=\pm h$$

From (44),

$$4\mu\psi' = -\frac{2}{\kappa^2} \frac{\cosh \kappa(z+h)}{\sinh 2\kappa h} J_0 \kappa R,$$

or, separating the odd and even parts in z ,

$$4\pi\psi' = -\frac{1}{\kappa^2} \frac{\cosh \kappa z}{\sinh \kappa h} J_0 \kappa R - \frac{1}{\kappa^2} \frac{\sinh \kappa z}{\cosh \kappa h} J_0 \kappa R$$

We also find easily

$$\left. \begin{aligned} 4\pi\phi' &= -\frac{\sinh \kappa h}{\kappa^2(\sinh 2\kappa h - 2\kappa h)} J_0 \kappa R \sinh \kappa z \\ &\quad - \frac{\cosh \kappa h}{\kappa^2(\sinh 2\kappa h + 2\kappa h)} J_0 \kappa R \cosh \kappa z \\ 4\mu\theta' &= \frac{2\kappa h \cosh \kappa h - \sinh \kappa h}{\kappa^2(\sinh 2\kappa h - 2\kappa h)} J_0 \kappa R \sinh \kappa z \\ &\quad + \frac{2\kappa h \sinh \kappa h - \cosh \kappa h}{\kappa^2(\sinh 2\kappa h + 2\kappa h)} J_0 \kappa R \cosh \kappa z \end{aligned} \right\} \quad (46)$$

Treating these expressions as in § 3, we find a solution for an element of X-traction at (x', y', h) of $8\pi\mu$ units given by (43) with

$$\begin{aligned} \psi' &= \int_0^z \left(-\frac{\sinh \kappa z}{\kappa^2 \cosh \kappa h} J_0 \kappa R + \frac{e^{-\kappa h} z}{\kappa} \right) d\kappa \\ &\quad + \int_0^\infty \left\{ -\frac{\cosh \kappa z}{\kappa^2 \sinh \kappa h} J_0 \kappa R + \frac{1}{\kappa^3 h} + \frac{e^{-\kappa h}}{\kappa h} \left(\frac{1}{2} z^2 - \frac{1}{6} h^2 - \frac{1}{4} R^2 \right) \right\} d\kappa \\ \phi' &= \int_0^\infty \left\{ -\frac{\sinh \kappa h \sinh \kappa z}{\kappa^2(\sinh 2\kappa h - 2\kappa h)} J_0 \kappa R + \frac{3z}{4\kappa^3 h^2} + \frac{e^{-\kappa h}}{8\kappa h} \left(z^3 - \frac{3}{2} R^2 z - \frac{1}{5} h^2 z \right) \right\} d\kappa \\ &\quad + \int_0^\infty \left\{ -\frac{\cosh \kappa h \cosh \kappa z}{\kappa^2(\sinh 2\kappa h + 2\kappa h)} J_0 \kappa R + \frac{1}{4\kappa^3 h} + \frac{e^{-\kappa h}}{8\kappa h} \left(z^2 - \frac{1}{2} R^2 + \frac{1}{3} h^2 \right) \right\} d\kappa \\ \theta' &= \int_0^\infty \left\{ \frac{(2\kappa h \cosh \kappa h - \sinh \kappa h) \sinh \kappa z}{\kappa^2(\sinh 2\kappa h - 2\kappa h)} J_0 \kappa R - \frac{3z}{4\kappa^3 h^2} - \frac{e^{-\kappa h}}{8\kappa h} \left(z^3 - \frac{3}{2} R^2 z + \frac{19}{5} h^2 z \right) \right\} d\kappa \\ &\quad + \int_0^\infty \left\{ \frac{(2\kappa h \sinh \kappa h - \cosh \kappa h) \cosh \kappa z}{\kappa^2(\sinh 2\kappa h + 2\kappa h)} J_0 \kappa R + \frac{1}{4\kappa^3 h} + \frac{e^{-\kappa h}}{8\kappa h} \left(z^2 - \frac{1}{2} R^2 - \frac{11}{3} h^2 \right) \right\} d\kappa \end{aligned} \quad (47)$$

These expressions may be transformed by the method of § 5, and a slight inspection of the relations between (16), (17) and (20), (21) will enable us to write down the

results at once. These are, if we separate the parts odd and even in z ,

$$\psi_0' = \sum_{\kappa} (-2) \frac{\sinh \kappa z}{\kappa^2 h \sinh \kappa h} \{ G_0(\kappa R) + z \nabla^2 \chi \} \quad (48)$$

where κ is a positive imaginary root of $\cosh \kappa h$,

$$\left. \begin{aligned} \phi_0' &= \sum_{\kappa} (-) \frac{\sinh \kappa h \sinh \kappa z}{\kappa^2 h (\cosh 2\kappa h - 1)} \left\{ G_0(\kappa R) - \frac{3}{4h} \left(z\chi - \frac{1}{6} z^3 \nabla^2 \chi \right) - \frac{1}{40} z \nabla^2 \chi \right. \\ \theta_0' &= \sum_{\kappa} \frac{(2\kappa h \cosh \kappa h - \sinh \kappa h) \sinh \kappa z}{\kappa^2 h (\cosh 2\kappa h - 1)} \left\{ G_0(\kappa R) + \frac{3}{4h} \left(z\chi - \frac{1}{6} z^3 \nabla^2 \chi - 2h^2 z \nabla^2 \chi \right) + \frac{1}{40} z \nabla^2 \chi + z \nabla^2 \chi \right\} \end{aligned} \right\} \quad (49)$$

where κ is a zero of $\sinh 2\kappa h - 2\kappa h$, with positive imaginary part.

$$\psi_e' = \sum_{\kappa} (-2) \frac{\cosh \kappa z}{\kappa^2 h \cosh \kappa h} \left\{ G_0(\kappa R) - \frac{1}{h} \left(\chi - \frac{1}{2} z^2 \nabla^2 \chi \right) - \frac{h}{6} \nabla^2 \chi \right\} \quad (50)$$

where κ is a positive imaginary root of $\sinh \kappa h$.

$$\left. \begin{aligned} \phi_e' &= \sum_{\kappa} (-) \frac{\cosh \kappa h \cosh \kappa z}{\kappa^2 h (\cosh 2\kappa h + 1)} \left\{ G_0(\kappa R) - \frac{1}{4h} \left(\chi - \frac{1}{2} z^2 \nabla^2 \chi \right) + \frac{h}{24} \nabla^2 \chi \right. \\ \theta_e' &= \sum_{\kappa} \frac{(2\kappa h \sinh \kappa h - \cosh \kappa h) \cosh \kappa z}{\kappa^2 h (\cosh 2\kappa h + 1)} \left\{ G_0(\kappa R) - \frac{1}{4h} \left(\chi - \frac{1}{2} z^2 \nabla^2 \chi \right) - \frac{11h}{24} \nabla^2 \chi \right\} \end{aligned} \right\} \quad (51)$$

where κ is a zero of $\sinh 2\kappa h + 2\kappa h$, with positive imaginary part.

The solution is defined by these equations with

$$\psi = \frac{d\psi'}{dy}, \quad \theta = \frac{d\theta'}{dx}, \quad \phi = \frac{d\phi'}{dx}.$$

17. Composition of the solution.

On examining the composition of the solution, we observe in the decaying parts of ϕ , θ , solutions of the class already obtained in (22), (25), and in the corresponding part of ψ , solutions of the type

$$\psi = \sin(2n+1) \frac{\pi z}{2h} F(x, y)$$

$$\psi = \cos \frac{n\pi z}{h} F(x, y),$$

each giving $\frac{d\psi}{dz} = 0$ on $z = \pm h$, and therefore zero tractions at the surface. As for the permanent terms, they may be arranged in the following groups, in each of which the surface stresses vanish.

$$(i) \quad \left. \begin{aligned} \phi &= -\frac{3}{4h^2} \frac{d}{dx} \left(z\chi - \frac{1}{6} z^3 \nabla^2 \chi \right) \\ \theta &= +\frac{3}{4h^2} \frac{d}{dx} \left(z\chi - \frac{1}{6} z^3 \nabla^2 \chi - 2h^2 z \nabla^2 \chi \right) \end{aligned} \right\}$$

$$(ii) \quad \left. \begin{aligned} \phi &= -\frac{1}{40} z \frac{d}{dx} \nabla^2 \chi \\ \theta &= \frac{1}{40} z \frac{d}{dx} \nabla^2 \chi \end{aligned} \right\}$$

These are of the types (23), (24).

$$(iii) \quad \left. \begin{aligned} \psi &= z \frac{d}{dy} \nabla^2 \chi \\ &= z \frac{d}{dx} \nabla^2 \chi \end{aligned} \right\}$$

This gives, by (4), since $\nabla^4 \chi = 0$,

$$\left. \begin{aligned} u &= z \frac{d^2}{dy^2} \nabla^2 \chi = - z \frac{d^2}{dx^2} \nabla^2 \chi \\ v &= - z \frac{d^2}{dxdy} \nabla^2 \chi \\ w &= \frac{d}{dx} \nabla^2 \chi \end{aligned} \right\}$$

and may therefore be considered as of the type (24). It is important to note, however, both here and in the cases immediately following, that the transformation on the value of u will not hold after the elementary solution has been integrated for the purposes of the general problem in (41).

$$(iv) \quad \left. \begin{aligned} \psi &= -\frac{1}{h} \frac{d}{dy} \left(\chi - \frac{1}{2} z^2 \nabla^2 \chi \right) \\ \phi &= -\frac{1}{4h} \frac{d}{dx} \left(\chi - \frac{1}{2} z^2 \nabla^2 \chi \right) \\ &= -\frac{1}{4h} \frac{d}{dx} \left(\chi - \frac{1}{2} z^2 \nabla^2 \chi \right) \end{aligned} \right\}$$

Now F being any function of x, y , satisfying $\nabla^4 F = 0$, the solution

$$\left. \begin{aligned} \psi &= 4 \frac{d}{dy} \left(F - \frac{1}{2} z^2 \nabla^2 F \right) \\ \phi = \theta &= \frac{d}{dx} \left(F - \frac{1}{2} z^2 \nabla^2 F \right) \end{aligned} \right\} \quad (52)$$

leads to

$$\left. \begin{aligned} u &= 8 \frac{d^2 F}{dy^2} + (\alpha + 1) \frac{d^2 F}{dx^2} + \frac{3 - \alpha}{2} z^2 \frac{d^2}{dx^2} \nabla^2 F \\ v &= -8 \frac{d^2 F}{dxdy} + (\alpha + 1) \frac{d^2 F}{dx dy} + \frac{3 - \alpha}{2} z^2 \frac{d^2}{dxdy} \nabla^2 F \\ w &= (\alpha - 3) z \frac{d}{dx} \nabla^2 F \end{aligned} \right\} \quad (53)$$

and $\widehat{zx} = \widehat{zy} = \widehat{zz} = 0$.

$$(v) \quad \psi = -\frac{h}{6} \frac{d}{dy} \nabla^2 \chi, \quad \phi = \frac{h}{24} \frac{d}{dx} \nabla^2 \chi, \quad \theta = -\frac{11h}{24} \frac{d}{dx} \nabla^2 \chi$$

which may be further decomposed into

$$\psi = 0, \quad \phi = \frac{h}{24} \frac{d}{dx} \nabla^2 \chi, \quad \theta = -\frac{h}{8} \frac{d}{dx} \nabla^2 \chi,$$

of the form (26), and

$$\psi = -\frac{h}{6} \frac{d}{dy} \nabla^2 \chi, \quad \phi = 0, \quad \theta = -\frac{h}{3} \frac{d}{dx} \nabla^2 \chi,$$

the displacements corresponding to which vanish.

18. *General solution. Comparison with the solution for normal traction.*

The solution of the general problem of (41) may be found by multiplying the expressions for ψ', θ', ϕ' in (48) (51) by $\frac{1}{8\pi\mu}f(x', y')$ and integrating term by term over the area A within which f is finite. As in the case of the problem of given normal traction, term by term differentiations of the resulting series will be legitimate just so far as the derivatives are required for the calculation of displacements and stresses. In order to see this, it should be noticed that, while an extra differentiation as to x or y will be required in virtue of (43), the series for ψ', θ', ϕ' have general terms of one order higher in $1/\kappa$ as compared with those of (20), (21). One effect, however, of this additional differentiation will be to increase the relative importance at the edge of the area A of that part of the displacement and stress which arises from the local perturbation, such displacement being of order h , and stress of order zero, as in the former problem, whereas the displacement and stress as a whole is of higher order in h than before.

The functions ψ', θ', ϕ' being symmetrical about the axis $R=0$, it is clear that the solution for an element of traction of $8\pi\mu$ units at (x', y', h) parallel to the axis of y is given by

$$\psi = -\frac{d\psi'}{dx}, \quad \theta = \frac{d\theta'}{dy}, \quad \phi = \frac{d\phi'}{dy}. \quad (54)$$

with ψ', θ', ϕ' as in (48) (51).

It will be seen presently that surface traction may be regarded as a special case of force applied in the body of the plate. We may therefore postpone any more extended development of the above solution, and in particular any more explicit comparison of the results with those of the accepted approximate theory of thin plates, until we have obtained the solutions of the problems relative to sources of strain situated in the interior of the solid.

19. *Normal force applied at a single internal point. Solution in definite integrals.*

We take first the case of a single force, say for convenience of $4\pi\mu(a+1)$ units, applied at (x', y', z') parallel to Oz , the faces of the plate being free from stress. Referring to (6), we see that the conditions of the problem may be taken to be

$$(i) \quad \left. \begin{aligned} u &= (z' - z) \frac{dr^{-1}}{dx} + U \\ v &= (z' - z) \frac{dr^{-1}}{dy} + V \\ w &= (z' - z) \frac{dr^{-1}}{dz} + ar^{-1} + W \end{aligned} \right\} \quad (55)$$

(ii) U, V, W , along with their derivatives as to (x, y, z) of the first order, are finite and continuous at every point of the solid at a finite distance, and have derivatives of

the second order satisfying

$$\mu \nabla^2 U + (\lambda + \mu) \frac{d\Delta}{dx} = 0; \quad \mu \nabla^2 V + (\lambda + \mu) \frac{d\Delta}{dy} = 0; \quad \mu \nabla^2 W + (\lambda + \mu) \frac{d\Delta}{dz} = 0;$$

where

$$\Delta = \frac{dU}{dx} + \frac{dV}{dy} + \frac{dW}{dz}.$$

$$(iii) \quad \widehat{zx} = \widehat{zy} = \widehat{z} = 0 \quad \text{on } z = \pm h$$

It is clear that these conditions do not completely define the solution, seeing that no condition to be satisfied at infinity is mentioned. But instead of laying down any such condition at infinity, it is simpler in the first instance to be content with *any* solution fulfilling (i), (ii), and (iii). The most general solution can then be obtained without difficulty, and with this before us, conditions at infinity can be discussed to much greater advantage than at present.

The problem is solved when U, V, W are found in the form (4), so as to give the same tractions on $z = \pm h$ as those due to (6), but reversed. These reversed tractions, as follows very readily from (5), (8), are given by

$$\begin{aligned} \frac{\widehat{zx}}{2\mu} &= \frac{d}{dx} \left\{ \frac{1-a}{2} r^{-1} + (z-z') \frac{dr^{-1}}{dz} \right\} \\ \frac{\widehat{zy}}{2\mu} &= \frac{d}{dy} \left\{ \frac{1-a}{2} r^{-1} + (z-z') \frac{dr^{-1}}{dz} \right\} \\ \frac{\widehat{z}}{2\mu} &= -\frac{1+a}{2} \frac{dr^{-1}}{dz} + (z-z') \frac{d^2 r^{-1}}{dz^2} \end{aligned}$$

Now when $z > z'$,

$$\begin{aligned} r^{-1} &= \int_0^\infty e^{-\kappa(z-z')} J_0 \kappa R d\kappa \\ \frac{dr^{-1}}{dz} &= \int_0^\infty (-\kappa) e^{-\kappa(z-z')} J_0 \kappa R d\kappa \\ \frac{d^2 r^{-1}}{dz^2} &= \int_0^\infty \kappa^2 e^{-\kappa(z-z')} J_0 \kappa R d\kappa \end{aligned}$$

but when $z < z'$,

$$\begin{aligned} r^{-1} &= \int_0^\infty e^{\kappa(z-z')} J_0 \kappa R d\kappa \\ \frac{dr^{-1}}{dz} &= \int_0^\infty \kappa e^{\kappa(z-z')} J_0 \kappa R d\kappa \\ \frac{d^2 r^{-1}}{dz^2} &= \int_0^\infty \kappa^2 e^{\kappa(z-z')} J_0 \kappa R d\kappa \end{aligned} \quad (56)$$

Hence if U, V, W be defined as in (4), the function ψ is not required, and the conditions to be satisfied by θ , ϕ are, if in the first instance we take integrands instead of integrals,

$$\begin{aligned} \frac{d\theta}{dz} + \frac{d\phi}{dz} + 2z \frac{d^2 \phi}{dz^2} &= \left\{ \frac{1-a}{2} - \kappa(h-z') \right\} e^{-\kappa(h-z')} J_0 \kappa R, \text{ on } z=h \\ &= \left\{ \frac{1-a}{2} - \kappa(h+z') \right\} e^{-\kappa(h+z')} J_0 \kappa R, \text{ on } z=-h \\ \frac{d^2 \theta}{dz^2} - \frac{d^2 \phi}{dz^2} + 2z \frac{d^3 \phi}{dz^3} &= \left\{ \frac{1+a}{2} \kappa + \kappa^2(h-z') \right\} e^{-\kappa(h-z')} J_0 \kappa R, \text{ on } z=h \\ &= \left\{ -\frac{1+a}{2} \kappa - \kappa^2(h+z') \right\} e^{-\kappa(h+z')} J_0 \kappa R, \text{ on } z=-h \end{aligned} \quad (57)$$

Assuming

$$\begin{aligned}\phi &= A \sinh \kappa z + B \cosh \kappa z \left\{ J_0 \kappa R \right. \\ \theta &= C \sinh \kappa z + D \cosh \kappa z \left. \right\}\end{aligned}$$

(57) give four equations to determine A, B, C, D. By addition and subtraction these are resolved into two equations for A, C, and other two for B, D. Thus we find

$$\begin{aligned}\phi &= \frac{\sinh \kappa z J_0 \kappa R}{\kappa(\sinh 2\kappa h - 2\kappa h)} \left\{ \kappa z' \sinh \kappa z' - \frac{1}{2} \cosh \kappa z' (e^{-2\kappa h} + a + 2\kappa h) \right\} \\ &+ \frac{\cosh \kappa z J_0 \kappa R}{\kappa(\sinh 2\kappa h + 2\kappa h)} \left\{ \kappa z' \cosh \kappa z' + \frac{1}{2} \sinh \kappa z' (e^{-2\kappa h} - a - 2\kappa h) \right\} \\ \theta &= \frac{\sinh \kappa z J_0 \kappa R}{\kappa(\sinh 2\kappa h - 2\kappa h)} \left\{ -\kappa z' \sinh \kappa z' (e^{-2\kappa h} + 2\kappa h) + \cosh \kappa z' \left(\frac{a}{2} e^{-2\kappa h} + \frac{1}{2} + a\kappa h + 2\kappa^2 h^2 \right) \right\} \\ &+ \frac{\cosh \kappa z J_0 \kappa R}{\kappa(\sinh 2\kappa h + 2\kappa h)} \left\{ \kappa z' \cosh \kappa z' (e^{-2\kappa h} - 2\kappa h) + \sinh \kappa z' \left(-\frac{a}{2} e^{-2\kappa h} + \frac{1}{2} + a\kappa h + 2\kappa^2 h^2 \right) \right\}\end{aligned} \quad (58)$$

If these expressions could be integrated with respect to κ from 0 to ∞ , the balancing displacements U, V, W of (55) would be determined. Near the upper limit the functions converge to zero exponentialwise, since both z and z' lie between $-h$ and $+h$. But for $\kappa=0$ both functions are infinite, and their expansions in ascending powers of κ contain terms of negative degree which must be removed after the manner of § 3. The integrals are then convergent, but a further modification of a different sort is necessary before they can be transformed into series as in §§ 5, 16. The possibility of this transformation in the former cases was intimately related to the fact that the functions in (13), (46) were odd in κ , which the functions in (58) obviously are not. However, when the odd and even parts are separated, the latter are found to have a very simple form, free from the denominators $\sinh 2\kappa h \pm 2\kappa h$, for we find

$$\begin{aligned}\phi &= \frac{1}{2\kappa} \sinh \kappa(z - z') J_0 \kappa R \\ &+ \frac{\sinh \kappa z J_0 \kappa R}{\kappa(\sinh 2\kappa h - 2\kappa h)} \left\{ \kappa z' \sinh \kappa z' - \frac{1}{2} (a + \cosh 2\kappa h) \cosh \kappa z' \right\} \\ &+ \frac{\cosh \kappa z J_0 \kappa R}{\kappa(\sinh 2\kappa h + 2\kappa h)} \left\{ \kappa z' \cosh \kappa z' + \frac{1}{2} (\cosh 2\kappa h - a) \sinh \kappa z' \right\} \\ \theta &= - \left\{ \frac{a}{2\kappa} \sinh \kappa(z - z') + z' \cosh \kappa(z - z') \right\} J_0 \kappa R \\ &+ \frac{\sinh \kappa z J_0 \kappa R}{\kappa(\sinh 2\kappa h - 2\kappa h)} \left\{ -\kappa z' \cosh 2\kappa h \sinh \kappa z' + \left(\frac{a}{2} \cosh 2\kappa h + \frac{1}{2} + 2\kappa^2 h^2 \right) \cosh \kappa z' \right\} \\ &+ \frac{\cosh \kappa z J_0 \kappa R}{\kappa(\sinh 2\kappa h + 2\kappa h)} \left\{ \kappa z' \cosh 2\kappa h \cosh \kappa z' + \left(-\frac{a}{2} \cosh 2\kappa h + \frac{1}{2} + 2\kappa^2 h^2 \right) \cosh \kappa z' \right\}\end{aligned} \quad (59)$$

The even terms in κ can be eliminated from these expressions by including the

values of ϕ , θ which define the source, given in (8). For these are

$$\begin{aligned}\phi &= \frac{1}{2} \int_0^\infty \left\{ e^{-\kappa(z-z')} J_0 \kappa R - e^{-\kappa h} \right\} \frac{d\kappa}{\kappa}, \text{ if } z > z' \\ &= -\frac{1}{2} \int_0^\infty \left\{ e^{\kappa(z-z')} J_0 \kappa R - e^{-\kappa h} \right\} \frac{d\kappa}{\kappa}, \text{ if } z < z'. \\ \theta &= -\frac{a}{2} \int_0^\infty \left\{ e^{-\kappa(z-z')} J_0 \kappa R - e^{-\kappa h} \right\} \frac{d\kappa}{\kappa} + z' \int_0^\infty e^{-\kappa(z-z')} J_0 \kappa R d\kappa, \text{ if } z > z' \\ &= \frac{a}{2} \int_0^\infty \left\{ e^{\kappa(z-z')} J_0 \kappa R - e^{-\kappa h} \right\} \frac{d\kappa}{\kappa} + z' \int_0^\infty e^{\kappa(z-z')} J_0 \kappa R d\kappa, \text{ if } z < z'.\end{aligned}\quad (60)$$

These will obviously be reproduced, after preparation by removal of terms of negative degree and integration, from

$$\begin{aligned}\phi &= \frac{1}{2\kappa} e^{-\kappa(z-z')} J_0 \kappa R, \text{ if } z > z' \\ &= -\frac{1}{2\kappa} e^{\kappa(z-z')} J_0 \kappa R, \text{ if } z < z'. \\ \theta &= \left(-\frac{a}{2\kappa} + z' \right) e^{-\kappa(z-z')} J_0 \kappa R, \text{ if } z > z' \\ &= \left(\frac{a}{2\kappa} + z' \right) e^{\kappa(z-z')} J_0 \kappa R, \text{ if } z < z'.\end{aligned}$$

When these last terms are taken in, the first lines of ϕ , θ in (59) become

$$\begin{aligned}\phi &= \pm \frac{1}{2\kappa} \cosh \kappa(z-z') J_0 \kappa R \\ \theta &= \mp \frac{a}{2\kappa} \cosh \kappa(z-z') J_0 \kappa R \mp z' \sinh \kappa(z-z') J_0 \kappa R,\end{aligned}$$

the upper or lower sign being taken in the ambiguities according as $z >$ or $< z'$. Hence, when the source is taken in, the following are the unprepared and unintegrated forms of ϕ , θ :—

$$\begin{aligned}\phi &= \pm \frac{1}{2\kappa} \cosh \kappa(z-z') J_0 \kappa R \\ &\quad + \frac{\sinh \kappa z J_0 \kappa R}{\kappa(\sinh 2\kappa h - 2\kappa l)} \left\{ \kappa z' \sinh \kappa z' - \frac{1}{2}(a + \cosh 2\kappa h) \cosh \kappa z' \right\} \\ &\quad + \frac{\cosh \kappa z J_0 \kappa R}{\kappa(\sinh 2\kappa h + 2\kappa l)} \left\{ \kappa z' \cosh \kappa z' + \frac{1}{2}(\cosh 2\kappa h - a) \sinh \kappa z' \right\} \\ \theta &= \mp \frac{a}{2\kappa} \cosh \kappa(z-z') J_0 \kappa R \mp z' \sinh \kappa(z-z') J_0 \kappa R \\ &\quad + \frac{\sinh \kappa z J_0 \kappa R}{\kappa(\sinh 2\kappa h - 2\kappa l)} \left\{ -\kappa z' \cosh 2\kappa h \sinh \kappa z' + \left(\frac{a}{2} \cosh 2\kappa h + \frac{1}{2} + 2\kappa^2 h^2 \right) \cosh \kappa z' \right\} \\ &\quad + \frac{\cosh \kappa z J_0 \kappa R}{\kappa(\sinh 2\kappa h + 2\kappa l)} \left\{ \kappa z' \cosh 2\kappa h \cosh \kappa z' + \left(-\frac{a}{2} \cosh 2\kappa h + \frac{1}{2} + 2\kappa^2 h^2 \right) \sinh \kappa z' \right\}\end{aligned}\quad (61)$$

In (61) the terms of negative degree in κ are of the forms :—in ϕ , $H/\kappa^3 + K/\kappa$; in θ , $L/\kappa^3 + M/\kappa$; and these terms, as in § 3, give $\widehat{zx} = \widehat{zy} = \widehat{zz} = 0$ at the faces of the plate.

Hence if from ϕ in (61) we subtract $H/\kappa^3 + Ke^{-\kappa h}/\kappa$, and from θ , $L/\kappa^3 + Me^{-\kappa h}/\kappa$, the resulting expressions, integrated with respect to κ from 0 to ∞ , will define a solution of the problem stated at the beginning of this article.

20. *Normal force applied at a single internal point. Solution in series.*

To the integrals thus obtained we can apply the transformation of § 5, but one remark should be made. From the synthesis which gave (61), it is sufficiently obvious, in view of the forms in (58) and (60), that the expressions of (61), with $G_0\kappa R$ substituted for $J_0(\kappa R)$, vanish effectively at infinity in the first quadrant of the κ plane; that they similarly vanish in the second quadrant follows at once from the fact that the functions of (61) are odd functions of κ .

A glance at the relation between (16), (17) and (20), (21) will again save us the necessity of writing down the details. Thus, let the values of H , K , L , M , when R is put equal to zero, be denoted by H_0 , K_0 , L_0 , M_0 . Then the persistent part of the transformed solution is given by

$$\begin{aligned} \phi &= H_0\chi(R) - K_0\nabla^2\chi(R) \\ \theta &= L_0\chi(R) - M_0\nabla^2\chi(R) \end{aligned} \quad (62)$$

The decaying part is given by

$$\begin{aligned} \phi &= \sum_{\kappa} \frac{G_0\kappa R \sinh \kappa z}{\kappa h (\cosh 2\kappa h - 1)} \left\{ \kappa z' \sinh \kappa z' - \frac{1}{2} (\alpha + \cosh 2\kappa h) \cosh \kappa z' \right\} \\ \theta &= \sum_{\kappa} \frac{G_0\kappa R \sinh \kappa z}{\kappa h (\cosh 2\kappa h - 1)} \left\{ -\kappa z' \cosh 2\kappa h \sinh \kappa z' + \left(\frac{\alpha}{2} \cosh 2\kappa h + \frac{1}{2} + 2\kappa^2 h^2 \right) \cosh \kappa z' \right\} \\ &= \sum_{\kappa} \frac{G_0\kappa R \sinh \kappa z}{\kappa h (\cosh 2\kappa h - 1)} \left\{ \kappa z' \sinh \kappa z' - \frac{1}{2} (\alpha + \cosh 2\kappa h) \cosh \kappa z' \right\} (-\cosh 2\kappa h) \end{aligned} \quad (63)$$

where κ is a zero of $\sinh 2\kappa h - 2\kappa h$, with positive imaginary part; with

$$\begin{aligned} \phi &= \sum_{\kappa} \frac{G_0\kappa R \cosh \kappa z}{\kappa h (\cosh 2\kappa h + 1)} \left\{ \kappa z' \cosh \kappa z' + \frac{1}{2} (\cosh 2\kappa h - \alpha) \sinh \kappa z' \right\} \\ \theta &= \sum_{\kappa} \frac{G_0\kappa R \cosh \kappa z}{\kappa h (\cosh 2\kappa h + 1)} \left\{ \kappa z' \cosh \kappa z' + \frac{1}{2} (\cosh 2\kappa h - \alpha) \sinh \kappa z' \right\} \cosh 2\kappa h \end{aligned} \quad (64)$$

where κ is a zero of $\sinh 2\kappa h + 2\kappa h$, with positive imaginary part.

When the values of H_0 , K_0 , L_0 , M_0 are obtained from (61), it will be found that (62) may be decomposed as follows:—

$$\begin{aligned} \text{(i)} \quad \phi &= -\frac{3(\alpha+1)}{8h^3} \left(z\chi - \frac{1}{6} z^3 \nabla^2 \chi \right) \\ \theta &= -\frac{3(\alpha+1)}{8h^3} \left(z\chi - \frac{1}{6} z^3 \nabla^2 \chi - 2h^2 z \nabla^2 \chi \right) \\ \text{(ii)} \quad \phi &= -\frac{z \nabla^2 \chi}{z \nabla^2 \chi} \left\{ \text{each multiplied by } \frac{3}{4h^3} (z^2 - h^2) + \frac{3(\alpha+1)}{8h^3} \left(\frac{1}{5} h^2 - \frac{1}{2} z^2 \right) \right\} \\ \text{(iii)} \quad \phi &= \frac{z(\alpha-3)}{8h} \nabla^2 \chi = \theta \\ \text{(iv)} \quad \phi &= \mp \frac{1}{2} \nabla^2 \chi; \quad \theta = \pm \frac{1}{2} \alpha \nabla^2 \chi, \quad \text{with upper or lower signs, as } z \text{ is greater} \end{aligned}$$

or less than z' .

(65)

There being no discontinuity of displacement or strain at the plane $z = z'$, except at the point where the force is applied, we are prepared to find that (iv) give no displacement at all if $R > 0$. So long, however, as we keep to the specification of the strain by the ϕ, θ functions, it is convenient to retain the terms in (iv). By so doing, we of course make the ϕ of the space above the plane $z = z'$ and the ϕ of the space below that plane two distinct potential functions, but we preserve the non-singular character of each of these functions at the axis $R = 0$.

If we take the limit of the above solution for $z' = h$, which obviously may be done by putting $z' = h$ in each term and using the lower signs in (iv), we obtain simply the solution of (20), (21) multiplied by $\frac{1}{2}(a+1)$. Since the present solution is for a force of $4\pi\mu(a+1)$ units, and the other for an element of traction of $8\pi\mu$ units, it follows that a unit element of traction may be regarded as simply the limiting case of a unit force, the point of application of which approaches indefinitely near the surface.

21. *Solution of a special problem of internal areal normal force.*

When the displacements due to a unit Z force at (x', y', z') , with the surface free, are known, the corresponding displacements for a body distribution of force, of amount $Z(x', y', z')$ per unit volume at (x', y', z') , can be found by multiplying by $Z(x', y', z')dx'dy'dz'$ and integrating through the space in which Z is finite. Certain peculiarities in the form of the solution given in § 20 make it convenient to take the integration with respect to z' last, or, as comes to the same thing, to begin by considering the solution for an *areal* distribution of force on the plane $z = z'$, of magnitude $Z(x', y', z')$ per unit area.

We take first a special problem analogous to that worked out in § 10, and suppose the Z force to be distributed over the area of a circle of radius a in the plane $z = z'$, with centre on Oz, the intensity per unit area being $4\pi\mu(a+1)J_m\beta\rho\cos m\omega$. It will be sufficient to attend to the value of ϕ , for when that is known, the corresponding value of θ can be written down at once.

The series deduced by integration from (63), (64), say ϕ_2 , fall naturally into two parts as in § 10, viz., (i) series defining a local perturbation at the cylinder $\rho = a$,

$$\phi = \sum_{\kappa} \frac{\sinh \kappa z}{\kappa h (\cosh 2\kappa h - 1)} \left\{ \kappa z' \sinh \kappa z' - \frac{1}{2} (a + \cosh 2\kappa h) \cosh \kappa z' \right\} \cdot \frac{2\pi}{\beta^2 - \kappa^2} P_{\kappa} \quad (\text{where } \sinh 2\kappa h - 2\kappa h = 0) \\ + \sum_{\kappa} \frac{\cosh \kappa z}{\kappa h (\cosh 2\kappa h + 1)} \left\{ \kappa z' \cosh \kappa z' + \frac{1}{2} (\cosh 2\kappa h - a) \sinh \kappa z' \right\} \cdot \frac{2\pi}{\beta^2 - \kappa^2} P_{\kappa} \quad (\text{where } \sinh 2\kappa h + 2\kappa h = 0)$$

with

$$P_{\kappa} = J_m \kappa \rho \cos m\omega (\kappa a G_m' \kappa a J_m \beta a - G_m \kappa a \beta a J_m' \beta a), \quad \text{if } \rho < a \\ = G_m \kappa \rho \cos m\omega (\kappa a J_m' \kappa a J_m \beta a - J_m \kappa a \cdot \beta a J_m' \beta a), \quad \text{if } \rho > a \quad (66)$$

(ii) When $\rho < a$, series which can be summed in finite terms,

$$\phi = 2\pi J_m \beta \rho \cos m\omega \left[\sum_{\kappa} \frac{\sinh \kappa z}{(\beta^2 - \kappa^2) \kappa h (\cosh 2\kappa h - 1)} \left\{ \kappa z' \sinh \kappa z' - \frac{1}{2} (a + \cosh 2\kappa h) \cosh \kappa z' \right\} (\text{where } \sinh 2\kappa h - 2\kappa h = 0) \right. \\ \left. + \sum_{\kappa} \frac{\cosh \kappa z}{(\beta^2 - \kappa^2) \kappa h (\cosh 2\kappa h + 1)} \left\{ \kappa z' \cosh \kappa z' + \frac{1}{2} (\cosh 2\kappa h - a) \sinh \kappa z' \right\} (\text{where } \sinh 2\kappa h + 2\kappa h = 0) \right] \quad (67)$$

In order to sum these last series, consider the function of κ ,

$$\begin{aligned} & \pm \frac{\cosh \kappa(z-z')}{2\kappa(\beta^2 - \kappa^2)} \quad \left(\pm \text{ according as } z > \text{ or } < z' \right) \\ & + \frac{\sinh \kappa z}{(\beta^2 - \kappa^2)\kappa(\sinh 2\kappa h - 2\kappa h)} \quad \left\{ \kappa z' \sinh \kappa z' - \frac{1}{2}(a + \cosh 2\kappa h) \cosh \kappa z' \right\} \\ & + \frac{\cosh \kappa z}{(\beta^2 - \kappa^2)\kappa(\sinh 2\kappa h + 2\kappa h)} \quad \left\{ \kappa z' \cosh \kappa z' + \frac{1}{2}(\cosh 2\kappa h - a) \sinh \kappa z' \right\} \quad . \quad . \quad . \quad (68) \end{aligned}$$

Looking back at (61) we assure ourselves that this function vanishes at infinity in such a way as to make the sum of its residues zero. Also, since the function is odd in κ , the residues at $\kappa = \pm \kappa_1$ are equal, and therefore the sum of the residues at the zeroes of $\sinh 2\kappa h \pm 2\kappa h$ is simply the coefficient of $2\pi J_m \beta \rho \cos m\omega$ in (67). The sum of the residues at $\kappa = \pm \beta$ is

$$\begin{aligned} & \mp \frac{\cosh \beta(z-z')}{2\beta^2} \\ & - \frac{\sinh \beta z}{\beta^2(\sinh 2\beta h - 2\beta h)} \quad \left\{ \beta z' \sinh \beta z' - \frac{1}{2}(a + \cosh 2\beta h) \cosh \beta z' \right\} \\ & - \frac{\cosh \beta z}{\beta^2(\sinh 2\beta h + 2\beta h)} \quad \left\{ \beta z' \cosh \beta z' + \frac{1}{2}(\cosh 2\beta h - a) \sinh \beta z' \right\} \quad . \quad . \quad . \quad (69) \end{aligned}$$

If this last expression near $\beta = 0$ be of the form $\frac{A}{\beta^4} + \frac{B}{\beta^2} + \dots$, the residue at $\kappa = 0$ of (68) is simply $-\frac{A}{\beta^4} - \frac{B}{\beta^2}$.

Hence the coefficient of $2\pi J_m \beta \rho \cos m\omega$ in (67) is simply (69) with sign changed and the terms of negative degree in β subtracted. These terms of negative degree, just as in § 12, are added on again when we take in the part of the solution coming from (65), which is obtained by writing F for χ in (65) where

$$\begin{aligned} F &= \iint \chi(R) J_m \beta \rho' \cos m\omega' \rho' d\rho' d\omega' \\ &= \frac{2\pi}{\beta^4} J_m \beta \rho \cos m\omega + F_1. \quad (\text{Introd. } (f).) \end{aligned}$$

The term $\frac{2\pi}{\beta^4} J_m \beta \rho \cos m\omega$ being taken in for the purpose just mentioned, we are left with F_1 instead of χ in (65). Since $\nabla^4 F_1 = 0$, these equations now define a combination of deformations of the persistent or permanent type, under no body force and no surface traction.

The solution therefore resolves itself into

- (i) this free deformation of the permanent mode ;
- (ii) a local perturbation ;
- (iii) a particular solution giving the proper discontinuity of stress corresponding to the applied areal force.

The particular solution is

$$\phi = 2\pi J_m \beta \rho \cos m\omega \left\{ \begin{aligned} &\pm \frac{1}{2\beta^2} \cosh \beta(z-z') \\ &+ \frac{\sinh \beta z}{\beta^2(\sinh 2\beta h - 2\beta h)} (\beta z' \sinh \beta z' - \frac{1}{2} \overline{a} + \cosh 2\beta h \cosh \beta z') \\ &+ \frac{\cosh \beta z}{\beta^2(\sinh 2\beta h + 2\beta h)} (\beta z' \cosh \beta z' + \frac{1}{2} \cosh 2\beta h - \overline{a} \sinh \beta z') \end{aligned} \right\} \quad (70)$$

with a corresponding expression for θ , obtainable from (61) by changing κ into β and then replacing $J_0 \beta R$ by $\frac{1}{\beta} 2\pi J_m \beta \rho \cos m\omega$.

It is easy to verify that this is actually a particular solution. Consider in the first place the analogous forms of ϕ , θ in (61), and for greater generality, suppose $J_0 \kappa R$ replaced by $f(x, y)$ where $(\nabla^2 + \kappa^2)f = 0$.

Then, from the method by which (61) were found, it is obvious that they give no stress across the planes $z = \pm h$. Let us examine the effect of the discontinuity in the forms of ϕ , θ at the plane $z = z'$, on the displacements and stresses as given in (4), (5).

If we take simply

$$\begin{aligned} \phi &= \frac{1}{2\kappa} \cosh \kappa(z-z') f \\ \theta &= \frac{-a}{2\kappa} \cosh \kappa(z-z') f - z' \sinh \kappa(z-z') f \end{aligned}$$

then we find at $z = z'$,

$$\begin{aligned} u &= v = w = 0 \\ \widehat{zx} &= \widehat{zy} = 0 \\ \widehat{zz} &= -\mu(a+1)\kappa f. \end{aligned}$$

Thus with the complete expression (61), the displacements are continuous, as also the stresses \widehat{zx} , \widehat{zy} , but $\widehat{zz}(z=z'+)$ exceeds $\widehat{zz}(z=z'-)$ by $-2\mu(a+1)\kappa f$. We thus see that in (70) the corresponding discontinuity in \widehat{zz} will be $-4\pi\mu(a+1)J_m\beta\rho \cos m\omega$. This continuity of displacement, and discontinuity in \widehat{zz} , are precisely as demanded by the conditions of equilibrium of the plate.

If we take (61) with $J_0 \kappa R$ unaltered, prepare them for integration as in § 19, multiply by $e^{-\kappa z}$ and integrate with respect to κ from 0 to ∞ , the discontinuity in \widehat{zz} at $z = z'$ will become

$$-2\mu(a+1) \int_0^\infty e^{-\kappa z} \kappa J_0 \kappa R d\kappa.$$

If further we multiply this by $Z(x', y', z') dx' dy'$ and integrate with respect to x', y' , and then take the limit for $\epsilon = 0$, the discontinuity becomes, in virtue of (11),

$$-4\pi\mu(a+1)Z(x, y, z').$$

We have thus a proof of the solution for an areal distribution of Z force, independent of the infinite solid solution (6), which might itself be found from the beginning by this method.

22. *The general problem of internal normal force. Approximate forms of displacements and stresses.*

The developments given in §§ 14, 15 may obviously be applied in the present case also.

Thus if in (61) we divide by $\kappa J_0 \kappa R$, expand in ascending powers of κ , put $-\nabla^2$, that is $-\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)$ for κ^2 , and operate on $Z(x, y, z')$, we obtain a form of solution which, with the interpretation of $\nabla^{-1}Z$ and $\nabla^{-2}Z$ given in (38), is simply the foregoing general solution for areal force of intensity $2\mu(a+1)Z$, arranged in terms of ascending order in h^2 . The solution in this form fails if at (x, y, z) , Z or any of its successive derivatives become discontinuous, but it has been shown in § 14 how the local perturbation in the neighbourhood of any surface of discontinuity may be calculated.

For the case when Z vanishes outside an area A , the principal part of the perturbation at the edge of A , when h is small, is found by substituting for $G_0 \kappa R$ in (63), (64),

$$-\frac{1}{\kappa^2} \int \left\{ Z(x', y') \frac{d}{dn} G_0 \kappa R - G_0(\kappa R) \frac{d}{dn} Z(x', y') \right\} ds,$$

where differentiations and integrations have reference to the accented coordinates.

Since the solution for the case when there are any finite number of surfaces at which Z or its derivatives become discontinuous can be found from this elementary case by simple summation, we see that discontinuity in the force itself gives rise to values of ϕ, θ in the perturbation terms of order h^2 at the surface, discontinuity in $\frac{dZ}{dn}$ to terms of order h^3 if Z itself is continuous. The next term is of order h^4 and depends on discontinuity of $\nabla^2 Z$, that is, of the second derivatives of Z , and so on.

The symbolical solution for Z force distributed on the plane $z = z'$ with intensity $2\mu(a+1)Z(x, y, z')$ per unit area at (x, y, z') is given by

$$\begin{aligned} \phi &= \pm \frac{1}{2\kappa^2} \cosh \kappa(z - z') \\ &+ \frac{\sinh \kappa z}{\kappa^2(\sinh 2\kappa h - 2\kappa h)} \left(\kappa z' \sinh \kappa z' - \frac{1}{2} \overline{a + \cosh 2\kappa h} \cosh \kappa z' \right) \\ &+ \frac{\cosh \kappa z}{\kappa^2(\sinh 2\kappa h + 2\kappa h)} \left(\kappa z' \cosh \kappa z' + \frac{1}{2} \overline{\cosh 2\kappa h - a} \sinh \kappa z' \right) \\ \theta &= \mp \frac{a}{2\kappa^2} \cosh \kappa(z - z') \mp \frac{z'}{\kappa} \sinh \kappa(z - z') \\ &+ \frac{\sinh \kappa h}{\kappa^2(\sinh 2\kappa h - 2\kappa h)} \left(-\kappa z' \cosh 2\kappa h \sinh \kappa z' + \frac{a}{2} \cosh 2\kappa h + \frac{1}{2} + 2\kappa^2 h^2 \cosh \kappa z' \right) \\ &+ \frac{\cosh \kappa z}{\kappa^2(\sinh 2\kappa h + 2\kappa h)} \left(\kappa z' \cosh 2\kappa h \cosh \kappa z' + \frac{1}{2} - \frac{a}{2} \cosh 2\kappa h + 2\kappa^2 h^2 \sinh \kappa z' \right) \\ &\quad \text{with } \kappa^2 = -\nabla^2, \text{ operating on } Z(x, y, z') \end{aligned} \quad (71)$$

The approximate solution is obtained by expanding in ascending powers of κ^2 . By retaining only the terms of negative degree in κ^2 , each of the displacements will be

given to a second approximation, and each of the stresses, except \widehat{zz} , to a first approximation. The result is obviously the same as that found by integrating the permanent terms (65) of the original source solution.

If we write

$$F = \iint Z(x', y', z') \chi(\mathbf{R}) dx' dy',$$

then the displacements for Z force of intensity $Z(x, y, z')$ per unit area on $z = z'$ are

$$\begin{aligned} u &= \frac{d}{dx} \left\{ \frac{3}{32\pi\mu h^3} \left[-(a+1)zF + \nabla^2 F \left(\frac{a+5}{6} z^3 + \frac{a-3}{2} z'^2 z - \frac{+1}{5} h^2 z + \frac{a-3}{3} h^2 z' \right) \right] \right\} \\ v &= \frac{d}{dy} \left\{ \frac{3}{32\pi\mu h^3} \left[-(a+1)zF + \nabla^2 F \left(\frac{a+5}{6} z^3 + \frac{a-3}{2} z'^2 z - \frac{+1}{5} h^2 z + \frac{a-3}{3} h^2 z' \right) \right] \right\} \\ w &= \frac{3}{32\pi\mu h^3} \left\{ (a+1)F + \nabla^2 F \left(\frac{3-a}{2} z^3 + \frac{3-a}{2} z'^2 z - 4h^2 z + \frac{a+1}{5} h^2 \right) \right\} \end{aligned} \quad (72)$$

The corresponding results for a volume distribution of force, $Z(x, y, z)$ per unit volume, are found by integrating these with respect to z' from $-h$ to $+h$.

In order to calculate the stress \widehat{zz} from displacements, we should need the value of w to a third approximation. It is therefore easier to find \widehat{zz} directly from (71) and (5). On dividing by $2\mu(a+1)$, we find, corresponding to (72),

$$\widehat{zz} = \mp \frac{1}{2} Z + \frac{3h^2 z - z^3}{4h^3} Z \quad (73)$$

When the force is $Z(x, y, z)$ per unit volume, this leads to

$$\widehat{zz} = \frac{3h^2 z - z^3}{4h^3} \int_{-h}^{+h} Z(x, y, z') dz' - \frac{1}{2} \int_{-h}^z Z(x, y, z') dz' + \frac{1}{2} \int_z^h Z(x, y, z') dz' \quad (74)$$

We can now find the stresses \widehat{xx} , \widehat{xy} , \widehat{yy} to a second approximation.

For

$$\begin{aligned} \widehat{xx} &= (\lambda + 2\mu) \frac{du}{dx} + \lambda \frac{dv}{dy} + \lambda \frac{dw}{dz} \\ \widehat{zz} &= \lambda \left(\frac{du}{dx} + \lambda \frac{dv}{dy} + (\lambda + 2\mu) \frac{dw}{dz} \right), \end{aligned}$$

whence, eliminating $\frac{dw}{dz}$,

$$\widehat{xx} = \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} \left(\frac{du}{dx} + \sigma \frac{dv}{dy} \right) + \frac{\lambda}{\lambda + 2\mu} \widehat{zz},$$

and similarly

$$\widehat{yy} = \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} \left(\sigma \frac{du}{dx} + \frac{dv}{dy} \right) + \frac{\lambda}{\lambda + 2\mu} \widehat{zz}, \quad \text{where } \sigma = \frac{1}{2} \lambda / (\lambda + \mu).$$

Also

$$\widehat{xy} = \mu \left(\frac{du}{dy} + \frac{dv}{dx} \right).$$

We have now only to put in the values of u, v from (72) and the value of \widehat{zz} from (73).

If we denote $\frac{3}{32\pi\mu h^3}(a+1)F$, the principal term in w in (72), by W , we have

$$\begin{aligned}\nabla^4 W &= \frac{3}{32\pi\mu h^3}(a+1)\nabla^4 F \\ &= \frac{3}{16\mu h^3} \cdot (a+1)Z \\ &= \frac{3}{8\mu h^3} \frac{\lambda+2\mu}{\lambda+\mu} Z\end{aligned}$$

$$\text{or} \quad C\nabla^4 W = Z, \quad \text{where} \quad C = \frac{3}{8}\mu h^3(\lambda+\mu)/(\lambda+2\mu) \quad (75)$$

In this notation, to a first approximation

$$\begin{aligned}\hat{x}x &= -\frac{3C}{2h^3} \left(\frac{d^2 W}{dx^2} + \sigma \frac{d^2 W}{dy^2} \right) \\ \hat{y}y &= -\frac{3C}{2h^3} \left(\sigma \frac{d^2 W}{dx^2} + \frac{d^2 W}{dy^2} \right) \\ \hat{x}y &= -\frac{3C}{2h^3} (1-\sigma) \frac{d^2 W}{dx dy} \quad (76)\end{aligned}$$

Again from (72),

$$\begin{aligned}\hat{z}z &= \mu \left(\frac{dw}{dz} + \frac{dw}{dx} \right) = \frac{3\mu}{32\pi\mu h^3} \cdot 4(z^2 - h^2) \frac{d}{dx} \nabla^2 F \\ &= \frac{3}{4h^3} C (z^2 - h^2) \frac{d}{dx} \nabla^2 W \quad (77)\end{aligned}$$

23. Normal force a function of z only.

It may be useful to put down here the next term in the development of w , of which the two principal terms are given in (72). This is

$$\begin{aligned}\frac{3}{32\pi\mu h^3} \nabla^4 F \left\{ \frac{a-7}{24}(z^4 + z'^4) + \frac{(a-3)^2}{4(a+1)} z^2 z'^2 + \left(\frac{3-a}{10} + \frac{2a-2}{a+1} \right) h^2 (z^2 + z'^2) + \left(\frac{4}{3a+3} - \frac{4}{5} + \frac{11a+11}{3 \cdot 5^2 \cdot 7} \right) h^4 \right. \\ \left. + \frac{(3-a)^2}{3a+3} h^2 z z' \right\} \\ \pm (1-a)(z-z') Z(x, y, z').\end{aligned} \quad (78)$$

In this, of course,

$$\nabla^4 F = 2\pi Z(x, y, z').$$

The terms which have to be added to (72), (78) in order to give the complete particular values of u, v, w , all contain x, y derivatives of $\nabla^4 F$ or Z . Hence, if $Z(x, y, z')$ is a function of z' alone, (72) and (78) give a complete particular solution of the problem. Further, Z may have one constant value in one region of the plane $z = z'$, and another constant value in another region of that plane. (72), (78) will still give a particular solution in each of those regions taken separately, or rather in the cylindrical spaces of which these regions are sections, but it ought to be carefully noticed that it is not in general an exact solution when the two regions are considered together as part of one body. The point of failure is, it need scarcely be said, the condition of synexis;

the two particular solutions do not fit, that is, they do not give the same values for displacements and strains on the two sides of the cylindrical surface or surfaces of discontinuity.

On the other hand, the supplementary terms required in order to make the solution synectic belong to what we have called the decaying type. They give rise to displacements and strains of infinitely high order, if we may so speak, in the small quantity h , except very near the surfaces of discontinuity. This being so, we need not be surprised to find that the solution (72), (78) is not necessarily the simplest particular solution in any one region within which Z is continuously constant.

Thus, for example, if we pick out the terms which contain z'^2 as a factor, we find displacements proportional to

$$\begin{aligned} u &= \frac{d}{dx} \left\{ -z \nabla^2 F \right. \\ v &= \frac{d}{dy} \left. \right\} \\ w &= \nabla^2 F + \nabla^4 F \left(\frac{3-\alpha}{\alpha+1} \frac{1}{2} z^2 + C_1 h^2 \right) \end{aligned}$$

which belong to the type (23), and contribute nothing to body force or face tractions. These terms might therefore be omitted in any problem where the condition of synexis is irrelevant, and in particular when the object is merely to obtain a particular solution for body force and face traction in a problem relating to a *finite* solid.

24. Internal force parallel to the faces.

We will now go on to consider the problem of force applied to the body in a direction parallel to the faces of the plate.

A force of $4\pi\mu(\alpha+1)$ units applied at (x', y', z') in the direction of Ox gives in an infinite solid displacements defined, according to (9), by

$$\left. \begin{aligned} \psi &= -\frac{\alpha+1}{2} \frac{d}{dy} \frac{d^{-2}r^{+1}}{dz^{-2}} \\ \theta &= -\frac{\alpha}{2} \frac{d}{dx} \frac{d^{-2}r^{-1}}{dz^{-2}} + z' \frac{d}{dx} \frac{d^{-1}r^{-1}}{dz^{-1}} \\ \phi &= -\frac{1}{2} \frac{d}{dx} \frac{d^{-2}r^{-1}}{dz^{-2}} \end{aligned} \right\}$$

Hence the tractions which such a force produces on $z = \pm h$ will be neutralised by a system ψ, θ, ϕ for which

$$\left. \begin{aligned} \frac{d\psi}{dz} &= \frac{d}{dy} \left(-\frac{\alpha+1}{2} \frac{d^{-1}r^{-1}}{dz^{-1}} \right) \\ \frac{d\theta}{dz} + \frac{d\phi}{dz} + 2z \frac{d^2\phi}{dz^2} &= \frac{d}{dx} \left(-\frac{\alpha+1}{2} \frac{d^{-1}r^{-1}}{dz^{-1}} + z - z' \frac{dr^{-1}}{dz} \right) \\ \frac{d^2\theta}{dz^2} - \frac{d^2\phi}{dz^2} + 2z \frac{d^3\phi}{dz^3} &= \frac{d}{dx} \left(-\frac{\alpha-1}{2} \frac{r^{-1} + z - z'}{dz} \frac{dr^{-1}}{dz} \right) \end{aligned} \right\} \text{ on } z = \pm h.$$

These conditions are satisfied if

$$\psi = \frac{d\psi'}{dy}, \quad \theta = \frac{d\theta'}{dx}, \quad \phi = \frac{d\phi'}{dx} \quad (79)$$

and

$$\left. \begin{aligned} \frac{d\psi}{dz} &= \frac{a+1}{2} \frac{d^{-1}r^{-1}}{dz^{-1}} \\ \frac{d\theta'}{dz} + \frac{d\phi'}{dz} + 2z \frac{d^2\phi'}{dz^2} &= \frac{a+1}{2} \frac{d^{-1}r^{-1}}{dz^{-1} + z - z'} r^{-1} \\ \frac{d^2\theta'}{dz^2} - \frac{d^2\phi'}{dz^2} + 2z \frac{d^3\phi'}{dz^3} &= \frac{a-1}{2} r^{-1} + \frac{d^{-1}r^{-1}}{dz} \end{aligned} \right\} \text{on } z = \pm h.$$

We may take

$$\begin{aligned} \frac{d^{-1}r^{-1}}{dz^{-1} + z - z'} &= \int_0^\infty \left\{ e^{\mp \kappa(z-z')} J_0 \kappa R - 1 \pm \kappa(z-z') e^{-\kappa h} \right\} \frac{d\kappa}{\kappa^2} \\ \frac{d^{-1}r^{-1}}{dz^{-1}} &= \int_0^z (\mp) \left\{ e^{\mp \kappa(z-z')} J_0 \kappa R - e^{-\kappa h} \right\} \frac{d\kappa}{\kappa} \\ r^{-1} &= \int_0^z e^{\mp \kappa(z-z')} J_0 \kappa R / d\kappa \end{aligned}$$

upper or lower signs being taken, as all along, according as $z > z'$ or $z < z'$. We therefore determine provisional values of ψ' , θ' , ϕ' , such that

$$\left. \begin{aligned} \frac{d\psi'}{dz} &= \mp \frac{a+1}{2\kappa} e^{\mp \kappa(z-z')} J_0 \kappa R \\ \frac{d\theta'}{dz} + \frac{d\phi'}{dz} + 2z \frac{d^2\phi'}{dz^2} &= \left(\mp \frac{a+1}{2\kappa} + z - z' \right) e^{\mp \kappa(z-z')} J_0 \kappa R \\ \frac{d^2\theta'}{dz^2} - \frac{d^2\phi'}{dz^2} + 2z \frac{d^3\phi'}{dz^3} &= \left(\frac{a-1}{2} \mp \kappa z - z' \right) e^{\mp \kappa(z-z')} J_0 \kappa R \end{aligned} \right\} \text{on } z = \pm h$$

These provisional values are easily found to be

$$\left. \begin{aligned} \psi' &= \frac{a+1}{2\kappa^2} \cosh \kappa(z-z') \\ &+ \frac{a+1}{2\kappa^2 \cosh \kappa h} \sinh \kappa h \sinh \kappa z' \sinh \kappa z \\ &- \frac{a+1}{2\kappa^2 \sinh \kappa h} \cosh \kappa h \cosh \kappa z' \cosh \kappa z \end{aligned} \right\} J_0 \kappa R$$

$$\left. \begin{aligned} \phi' &= \frac{1}{2\kappa^2} \cosh \kappa(z-z') \\ &+ \frac{\sinh \kappa z}{\kappa^2 (\sinh 2\kappa h - 2\kappa h)} \left\{ -\kappa z' \cosh \kappa z' + \frac{1}{2} (\cosh 2\kappa h - a) \sinh \kappa z' \right\} \\ &- \frac{\cosh \kappa z}{\kappa^2 (\sinh 2\kappa h + 2\kappa h)} \left\{ \kappa z' \sinh \kappa z' + \frac{1}{2} (\cosh 2\kappa h + a) \cosh \kappa z' \right\} \end{aligned} \right\} J_0 \kappa R$$

$$\left. \begin{aligned} \theta' &= \frac{a}{2\kappa^2} \cosh \kappa(z-z') - \frac{z'}{\kappa} \sinh \kappa(z-z') \\ &+ \frac{\sinh \kappa z}{\kappa^2 (\sinh 2\kappa h - 2\kappa h)} \left\{ \kappa z' \cosh 2\kappa h \cosh \kappa z' + \frac{1}{2} (a \cosh 2\kappa h - 1 - 4\kappa^2 h^2) \sinh \kappa z' \right\} \\ &- \frac{\cosh \kappa z}{\kappa^2 (\sinh 2\kappa h + 2\kappa h)} \left\{ \kappa z' \cosh 2\kappa h \sinh \kappa z' + \frac{1}{2} (a \cosh 2\kappa h + 1 + 4\kappa^2 h^2) \cosh \kappa z' \right\} \end{aligned} \right\} J_0 \kappa R \quad (80)$$

The source itself is similarly given by the temporary values

$$\left. \begin{aligned} \psi' &= -\frac{\alpha+1}{2\kappa^2} \\ \theta' &= -\frac{\alpha}{2\kappa^2} \mp \frac{z'}{\kappa} \\ \phi' &= -\frac{1}{2\kappa^2} \end{aligned} \right\} \epsilon^{\mp \kappa(z-z')} J_0 \kappa R$$

Thus, when the source is included, the provisional values of ψ' , θ' , ϕ' are as in (80), but with the first lines altered,

$$\left. \begin{aligned} \text{in } \psi' \text{ to } &\pm \frac{\alpha+1}{2\kappa^2} \sinh \kappa(z-z') \\ \text{,, } \phi' \text{ ,, } &\pm \frac{1}{2\kappa^2} \sinh \kappa(z-z') \\ \text{,, } \theta' \text{ ,, } &\pm \frac{\alpha}{2\kappa^2} \sinh \kappa(z-z') \mp \frac{z'}{\kappa} \cosh \kappa(z-z') \end{aligned} \right\} J_0 \kappa R \quad . \quad . \quad . \quad (81)$$

25. *Solution of the problem of internal force parallel to the faces.*

From these expressions the solution in the form of definite integrals, and finally of series, is obtained as in the previous cases. After the explanations already given, it will be sufficient to write down the final results. For the transitory part of the solution,

$$\begin{aligned} \psi' &= \sum_{\kappa} (\alpha+1) \frac{1}{\kappa^2 h} \sinh \kappa z' \sinh \kappa z G_0 \kappa R, (\kappa \text{ a pos.-imag. root of } \cosh \kappa h) \\ &\quad - \sum_{\kappa} (\alpha+1) \frac{1}{\kappa^2 h} \cosh \kappa z' \cosh \kappa z G_0 \kappa R, (\kappa \text{ a pos. imag. root of } \sinh \kappa h). \\ \phi' &= \sum_{\kappa} \frac{\sinh \kappa z G_0 \kappa R}{\kappa^2 h (\cosh 2\kappa h - 1)} \left\{ -\kappa z' \cosh \kappa z' + \frac{1}{2} (\cosh 2\kappa h - \alpha) \sinh \kappa z' \right\} \\ \theta' &= \sum_{\kappa} \text{ same as previous line multiplied by } (-\cosh 2\kappa h) \end{aligned}$$

where κ is a zero of $\sinh 2\kappa h - 2\kappa h$, with pos. imag. part.

With

$$\begin{aligned} \phi' &= \sum_{\kappa} (-) \frac{\cosh \kappa z G_0 \kappa R}{\kappa^2 h (\cosh 2\kappa h + 1)} \left\{ \kappa z' \sinh \kappa z' + \frac{1}{2} (\cosh 2\kappa h + \alpha) \cosh \kappa z' \right\} \\ \theta' &= \sum_{\kappa} \text{ same as previous line multiplied by } \cosh 2\kappa h \end{aligned}$$

where κ is a zero of $\sinh 2\kappa h + 2\kappa h$, with pos. imag. part

(82)

We may recall the method of obtaining the permanent terms. Taking any one of the functions of (80), altered as in (81), we omit the factor $J_0 \kappa R$, and then find its expansion near $\kappa = 0$ to contain terms of negative degree in κ , say $A/\kappa^3 + B/\kappa$. The

permanent part of this function is then $A\chi(R) - B\nabla^2\chi(R)$. We thus find for this part of the solution

$$\begin{aligned}
 \psi' &= -\frac{\alpha+1}{2h}\left(\chi - \frac{1}{2}z^2\nabla^2\chi\right) + \frac{\alpha+1}{2h}\left(\frac{1}{2}z'^2 + \frac{1}{3}h^2\right)\nabla^2\chi \mp \frac{\alpha+1}{2}(z-z')\nabla^2\chi \\
 \phi' &= -\frac{\alpha+1}{8h}\left(\chi - \frac{1}{2}z^2\nabla^2\chi\right) + \frac{1}{4h}\left(\frac{\alpha+5}{4}z^2 + \frac{5-\alpha}{6}h^2\right)\nabla^2\chi \\
 &\quad - \frac{3(\alpha+1)}{8h^3}z'\left(z\chi - \frac{1}{6}z^3\nabla^2\chi\right) + \frac{3}{4h}\left(\frac{\alpha+5}{12}z^3 - h^2z' - \frac{\alpha+1}{10}h^2z'\right)z\nabla^2\chi \\
 &\quad \mp \frac{1}{2}(z-z')\nabla^2\chi \\
 \theta' &= -\frac{\alpha+1}{8h}\left(\chi - \frac{1}{2}z^2\nabla^2\chi\right) + \frac{1}{4h}\left(\frac{\alpha+5}{4}z^2 + \frac{5-\alpha}{6}h^2 + \alpha + 1h^2\right)\nabla^2\chi \\
 &\quad + \frac{3(\alpha+1)}{8h^3}z'\left(z\chi - \frac{1}{6}z^3\nabla^2\chi - 2h^2z\nabla^2\chi\right) - \frac{3}{4h^3}\left(\frac{\alpha+5}{12}z^3 - h^2z' - \frac{\alpha+1}{10}h^2z'\right)z\nabla^2\chi \\
 &\quad \mp \frac{\alpha}{2}(z-z')\nabla^2\chi \pm z'\nabla^2\chi
 \end{aligned} \tag{83}$$

When z' is put equal to h in the above values of ψ' , ϕ' , θ' it will be found, with very little trouble, that they reduce to those of (48) . . . (51), multiplied by $\frac{1}{2}(\alpha+1)$. (*Cf.* § 20.) As in § 20, the displacements due to the ambiguous terms in (83) are null if $R > 0$. But there is this difference in the present case, that they do not continue to vanish in the corresponding solution for an areal distribution of force on $z = z'$.

If the intensity of the distribution is $X(x, y, z')$ per unit area at (x, y) , this solution is defined as in (79), ψ' , θ' , ϕ' being obtained from (82), (83) by multiplying by

$$\frac{1}{4\pi\mu(\alpha+1)}X(x', y', z')dx'dy',$$

and integrating over the area within which X is finite.

When this is done we find that the ambiguous terms lead to

$$\left. \begin{aligned} u &= \mp \frac{1}{2\mu}(z-z')X \\ v &= 0 \\ w &= 0 \end{aligned} \right\} \tag{84}$$

In verification, we observe that these displacements are continuous above and below the plane $z = z'$, and that the corresponding stresses are also continuous with the exception of \widehat{zx} , the value of which just below $z = z'$ exceeds its value just above by X . The value of \widehat{zx} being $\mp \frac{1}{2}X$, we have for the contribution of (84) to the resultant $\int_{-h}^{+h} \widehat{zx} dz$,

$$\frac{1}{2}X\left(\int_{-h}^{z'} dz - \int_{z'}^h dz\right) = z'X \tag{85}$$

26. *Approximate values of the displacements. Lagrange's equation for flexure to a second approximation.*

The unambiguous terms in (83), as in (82), fall naturally into two classes, in the first of which ψ', θ', ϕ' are odd functions of z , while in the second they are even. Of the displacements derived from the first class, u and v are odd, and w even in z , and the strain may be described as *flexural*. In the other class u, v are even, and w odd in z , and the strain may be described as *extensional*. A force X at (x', y', z') acting along with a parallel but oppositely directed force X at $(x', y', -z')$ would give rise to flexural strain only; equal and similarly directed X forces at these two points to extensional strain only. This follows at once from the fact that the terms of ψ', θ', ϕ' , which are odd in z , are also odd in z' , and *vice versa*.

The distribution of force being $X(x, y, z')$ per unit area at (x, y) on $z = z'$, let

$$F = \frac{d}{dx} \iint X(x', y', z') \chi(R) dx' dy'.$$

Then from the flexural part of (83),

$$\left. \begin{aligned} \phi &= -\frac{3}{32\pi\mu h^3} z' \left(zF - \frac{1}{6} z^3 \nabla^2 F \right) \\ \theta &= \frac{3}{32\pi\mu h^3} z' \left(zF - \frac{1}{6} z^3 \nabla^2 F - 2h^2 z \nabla^2 F \right) \end{aligned} \right\} \text{with}$$

$$\left. \begin{aligned} \phi &= -z \nabla^2 F \\ \theta &= z \nabla^2 F \end{aligned} \right\} \text{each multiplied by } \frac{3}{32\pi\mu h^3} \cdot \frac{2}{\alpha+1} \left(\frac{\alpha+11}{10} h^2 z' - \frac{\alpha+5}{12} z'^3 \right)$$

These lead to

$$\left. \begin{aligned} u &= \frac{d}{dx} \left\{ \frac{3}{32\pi\mu h^3} \left\{ -(\alpha+1) z' z F + \nabla^2 F \left(\frac{\alpha+5}{6} z'^3 + \frac{\alpha+5}{6} z'^3 z - \frac{\alpha+21}{5} h^2 z' z \right) \right\} \right. \\ v &= \frac{d}{dy} \left\{ \frac{3}{32\pi\mu h^3} \left\{ -(\alpha+1) z' z F + \nabla^2 F \left(\frac{\alpha+5}{6} z'^3 + \frac{\alpha+5}{6} z'^3 z - \frac{\alpha+21}{5} h^2 z' z \right) \right\} \right. \\ w &= \frac{3}{32\pi\mu h^3} \left\{ (\alpha+1) z' z F + \nabla^2 F \left(\frac{3-\alpha}{2} z'^3 + \frac{\alpha+1}{5} h^2 z' - \frac{\alpha+5}{6} z'^3 \right) \right\} \end{aligned} \right\} \quad (86)$$

For Y force the same expressions hold if we take $F = \frac{d}{dy} \iint Y(x', y', z') \chi(R) dx' dy'$.

These formulæ, with all of (72) but the last terms of u, v , and with the odd parts in z arising from the ambiguous terms, give to a second approximation the displacements of the flexural mode under any forces. The differential equation satisfied by \bar{w} , the normal displacement of the mid plane, or value of w , for z equal to zero, is important in the history of the approximate theory. We can now write it down to a second approximation, namely, with C as in (75),

$$\begin{aligned} CV^4 \bar{w} &= Z + z' \left(\frac{dX}{dx} + \frac{dY}{dy} \right) \\ &+ \left(\frac{\alpha-19}{\alpha+1} \frac{h^2}{5} + \frac{3-\alpha}{\alpha+1} \frac{z'^2}{2} \right) \nabla^2 Z + \left(\frac{1}{6} h^2 z' - \frac{\alpha+5}{\alpha+1} \frac{z'^3}{6} \right) \nabla^2 \left(\frac{dX}{dx} + \frac{dY}{dy} \right). \end{aligned} \quad (87)$$

This equation gives the result for an areal distribution X, Y, Z on $z = z'$. For traction on $z = \pm h$ replace z' by $\pm h$; for a volume distribution X, Y, Z replace

$$Z \text{ by } \int_{-h}^h Z(x, y, z') dz'$$

$$z' \left(\frac{dX}{dx} + \frac{dY}{dy} \right) \text{ by } \frac{d}{dx} \int_{-h}^h z' X(x, y, z') dz' + \frac{d}{dy} \int_{-h}^h z' Y(x, y, z') dz',$$

and so on.

27. *Extensional strain. Differential equations of the principal mode.*

The unambiguous extensional terms of (83) remain to be considered.

$$\text{Write } E = \frac{1}{32\pi\mu h} \iint X(x', y', z') \chi(R) dx' dy'.$$

Then for an areal distribution X , these terms are

$$\begin{aligned} \psi' &= -4(E - \frac{1}{2}z^2 \nabla^2 E) + 4(\frac{1}{2}z^2 + \frac{1}{3}h^2) \nabla^2 E \\ \theta' &= -(E - \frac{1}{2}z^2 \nabla^2 E) + \frac{2}{\alpha+1} \left(\frac{\alpha+5}{4} z^2 + \frac{5-\alpha}{6} h^2 + \frac{1}{\alpha+1} h^2 \right) \nabla^2 E \\ \phi' &= -(E - \frac{1}{2}z^2 \nabla^2 E) + \frac{2}{\alpha+1} \left(\frac{\alpha+5}{4} z^2 + \frac{5-\alpha}{6} h^2 \right) \nabla^2 E \end{aligned}$$

The second parts of these expressions give

$$\begin{aligned} u &= (\alpha-3) \left(\frac{1}{2}z^2 - \frac{1}{3}h^2 \right) \frac{d^2}{dx^2} \nabla^2 E + 4 \left(z^2 + \frac{2}{3}h^2 \right) \nabla^4 E \\ v &= (\alpha-3) \left(\frac{1}{2}z^2 - \frac{1}{3}h^2 \right) \frac{d^2}{dx dy} \nabla^2 E \\ w &= 0 \end{aligned}$$

The first parts give

$$\begin{aligned} u &= -(\alpha+1) \frac{d^2 E}{dx^2} - 8 \frac{d^2 E}{dy^2} + z^2 \left(\frac{\alpha-3}{2} \frac{d^2}{dx^2} \nabla^2 E + 4 \nabla^4 E \right) \\ v &= -(\alpha+1) \frac{d^2 E}{dx dy} + 8 \frac{d^2 E}{dx dy} + z^2 \frac{\alpha-3}{2} \frac{d^2}{dx dy} \nabla^2 E \\ w &= (3-\alpha) z \frac{d}{dx} \nabla^2 E \end{aligned}$$

$$\text{If now further we write } K = \frac{1}{32\pi\mu h} \iint Y(x', y', z') \chi(R) dx' dy'$$

the corresponding displacements for a distribution of Y force on $z = z'$ can at once be written down from symmetry. The results for X and Y force combined cannot conveniently be expressed in terms of one function, as in the case of the flexural mode, and the best plan is probably to put everything in terms of the principal values of u, v , namely,

$$\begin{aligned} U &= -(\alpha+1) \frac{d^2 E}{dx^2} - 8 \frac{d^2 E}{dy^2} - (\alpha+1) \frac{d^2 K}{dx dy} + 8 \frac{d^2 K}{dx dy} \\ V &= -(\alpha+1) \frac{d^2 E}{dx dy} + 8 \frac{d^2 E}{dx dy} - (\alpha+1) \frac{d^2 K}{dy^2} - 8 \frac{d^2 K}{dx^2} \end{aligned} \quad (88)$$

We have then

$$\left. \begin{aligned} u &= U + \frac{3-\alpha}{\alpha+1} \left(\frac{1}{2}z^2 + \frac{1}{2}z'^2 - \frac{1}{3}h^2 \right) \frac{d}{dx} \left(\frac{dU}{dx} + \frac{dV}{dy} \right) + \left(\frac{1}{2}z^2 + \frac{1}{2}z'^2 + \frac{1}{3}h^2 \right) \frac{X}{2\mu h} \\ v &= V + \frac{3-\alpha}{\alpha+1} \left(\frac{1}{2}z^2 + \frac{1}{2}z'^2 - \frac{1}{3}h^2 \right) \frac{d}{dy} \left(\frac{dU}{dx} + \frac{dV}{dy} \right) + \left(\frac{1}{2}z^2 + \frac{1}{2}z'^2 + \frac{1}{3}h^2 \right) \frac{Y}{2\mu h} \\ w &= \frac{\alpha-3}{\alpha+1} z \left(\frac{dU}{dx} + \frac{dV}{dy} \right) \end{aligned} \right\} \quad (89)$$

The ordinary approximate theory obtains differential equations to determine U, V . These are easily found by eliminating E and K in turn from (88).

Thus

$$\begin{aligned} \frac{dU}{dx} + \frac{dV}{dy} &= -(\alpha+1) \left(\frac{d}{dx} \nabla^2 E + \frac{d}{dy} \nabla^2 K \right) \\ \frac{dU}{dy} - \frac{dV}{dx} &= -8 \left(\frac{d}{dy} \nabla^2 E - \frac{d}{dx} \nabla^2 K \right) \end{aligned}$$

and

$$\left. \begin{aligned} \frac{1}{\alpha+1} \frac{d}{dx} \left(\frac{dU}{dx} + \frac{dV}{dy} \right) + \frac{1}{8} \frac{d}{dy} \left(\frac{dU}{dy} - \frac{dV}{dx} \right) &= -\frac{X}{16\mu h} \\ \frac{1}{\alpha+1} \frac{d}{dy} \left(\frac{dU}{dx} + \frac{dV}{dy} \right) - \frac{1}{8} \frac{d}{dx} \left(\frac{dU}{dy} - \frac{dV}{dx} \right) &= -\frac{Y}{16\mu h} \end{aligned} \right\} \quad (90)$$

The principal parts of the contribution of Z force to extensional displacements appear in (72), (78). In the notation of those formulæ

$$\left. \begin{aligned} u &= \frac{d}{dx} \nabla^2 F \\ v &= \frac{d}{dy} \nabla^2 F \\ w &= \frac{\alpha-3}{\alpha+1} z \nabla^4 F \end{aligned} \right\} \quad \frac{\alpha-3}{32\pi\mu h} z$$

with in addition, w = the odd part in z of the ambiguous term in (78).

If these last values of u, v be included in the principal values U, V , then the right-hand members of (90) will become respectively

$$\frac{1}{16\mu h} \left(-X + \frac{\alpha-3}{\alpha+1} z \frac{dZ}{dx} \right), \quad \frac{1}{16\mu h} \left(-Y + \frac{\alpha-3}{\alpha+1} z \frac{dZ}{dy} \right) \quad (91)$$

28. Approximate values of the stresses across a plane parallel to the faces.

For any distribution of force parallel to the faces of the plate, the formulæ of §§ 26, 27 give the terms of the two lowest orders in the values of u, v , and the term of lowest order in w .* From these terms we can calculate all the stresses but \widehat{zz} to a first approximation, and as in § 22, when the first term of \widehat{zz} is known, we can find two terms of $\widehat{xx}, \widehat{xy}$ and \widehat{yy} . This first term of \widehat{zz} we may get very easily from the symbolical form of the solution corresponding to (80), (81). Thus for areal force X

$$\widehat{zz} = \pm \frac{1}{2}(z-z') \frac{dX}{dx} + \frac{3}{4h^3} z'(h^2 z - \frac{1}{3} z'^2) \frac{dX}{dx} - \frac{1}{4h} (z^2 + h^2) \frac{dX}{dx} \quad (92)$$

* There should be added from (84) the terms

$$u = \mp(z-z')X/2\mu, \quad v = \mp(z-z')Y/2\mu, \quad w = 0.$$

From (84), (86), (89) for areal force X

$$\left. \begin{aligned} \widehat{z} &= \mp \frac{1}{2}X + \frac{3}{4h^3}z'(z^2 - h^2) \frac{1}{2\pi} \frac{d^2}{dx^2} \iint X(x', y', z') \nabla^2 \chi(R) dx' dy' + \frac{z}{2h} X \\ \widehat{y} &= \frac{3}{4h^3} z'(z^2 - h^2) \frac{1}{2\pi} \frac{d^2}{dx dy} \iint X(x', y', z') \nabla^2 \chi(R) dx' dy' \end{aligned} \right\} \quad (93)$$

It may be verified that these give zero stress on $z = \pm h$, and $\frac{d\widehat{z}}{dx} + \frac{d\widehat{y}}{dy} + \frac{d\widehat{z}}{dz} = 0$.

From the formulæ we have given, it is of course merely a matter of the simplest algebra to calculate any of the stresses to whatever order of approximation may be required, but it may be worth while to remark here that the fundamental equations of equilibrium (1) may be used with great advantage in obtaining the principal results. If, for example, we know only the first terms of \widehat{xx} , \widehat{xy} , \widehat{yy} , the two first of these equations would give the first terms of \widehat{zx} , \widehat{zy} by a simple integration with respect to z , and then the last equation would give the first term of \widehat{zz} . Similarly, when the first two terms of \widehat{xx} , \widehat{xy} , \widehat{yy} are known (as above), we may find the first two terms of the other stresses.

29. *Transmission of force to a distance. Expansions in polar coordinates.*

We have up to this point been considering mainly the particular solution to which our general source solutions lead for any given distribution of force; or, as we may say, we have been investigating the effect of any given force system on that part of the solid to which the force is applied. But it is also of great interest to inquire what is the effect of this force at points of the solid remote from its region of application. It is obvious that we obtain a sufficient answer to this question by retaining only the permanent terms in the source solutions, those terms, namely, which are given in (65) and (83).

For force applied only at points on a given normal to the plate, these formulæ are all that we require. They show at a glance that the distant effect depends chiefly on *resultant* forces and couples, but not entirely, since z' and z'^2 occur in the formulæ for Z force, and z'^2 , z'^3 in those for X force. When the force is not confined to a line, but is distributed over a finite volume of the solid, the result is obtained in more intelligible form if before integration the function χ is suitably expanded so as to yield a series of solutions in which accented and unaccented coordinates are explicitly separated. The most convenient expansion of χ is in terms of polar coordinates as given in (e) of the introductory section.

Suppose, then, a single force applied at the point (x_1, y_1, z_1) or (ρ_1, ω_1, z_1) , the components of the force being X_1, Y_1, Z_1 , parallel to the rectangular axes, or P_1, Ω_1, Z_1 parallel to radius vector, transverse, and axis of z . We have to find the displacements at (ρ, ω, z) where we suppose $\rho > \rho_1$.

For an X force, the value of ψ is $\frac{d\psi'}{dy}$ with ψ' given in (83), the coefficient depending on the magnitude of X being for the moment suppressed.

This is the same as $-\frac{d\psi'}{dy}$ or $(-)$ rate of variation of ψ' in the direction perpendicular to the force at its point of application.

Similarly $\phi = -\frac{d\phi'}{dx'} = (-)$ rate of variation of ϕ' in direction of force.

Hence for
$$\left. \begin{aligned} P_1, \psi &= -\frac{1}{\rho_1} \frac{d\psi'}{d\omega_1} \\ \theta &= -\frac{d\theta'}{d\rho_1} \\ \phi &= -\frac{d\phi'}{d\rho_1} \end{aligned} \right\}, \text{ and for } \Omega_1, \left. \begin{aligned} \psi &= \frac{d\psi'}{d\rho_1} \\ \theta &= -\frac{1}{\rho_1} \frac{d\theta'}{d\omega_1} \\ \phi &= -\frac{1}{\rho_1} \frac{d\phi'}{d\omega_1} \end{aligned} \right\}$$

We shall take separately the extensional and flexural parts of the solution.

Also in the following u, v are the displacements along radius vector and transverse.

I. Extensional terms.

The following solutions occur.

$$\begin{aligned} \text{(i)} \quad u &= \left(\frac{7-\alpha}{4} - \frac{9+\alpha}{2} \log \frac{\rho}{2h} + \frac{3-\alpha}{2} z^2 \rho^{-2} \right) \cos \omega \\ v &= \left(\frac{7-\alpha}{4} + \frac{9+\alpha}{2} \log \frac{\rho}{2h} + \frac{3-\alpha}{2} z^2 \rho^{-2} \right) \sin \omega \\ w &= (3-\alpha) z \rho^{-1} \cos \omega \end{aligned}$$

(ii) Same as (i) with $\cos \omega$ changed into $\sin \omega$, and $\sin \omega$ into $-\cos \omega$

(iii) When $m > 1$,

$$\begin{aligned} u &= \left\{ \frac{8m - (m-2)(\alpha+1)}{4(m-1)} \rho^{-m+1} + \frac{3-\alpha}{2} m z^2 \rho^{-m-1} \right\} \cos m\omega \\ v &= \left\{ \frac{8(m-2) - m(\alpha+1)}{4(m-1)} \rho^{-m+1} + \frac{3-\alpha}{2} m z^2 \rho^{-m-1} \right\} \sin m\omega \\ w &= (3-\alpha) z \rho^{-m} \cos m\omega \end{aligned}$$

(iv) Same as (iii) with $\cos m\omega$ changed into $\sin m\omega$, and $\sin m\omega$ into $-\cos m\omega$.

$$\begin{aligned} \text{(v)} \quad u &= \rho^{-m-1} \cos m\omega \\ v &= \rho^{-m-1} \sin m\omega \end{aligned}$$

$$\begin{aligned} \text{(vi)} \quad u &= \rho^{-m-1} \sin m\omega \\ v &= -\rho^{-m-1} \cos m\omega \end{aligned}$$

For the force with components P_1, Ω_1, Z_1 , the coefficients of the above solutions are the following, in each case divided by $32\pi\mu h$.

$$\text{(i)} \quad P_1 \cos \omega_1 - \Omega_1 \sin \omega_1 = X_1$$

$$\text{(ii)} \quad P_1 \sin \omega_1 + \Omega_1 \cos \omega_1 = Y_1$$

$$\text{(iii)} \quad \rho_1^{m-1} \cos m\omega_1 P_1 - \rho_1^{m-1} \sin m\omega_1 \Omega_1$$

$$\text{(iv)} \quad \rho_1^{m-1} \sin m\omega_1 P_1 + \rho_1^{m-1} \cos m\omega_1 \Omega_1$$

$$\begin{aligned} \text{(v)} \quad & \left\{ \frac{-8m + (m+2)(\alpha+1)}{4(m+1)} \rho_1^{m+1} + (3-\alpha) \left(\frac{1}{2} z_1^2 - \frac{1}{3} h^2 \right) m \rho_1^{m-1} \right\} \cos m\omega_1 P_1 \\ & + \left\{ \frac{8(m+2) - m(\alpha+1)}{4(m+1)} \rho_1^{m+1} - (3-\alpha) \left(\frac{1}{2} z_1^2 - \frac{1}{3} h^2 \right) m \rho_1^{m-1} \right\} \sin m\omega_1 \Omega_1 \\ & + (\alpha-3) \rho_1^{m-1} \cos m\omega_1 Z_1 \end{aligned}$$

(vi) Same as (v) with $\cos m\omega_1$ changed into $\sin m\omega_1$, and $\sin m\omega_1$ into $-\cos m\omega_1$.

II. Flexural terms.

The flexural solutions are of the form given in (23), or in polar coordinates

$$\left. \begin{aligned} u &= \frac{d}{d\rho} \left[\begin{aligned} &-(\alpha+1)(z^2 - \frac{1}{6}h^2 \nabla^2 F) + 2(\frac{1}{3}z^3 - h^2 z) \nabla^2 F \\ &(\alpha+1)(F - \frac{1}{2}z^2 \nabla^2 F) + 2(z^2 - h^2) \nabla^2 F \end{aligned} \right] \\ v &= \frac{1}{\rho} \frac{d}{d\omega} \left[\begin{aligned} &-(\alpha+1)(z^2 - \frac{1}{6}h^2 \nabla^2 F) + 2(\frac{1}{3}z^3 - h^2 z) \nabla^2 F \\ &(\alpha+1)(F - \frac{1}{2}z^2 \nabla^2 F) + 2(z^2 - h^2) \nabla^2 F \end{aligned} \right] \\ w &= \end{aligned} \right\} \quad (94)$$

$$\left. \begin{aligned} \text{(i)} \quad F &= \chi(\rho) = \frac{1}{4}\rho^2 \log \frac{\rho}{2h} - \frac{1}{4}\rho^2 \\ \text{(ii)} \quad F &= \frac{1}{4}\left(\rho - 2\rho \log \frac{\rho}{2h}\right) \cos \omega \\ \text{(iii)} \quad F &= \frac{1}{4}\left(\rho - 2\rho \log \frac{\rho}{2h}\right) \sin \omega \\ \text{(iv)} \quad F &= \frac{1}{4m(m-1)}\rho^{-m+2} \cos m\omega \\ \text{(v)} \quad F &= \frac{1}{4m(m-1)}\rho^{-m+2} \sin m\omega \\ \text{(vi)} \quad F &= \log \frac{\rho}{2h} \\ \text{(vii)} \quad F &= \rho^{-m} \cos m\omega \\ \text{(viii)} \quad F &= \rho^{-m} \sin m\omega \end{aligned} \right\} \quad m > 1.$$

$$\left. \begin{aligned} \text{(vii)} \quad F &= \rho^{-m} \cos m\omega \\ \text{(viii)} \quad F &= \rho^{-m} \sin m\omega \end{aligned} \right\} \quad m > 0.$$

For the force with components P_1, Ω_1, Z_1 the coefficients of the above solutions are the following, in each case divided by $\frac{2}{3}\pi\mu h^3$.

$$\left. \begin{aligned} \text{(i)} \quad Z_1 \\ \text{(ii)} \quad -z_1 \cos \omega_1 P_1 + z_1 \sin \omega_1 \Omega_1 + Z_1 \rho_1 \cos \omega_1 &= -X_1 z_1 + Z_1 y_1 \\ \text{(iii)} \quad -z_1 \sin \omega_1 P_1 - z_1 \cos \omega_1 \Omega_1 + Z_1 \rho_1 \sin \omega_1 &= -Y_1 z_1 + Z_1 x_1 \\ \text{(iv)} \quad -z_1 m \rho_1^{m-1} \cos m\omega_1 P_1 + z_1 m \rho_1^{m-1} \sin m\omega_1 \Omega_1 + \rho_1^m \cos m\omega_1 Z_1 \\ \text{(v)} \quad \text{Same as (iv) with } \cos m\omega_1 \text{ changed to } \sin m\omega_1, \text{ and } \sin m\omega_1 \text{ to } -\cos m\omega_1 \\ \text{(vi)} \quad -\frac{1}{2}z_1 \rho_1 P_1 + \left\{ \frac{1}{4}\rho_1^2 + \frac{1}{3}h^2 - \frac{1}{2}z_1^2 + \frac{2}{a+1}(z_1^2 - h^2) \right\} Z_1 \\ \text{(vii)} \quad \left[z_1 \frac{m+2}{4m(m+1)}\rho_1^{m+1} - \left\{ \frac{1}{6}z_1^3 - \frac{1}{3}h^2 z_1 + \frac{2}{a+1}(\frac{1}{3}z_1^3 - h^2 z_1) \right\} \rho_1^{m-1} \right] \cos m\omega_1 P \\ \quad - \left[z_1 \frac{1}{4(m+1)}\rho_1^{m+1} - \left\{ \begin{array}{c} \text{ " } \\ \text{ " } \end{array} \right\} \rho_1^{m-1} \right] \sin m\omega_1 \Omega_1 \\ \quad - \left[\frac{1}{4m(m+1)}\rho_1^{m+2} + \left\{ \frac{1}{3}h^2 - \frac{1}{2}z_1^2 + \frac{2}{a+1}(z_1^2 - h^2) \right\} \frac{1}{m}\rho_1^m \right] \cos m\omega_1 Z_1 \\ \text{(viii)} \quad \text{Same as (vii) with } \cos m\omega_1 \text{ changed to } \sin m\omega_1 \text{ and } \sin m\omega_1 \text{ to } -\cos m\omega_1. \end{aligned} \right\}$$

30. *Types of deformation conveying a given resultant stress.*

In these formulæ we remark at once a striking relation between the forms of the displacements u, v, w in the various solutions, and the multipliers of P_1, Ω_1, Z_1 in the coefficients of the solutions.

In I. (iii), *e.g.*, these multipliers are $\rho_1^{m-1} \cos m\omega_1, -\rho_1^{m-1} \sin m\omega_1, 0$, which are simply the displacements of I. (v) with sign of m changed, and consequently suitable for space containing the origin.

Similarly in I. (v) the multipliers of P_1, Ω_1, Z_1 are displacements compounded of the types (iii), (v), with sign of m changed, and so on.

The full explanation of this peculiarity will be given presently, when it will be seen that an independent verification of all the results may be obtained by means of the important principle known as Betti's Theorem.

In the meantime we may examine the scheme of solutions from another very important point of view.

With reference to any individual solution, the following questions are obviously of prime importance:—

(1) What is the resultant stress transmitted?

(2) Is the whole potential energy of the part of the solid bounded internally by a given cylindrical surface, finite or infinite?

Now, in order to single out those solutions which convey a finite resultant stress across any cylinder (or other surface) surrounding the origin, we have merely to look at the table of coefficients. Thus, for instance, I. (i) appears with coefficient $X_1/32\pi\mu h$, from which we may infer (as verified below) that this solution conveys a stress with resultant a force of $32\pi\mu h$ units parallel to the axis of x , and passing through the origin.

In this way we find that the six solutions, corresponding to the six elements which specify the resultant of a force system, are

I. (i), (ii), (vi) with $m = 0$; II. (i), (ii), (iii).

For these we shall write down the values of the stresses $\widehat{\rho\rho}, \widehat{\rho\omega}, \widehat{\rho z}$, the components of the stress across the cylinder $\rho = \text{constant}$.

In all, of course, we have $\widehat{zz} = 0$, and in I. in addition $\widehat{z\rho} = \widehat{z\omega} = 0$.

$$\text{I. (i)} \quad \left. \begin{aligned} \frac{\widehat{\rho\rho}}{2\mu} &= \left(\frac{\alpha-15}{2} \rho^{-1} + \alpha - 3z^2\rho^{-3} \right) \cos \omega \\ \frac{\widehat{\rho\omega}}{2\mu} &= \left(\frac{\alpha+1}{2} \rho^{-1} + \alpha - 3z^2\rho^{-3} \right) \sin \omega \end{aligned} \right\}$$

The resultant is a force along Ox , of magnitude

$$\begin{aligned} & \int \int (\widehat{\rho\rho} \cos \omega - \widehat{\rho\omega} \sin \omega) \rho d\omega dz, \quad \text{taken over the cylinder } \rho, \\ &= \left(\frac{\alpha-15}{2} - \frac{\alpha+1}{2} \right) \pi \cdot 2h \cdot 2\mu = -32\pi\mu h. \end{aligned}$$

$$\text{I. (ii)} \quad \left. \begin{aligned} \frac{\widehat{\rho\rho}}{2\mu} &= \left(\frac{\alpha-15}{2} \rho^{-1} + \alpha - 3z^2\rho^{-3} \right) \sin \omega \\ \frac{\widehat{\rho\omega}}{2\mu} &= - \left(\frac{\alpha+1}{2} \rho^{-1} + \alpha - 3z^2\rho^{-3} \right) \cos \omega \end{aligned} \right\}$$

The resultant is a force along Oy of magnitude $-32\pi\mu h$.

$$\text{I. (vi), } m = 0. \quad \left. \begin{aligned} u &= 0 \\ v &= -\rho^{-1} \\ w &= 0 \end{aligned} \right\}, \quad \left. \begin{aligned} \widehat{\rho\rho} &= 0 \\ \widehat{\rho\omega} &= 2\mu\rho^{-2} \end{aligned} \right\}$$

The resultant is a couple in the plane xy , of magnitude $8\pi\mu h$, and we observe that the solution occurs with coefficient $-\Omega_1\rho_1/8\pi\mu h$.

II. The stresses in the general flexural solution (94) are

$$\left. \begin{aligned} \frac{\widehat{\rho\rho}}{2\mu} &= (a-3)z\nabla^2 F + \frac{du}{d\rho} \\ \frac{\widehat{\rho\omega}}{2\mu} &= \frac{dv}{d\rho} \\ \frac{\widehat{\rho z}}{2\mu} &= 2(z^2 - h^2) \frac{d}{d\rho} \nabla^2 F. \end{aligned} \right\} \quad (95)$$

$$\text{II. (i)} \quad \left. \begin{aligned} \frac{\widehat{\rho\rho}}{2\mu} &= \frac{a-7}{2} z \log \frac{\rho}{2h} - \frac{a+1}{4} z - \left(\frac{a+5}{6} z^3 - 2h^2 z \right) \rho^{-2} \\ \frac{\widehat{\rho\omega}}{2\mu} &= 0 \\ \frac{\widehat{\rho z}}{2\mu} &= 2(z^2 - h^2) \rho^{-1} \end{aligned} \right\}$$

The resultant is a force along Oz , of magnitude $-\frac{32}{3}\pi\mu h^3$.

$$\text{II. (ii)} \quad \left. \begin{aligned} \frac{\widehat{\rho\rho}}{2\mu} &= \frac{7-a}{2} z \rho^{-1} \cos \omega + 2 \left(\frac{a+5}{6} z^3 - 2h^2 z \right) \rho^{-3} \cos \omega \\ \frac{\widehat{\rho\omega}}{2\mu} &= -\frac{a+1}{2} z \rho^{-1} \sin \omega + 2 \left(\frac{a+5}{6} z^3 - 2h^2 z \right) \rho^{-3} \sin \omega \\ \frac{\widehat{\rho z}}{2\mu} &= 2(z^2 - h^2) \rho^{-2} \cos \omega \end{aligned} \right\}$$

The resultant is a couple about Oy , of magnitude

$$\begin{aligned} & \iint \{ z(\widehat{\rho\rho} \cos \omega - \widehat{\rho\omega} \sin \omega) - \rho \cos \omega \widehat{\rho z} \} \rho^2 d\omega dz, \quad \text{taken over the cylinder } \rho, \\ &= \frac{32}{3} \pi \mu h^3. \end{aligned}$$

$$\text{II. (iii)} \quad \left. \begin{aligned} \frac{\widehat{\rho\rho}}{2\mu} &= \frac{7-a}{2} z \rho^{-1} \sin \omega + 2 \left(\frac{a+5}{6} z^3 - 2h^2 z \right) \rho^{-3} \sin \omega \\ \frac{\widehat{\rho\omega}}{2\mu} &= \frac{a+1}{2} z \rho^{-1} \cos \omega + 2 \left(\frac{a+5}{6} z^3 - 2h^2 z \right) \rho^{-3} \cos \omega \\ \frac{\widehat{\rho z}}{2\mu} &= 2(z^2 - h^2) \rho^{-2} \sin \omega. \end{aligned} \right\}$$

The resultant is a couple about Ox , of magnitude

$$\iint \{ -z(\widehat{\rho\rho} \sin \omega + \widehat{\rho\omega} \cos \omega) + \rho \sin \omega \widehat{\rho z} \} \rho^2 d\omega dz = -\frac{32}{3} \pi \mu h^3.$$

31. *Conditions for the existence of a solution with finite potential energy. Elastic equivalence of statically equipollent loads.*

The corresponding results for any distribution of body force, or of traction on the faces of the plate, may be deduced at once from the above by integration with respect to ρ_1 , ω_1 , z_1 or ρ_1 , ω_1 with $z_1 = \pm h$.

If the region within which the force is applied be entirely enclosed by a cylinder $\rho = a$, the results are valid for all points exterior to this cylinder.

For a distribution of force of finite intensity per unit area or per unit volume, the potential energy of that part of the solid within the cylinder is clearly finite. The energy of the remaining part of the solid can be determined from the forms of § 29.

Now, the energy between the cylinders $\rho = a$, $\rho = \rho'$ is the integral of

$$\frac{1}{2}(u \widehat{\rho\rho} + v \widehat{\rho\omega} + w \widehat{\rho z})$$

taken over the belt of the cylinder $\rho = \rho'$ cut off by the plate, diminished by the corresponding integral for $\rho = a$. Hence the condition of finiteness of the whole potential energy is simply that the value of the integral for the surface $\rho = \rho'$ tends to zero as ρ' tends to infinity. This condition is obviously satisfied by all the partial solutions of § 29, except those which have been already singled out as conveying a finite resultant stress. It is also satisfied by one of the latter class, namely, that which conveys a couple in the plane of the plate.

Hence, when force is applied to a circumscribed portion of the solid, a solution giving finite potential energy will exist provided the force either constitutes an equilibrating system, or reduces to a couple in the plane of the plate. It does not follow, however, even for an equilibrating application of force, that a solution will exist giving vanishing displacements at infinity. We need only point to the solutions of § 29, II. (vi) and (iv), (v) with $m = 2$. This being so, it may be of interest to write down a few more details of those solutions which rank in importance next to the solutions of finite resultant stress.

I. (iii) with $m = 2$.

$$\left. \begin{aligned} u &= \left\{ 4\rho^{-1} + (3-\alpha)z^2\rho^{-3} \right\} \cos 2\omega \\ v &= \left\{ -\frac{\alpha+1}{2}\rho^{-1} + (3-\alpha)z^2\rho^{-3} \right\} \sin 2\omega \\ w &= (3-\alpha)z\rho^{-2} \cos 2\omega \end{aligned} \right\} \cdot \left\{ \begin{aligned} \widehat{\rho\rho} &= \left\{ (a-7)\rho^{-2} + (a-3)3z^2\rho^{-4} \right\} \cos 2\omega \\ \widehat{\rho\omega} &= \left\{ \frac{a-7}{2}\rho^{-2} + (a-3)3z^2\rho^{-4} \right\} \sin 2\omega \end{aligned} \right.$$

This solution occurs with coefficient $(X_1x_1 - Y_1y_1)/32\pi\mu h$.

I. (iv) with $m = 2$ is obtained by writing $\sin 2\omega, -\cos 2\omega$ for $\cos 2\omega, \sin 2\omega$ in the preceding, and the coefficient is $(X_1y_1 + Y_1x_1)/32\pi\mu h$.

I. (v) with $m = 0$.

$$\left. \begin{aligned} u &= \rho^{-1} \\ v &= 0 \\ w &= 0 \end{aligned} \right\} \left\{ \begin{aligned} \widehat{\rho\rho} &= -2\mu\rho^{-2} \\ \widehat{\rho\omega} &= 0 \end{aligned} \right\} \text{ Coefficient} = \left\{ \frac{\alpha+1}{2}(X_1x_1 + Y_1y_1) + (a-3)z_1Z_1 \right\} / 32\pi\mu h.$$

II. (iv) with $m = 2$. $F = \frac{1}{8} \cos 2\omega$.

$$\left. \begin{aligned} u &= \left(\frac{\alpha+5}{6}z^3 - 2h^2z \right) \rho^{-3} \cos 2\omega \\ v &= \frac{\alpha+1}{4}z\rho^{-1} \sin 2\omega + \left(\frac{\alpha+5}{6}z^3 - 2h^2z \right) \rho^{-3} \sin 2\omega \\ w &= \frac{\alpha+1}{8} \cos 2\omega + \left(\frac{\alpha-3}{4}z^2 + h^2 \right) \rho^{-2} \cos 2\omega \end{aligned} \right\} \cdot \left\{ \begin{aligned} \widehat{\rho\rho} &= \frac{3-\alpha}{2}z\rho^{-2} \cos 2\omega + \text{a term in } \rho^{-4} \\ \widehat{\rho\omega} &= -\frac{\alpha+1}{4}z\rho^{-2} \sin 2\omega + \dots \\ \widehat{\rho z} &= 2(z^2 - h^2)\rho^{-3} \cos 2\omega. \end{aligned} \right.$$

The coefficient is

$$\frac{3}{32\pi\mu h^3} \left\{ -2z_1x_1X_1 + 2z_1y_1Y_1 + (x_1^2 - y_1^2)Z_1 \right\}$$

II. (v) with $m=2$ is the above with $\sin 2\omega, -\cos 2\omega$ for $\cos 2\omega, \sin 2\omega$ and coefficient

$$\frac{3}{32\pi\mu h^3} \left\{ -2z_1y_1X_1 - 2z_1x_1Y_1 + 2x_1y_1Z_1 \right\}$$

II. (vi). $F = \log \frac{\rho}{2h}$.

$$\left. \begin{aligned} u &= -(a+1)z\rho^{-1} \\ v &= 0 \\ w &= (a+1) \log \frac{\rho}{2h} \end{aligned} \right\} \begin{aligned} \widehat{\rho} &= (a+1)z\rho^{-2} \\ \frac{\rho\rho}{2\mu} &= (a+1)z\rho^{-2} \\ \widehat{\rho\omega} &= \widehat{\rho} = 0 \end{aligned}$$

Coefficient is

$$\left\{ -\frac{1}{2}z_1(x_1X_1 + y_1Y_1) + \left(\frac{x_1^2 + y_1^2}{4} + \frac{h^2}{5} - \frac{z_1^2}{2} + \frac{2}{a+1} \frac{z_1^2}{1+z_1^2/h^2} \right) Z_1 \right\} \frac{3}{32\pi\mu h^3}.$$

For all the remaining solutions, the stresses are of the third or higher order in $1/\rho$. The results of this and the preceding article bear directly upon a principle of fundamental importance in theories of approximation, generally referred to as the principle of the *elastic equivalence of statically equipollent systems of load*, and a study of these results will be found of service in imparting precision and definiteness to one's view of the principle in its application to the theory of plates.

It may be noted here, with reference to the occurrence of the function $\log (\rho/2h)$ in some of the principal solutions of § 30, that it would make no essential difference if this function were replaced throughout by $\log (\rho/c)$, c being any length whatever, the unit of length for example. The change would be equivalent to adding a solution of the permanent type, giving no body force or traction on the faces, and it will be observed that the addition would disappear altogether when the applied forces are in equilibrium.

We have here, in fact, an instance of the indeterminateness that of necessity arises in the absence of conditions at infinity, and we are thus brought to the question, what is the exact extent of this indeterminateness? or, as it may be put, given one solution of a problem satisfying the conditions at a finite distance, what is the most general solution satisfying such conditions?

For the investigation of this question we have at hand a powerful instrument in *Betti's Theorem*, which occupies in the theory of elastic solids the place held by Green's Theorem in the Theory of the Potential.

32. *Betti's reciprocal theorem. Verification of preceding solutions.*

Betti's Theorem may be thus stated:—Given two sets of displacements of an elastic solid, with the two corresponding sets of forces maintaining these displacements (including body forces, surface tractions, and kinetic reactions), then the work done by the forces of the first set acting on the displacements of the second set is

equal to the work done by the forces of the second set acting over the displacements of the first.

In potential theory one of the chief applications of Green's Theorem is to the case when one of the potential systems includes a mass concentrated at a single point, and in the present subject Betti's Theorem finds an application of like importance when one of the displacement systems contains a finite force applied at one point, or, in analytical language, includes a point singularity of the first order, that is to say, of one of the three forms indicated in (6).

Thus, let us suppose the solid to be bounded by a surface S , and in the first set let the displacements be u, v, w ; the components of body force per unit volume X, Y, Z ; and the components of the traction on S, F, G, H ; in the second set let the displacements be u', v', w' ; the only internal force a force X', Y', Z' at (x', y', z') , and the tractions on S, F', G', H' .

We may apply Betti's Theorem to the space bounded by S and a sphere S' of radius ϵ drawn round (x', y', z') as centre. Thus we have

$$\begin{aligned} & \iiint (Xu' + Yv' + Zw')dV + \iint (Fu' + Gv' + Hw')dS + \iint (F'u + G'v + H'w)dS' \\ &= \iint (F'u + G'v + H'w)dS + \iint (F'u + G'v + H'w)dS'. \end{aligned}$$

Now take the limits of both members of this equality for $\epsilon = 0$.

Since near the centre of the sphere S' , u', v', w' are of order $1/\epsilon$, F', G', H' of order $1/\epsilon^2$, and dS' of order ϵ^2 , the effect on the volume integral is simply to extend it to the whole volume within S ; the surface integral $\iint (Fu' + Gv' + Hw')dS'$ vanishes, and the surface integral $\iint (F'u + G'v + H'w)dS'$ has the same limit as

$$u(x', y', z') \iint F'dS' + v(x', y', z') \iint G'dS' + w(x', y', z') \iint H'dS',$$

namely,

$$u(x', y', z')X' + v(x', y', z')Y' + w(x', y', z')Z',$$

the tractions F', G', H' on S' being statically equivalent to the force X', Y', Z' at its centre.

It is thus apparent, and might indeed have been anticipated, that Betti's Theorem may legitimately be applied when one of the systems contains a force acting at a single point, provided the work done by this force on the other system of displacements be taken into account.

The theorem thus becomes

$$\begin{aligned} & \iiint (Xu' + Yv' + Zw')dV + \iint (Fu' + Gv' + Hw')dS - \iint (F'u + G'v + H'w)dS \\ &= u(x', y', z')X' + v(x', y', z')Y' + w(x', y', z')Z' \end{aligned} \quad (96)$$

In order to apply the theorem to the plate problems under discussion, take for the solid a portion of the plate bounded externally by any orthogonal cylinder. Let us

also suppose that the system u, v, w is maintained solely by tractions on the cylindrical edge, and the system u', v', w' by such tractions along with the force at (x', y', z') . Further, it will be convenient to decompose the latter system, and take u_1, v_1, w_1 as due to a unit X force, u_2, v_2, w_2 to a unit Y force, and u_3, v_3, w_3 to a unit Z force. The corresponding tractions on the edge we will denote by $X, Y, Z; X_1, Y_1, Z_1; X_2, Y_2, Z_2; X_3, Y_3, Z_3$. The theorem (96) then gives

$$\left. \begin{aligned} u(x', y', z') &= \iint (Xu_1 + Yv_1 + Zw_1 - X_1u - Y_1v - Z_1w) dS \\ v(x', y', z') &= \iint (Xu_2 + Yv_2 + Zw_2 - X_2u - Y_2v - Z_2w) dS \\ w(x', y', z') &= \iint (Xu_3 + Yv_3 + Zw_3 - X_3u - Y_3v - Z_3w) dS \end{aligned} \right\} \quad (97)$$

the integrals being taken over the edge.

As one application of these forms, we may indicate briefly how they can be used to verify the single force solutions already obtained.

Take, for example, the case of a Z force, and let u_3, v_3, w_3 have the values defined in (63), (64), (65). Also let the edge be the cylinder $R = \text{constant}$.

(i) The coefficient of the principal flexural term, in which, with the notation of (94) $F \propto \chi(R)$, is determined from the condition that the resultant of the stress $\hat{\epsilon}R$ must balance the applied force.

It is interesting to note that the conditions of equilibrium of applied forces and surface tractions may be regarded as special cases of Betti's Theorem. We have only to take for auxiliary systems the rigid body displacements $u = 0, v = 0, w = 1; u = y, v = -x, w = 0$, etc.

(ii) In the third of equations (97) take for u, v, w the values of (94) with $F = R^2$. Only the two flexural terms of (65) contribute to the surface integral; the contribution from the particular solution $\phi = G_0\kappa R \sinh \kappa z, \theta = -\cosh 2\kappa h \cdot \phi$ must vanish, as we see by pushing the edge to infinity.

This, with the result of (i), gives the coefficient of the second flexural term of (65).

(iii) The principal extensional term is verified by taking

$$u = \frac{a+1}{2}(x-x'), \quad v = \frac{a+1}{2}(y-y'), \quad w = (a-3)z.$$

(iv) The coefficient of the particular solution $\phi = G_0\kappa R \sinh \kappa z, \theta = -\cosh 2\kappa h \cdot \phi$ in (63) is verified by taking for u, v, w the values defined by $\phi = J_0\kappa R \sinh \kappa z, \theta = -\cosh 2\kappa h \cdot \phi$.

None of the solutions corresponding to the other roots of $\sinh 2\kappa h - 2\kappa h$ contribute to the surface integral. In fact, the partial contribution from a root κ' being independent of the radius of the cylinder, must vanish identically, since the Bessel Functions supply a factor tending to zero or infinity when R is made infinite, according as κ' is a higher or lower root than κ .

(v) The coefficient of the particular solution $\phi = G_0\kappa R \cosh \kappa z, \theta = \cosh 2\kappa h \cdot \phi$, may be verified in the same way.

It is now easy to see the significance of the forms of the coefficients in the solutions of § 29 and the confirmation of the values there given would obviously present no difficulties.

33. Finite plate under edge tractions. Form of the solution deduced by means of Betti's Theorem.

We pass, however, to a more important application of the theorems (97). The system u, v, w we still suppose maintained by edge tractions alone, but in addition to the external edge the solid may now be bounded by one or more internal edges. For u_1, v_1, w_1 , etc., we take the definite values defined in (79), (82), (83), and in (63), (64), (65).

Thus in (97) $u_1, v_1, w_1, X_1, Y_1, Z_1$, and the other displacements and tractions marked with suffixes, are known functions of x', y', z' , and the equations give explicitly the values of the displacements at any internal point in terms of the displacement and stress at the edge or edges.

The ideal solution would give the internal displacement in terms of edge displacement alone, or of edge stress alone, but the analytical difficulties are such that we are unable to solve the problem thus completely even for the simplest case, that of a single infinite plane edge. Meantime, however, we may derive valuable information from the expressions of (97), and in the first place as to the *form* into which any solution due to edge tractions alone may be thrown.

Just as in the case of the original source solutions, we find that the solution, in which, of course, the accented letters are now the variables, may be decomposed into an extensional and a flexural part, while in each of those parts we may separate a permanent mode from an infinite series of transitory or decaying modes of two types, the ψ type, characterised by no dilatation or normal displacement, and the θ, ϕ type, in which there is no molecular rotation in the plane of the plate.

In the following analysis integrals of the same form as those in (97) occur frequently; the system u, v, w appearing in each case, but associated with various other systems. For conciseness we shall refer to the first integral of (97) as the *work difference from* u_1, v_1, w_1 , and similarly in other cases.

I. Extensional part of the solution.

(i) Permanent mode.

In u_1, v_1, w_1 , the terms which relate to this mode are the unambiguous terms, even in z , of (83), after these have been divided by $4\pi\mu(a+1)$. These, as may be seen from a glance at the beginning of § 27, are equivalent to

$$\left[\begin{aligned} \psi &= \frac{1}{8\pi\mu h} \frac{d}{dy} \left(\chi - \frac{1}{2} z^2 \nabla^2 \chi \right) \\ \theta = \phi &= \frac{1}{32\pi\mu h} \left\{ \frac{d}{dx} \left(\chi - \frac{1}{2} z^2 \nabla^2 \chi \right) + \frac{3-a}{a+1} \left(\frac{1}{2} z^2 - \frac{1}{3} h^2 \right) \frac{d}{dx} \nabla^2 \chi \right\} \end{aligned} \right]$$

∇^2 standing for $\frac{d^2}{dx'^2} + \frac{d^2}{dy'^2}$.

Now let the work difference from the system $\psi = \frac{1}{8\pi\mu h}(\chi - \frac{1}{2}z^2\nabla^2\chi)$ be denoted by E_1 , and that from the system $\theta = \phi = \frac{1}{32\pi\mu h}(\chi - \frac{1}{2}z^2\nabla^2\chi)$ by E_2 ; then obviously the work differences from the two systems immediately preceding are respectively

$$\frac{dE_1}{dy'} \quad \text{and} \quad \frac{dE_2}{dx'} + \frac{3-a}{a+1}\left(\frac{1}{2}z^2 - \frac{1}{3}h^2\right)\frac{d}{dx'}\nabla^2E_2.$$

Hence (97) gives

$$u(x', y', z') = \frac{dE_1}{dy'} + \frac{dE_2}{dx'} + \frac{3-a}{a+1}\left(\frac{1}{2}z^2 - \frac{1}{3}h^2\right)\frac{d}{dx'}\nabla^2E_2$$

In the same way from u_2, v_2, w_2 and u_3, v_3, w_3 we obtain

$$\left. \begin{aligned} v(x', y', z') &= -\frac{dE_1}{dx'} + \frac{dE_2}{dy'} + \frac{3-a}{a+1}\left(\frac{1}{2}z^2 - \frac{1}{3}h^2\right)\frac{d}{dy'}\nabla^2E_2 \\ w(x', y', z') &= \frac{a-3}{a+1}z'\nabla^2E_2 \end{aligned} \right\}$$

Moreover, it can be seen in a moment that the displacements due to

$$\psi = \frac{d}{dy'}\nabla^2\chi \quad \text{and to} \quad \theta = \phi = -\frac{2}{a+1}\frac{d}{dx'}\nabla^2\chi$$

are in reality the same; as also those due to

$$\psi = \frac{d}{dx'}\nabla^2\chi \quad \text{and to} \quad \theta = \phi = \frac{2}{a+1}\frac{d}{dy'}\nabla^2\chi.$$

It follows that

$$\text{and} \quad \left. \begin{aligned} \frac{d}{dy'}\nabla^2E_1 + \frac{8}{a+1}\frac{d}{dx'}\nabla^2E_2 &= 0 \\ \frac{d}{dx'}\nabla^2E_1 - \frac{8}{a+1}\frac{d}{dy'}\nabla^2E_2 &= 0 \end{aligned} \right\}$$

If we write U for

$$\frac{dE_1}{dy'} + \frac{dE_2}{dx'} - \frac{3-a}{a+1}\frac{1}{3}h^2\frac{d}{dx'}\nabla^2E_2,$$

and V for

$$-\frac{dE_1}{dx'} + \frac{dE_2}{dy'} - \frac{3-a}{a+1}\frac{1}{3}h^2\frac{d}{dy'}\nabla^2E_2,$$

we obtain the form which it is convenient to take as the standard for this kind of strain, namely,

$$\left. \begin{aligned} u(x', y', z') &= U + \frac{3-a}{a+1}\frac{1}{2}z^2\frac{d}{dx'}\left(\frac{dU}{dx'} + \frac{dV}{dy'}\right) \\ v(x', y', z') &= V + \frac{3-a}{a+1}\frac{1}{2}z^2\frac{d}{dy'}\left(\frac{dU}{dx'} + \frac{dV}{dy'}\right) \\ w(x', y', z') &= \frac{a-3}{a+1}z'\left(\frac{dU}{dx'} + \frac{dV}{dy'}\right) \end{aligned} \right\} \quad (98)$$

with

$$\left. \begin{aligned} \frac{d}{dy'}\left(\frac{dU}{dy'} - \frac{dV}{dx'}\right) + \frac{8}{a+1}\frac{d}{dx'}\left(\frac{dU}{dx'} + \frac{dV}{dy'}\right) &= 0 \\ \frac{d}{dx'}\left(\frac{dU}{dy'} - \frac{dV}{dx'}\right) - \frac{8}{a+1}\frac{d}{dy'}\left(\frac{dU}{dx'} + \frac{dV}{dy'}\right) &= 0 \end{aligned} \right\}$$

(ii) Transitory modes, ψ or rotational type.

Referring to the expression for ψ' in (82), put

$$E_3 \equiv \text{work difference from the system } \psi = \frac{1}{8\pi\mu\kappa^2h} \cosh \kappa z G_0 \kappa R.$$

Then for this part of the solution

$$\left. \begin{aligned} u(x', y', z') &= \sum_{\kappa} 2 \frac{dE_3}{dy'} \cosh \kappa z' \\ v(x', y', z') &= \sum_{\kappa} (-2) \frac{dE_3}{dx'} \cosh \kappa z' \\ w(x', y', z') &= 0 \end{aligned} \right\} \quad (99)$$

where κ is a pos. imag. root of $\sinh \kappa h$, and $\frac{d^2 E_3}{dx'^2} + \frac{d^2 E_3}{dy'^2} + \kappa^2 E_3 = 0$.

The solutions here are obviously of the type $\psi = \cosh \kappa z' E_3(x', y')$.

(iii) Transitory modes, θ - ϕ or dilatational type.

Looking to (64), (82), put

$$E_4 \equiv \text{work difference from the system } \begin{cases} \phi = \frac{\cosh \kappa z G_0 \kappa R}{8\pi\mu(\alpha+1)\kappa^2h(\cosh 2\kappa h + 1)} \\ \theta = \cosh 2\kappa h \cdot \phi \end{cases}$$

Then

$$\left. \begin{aligned} u(x', y', z') &= \sum_{\kappa} \frac{dE_4}{dx'} \left\{ 2\kappa z' \sinh \kappa z' + (\cosh 2\kappa h + \alpha) \cosh \kappa z' \right\} \\ v(x', y', z') &= \sum_{\kappa} \frac{dE_4}{dy'} \left\{ 2\kappa z' \sinh \kappa z' + (\cosh 2\kappa h + \alpha) \cosh \kappa z' \right\} \\ w(x', y', z') &= \sum_{\kappa} \kappa E_4 \{ 2\kappa z' \cosh \kappa z' + (\cosh 2\kappa h - \alpha) \sinh \kappa z' \} \end{aligned} \right\} \quad (100)$$

where κ is a zero of $\sinh 2\kappa h + 2\kappa h$ with pos. imag. part, and $\frac{d^2 E_4}{dx'^2} + \frac{d^2 E_4}{dy'^2} + \kappa^2 E_4 = 0$.

The solutions are of the type $\phi = \cosh \kappa z' E_4(x', y')$, $\theta = \cosh 2\kappa h \cdot \phi$.

II. Flexural part of the solution.

(i) Permanent mode.

$$\text{Let } F_1 \equiv \text{work difference from the system } \begin{cases} \phi = -\frac{3}{32\pi\mu h^3} \left(z\chi - \frac{1}{6} z^3 \nabla^2 \chi \right) \\ \theta = \frac{3}{32\pi\mu h^3} \left(z\chi - \frac{1}{6} z^3 \nabla^2 \chi - 2h^2 z \nabla^2 \chi \right) \end{cases}$$

$$\text{Then } \nabla^2 F_1 = \text{work difference from } \phi = -\frac{3}{32\pi\mu h^3} z \nabla^2 \chi = -\theta,$$

$$\text{and } \left. \begin{aligned} u(x', y', z') &= -z' \frac{dF_1}{dx'} + \frac{2}{\alpha+1} \left(\frac{\alpha+5}{12} z'^3 - \frac{\alpha+11}{10} h^2 z' \right) \frac{d}{dx'} \nabla^2 F_1 \\ v(x', y', z') &= -z' \frac{dF_1}{dy'} + \frac{2}{\alpha+1} \left(\frac{\alpha+5}{12} z'^3 - \frac{\alpha+11}{10} h^2 z' \right) \frac{d}{dy'} \nabla^2 F_1 \\ w(x', y', z') &= F_1 + \left\{ \frac{1}{5} h^2 - \frac{1}{2} z'^2 + \frac{2}{\alpha+1} (z'^2 - h^2) \right\} \nabla^2 F_1 \end{aligned} \right\} \quad (101)$$

Here $\nabla^4 F_1 = 0$, and if we write $(\alpha + 1)F$ for $F_1 + \frac{1}{5}h' \nabla'^2 F_1$, these expressions reduce to the form which we have taken throughout for this kind of strain, namely,

$$\left. \begin{aligned} u(x', y', z') &= \frac{d}{dx'} \left\{ -(\alpha + 1)(z'F - \frac{1}{6}z'^3 \nabla'^2 F) + 2(\frac{1}{3}z'^3 - h^2 z') \nabla' F \right\} \\ v(x', y', z') &= \frac{d}{dy'} \left\{ -(\alpha + 1)(z'F - \frac{1}{6}z'^3 \nabla'^2 F) + 2(\frac{1}{3}z'^3 - h^2 z') \nabla' F \right\} \\ w(x', y', z') &= (\alpha + 1)(F - \frac{1}{2}z'^2 \nabla'^2 F) + 2(z'^2 - h^2) \nabla'^2 F \end{aligned} \right\} \quad (101')$$

where $\nabla'^4 F = 0$.

(ii) Transitory modes, ψ or rotational type.

Put $F_2 \equiv$ work difference from the system $\psi = -\frac{1}{8\pi\mu\kappa^2 h} \sinh \kappa z' G_0 \kappa R$.

$$\left. \begin{aligned} \text{Then } u(x', y', z') &= \sum_{\kappa} 2 \frac{dF_2}{dy'} \sinh \kappa z' \\ v(x', y', z') &= \sum_{\kappa} (-2) \frac{dF_2}{dx'} \sinh \kappa z' \\ w(x', y', z') &= 0 \end{aligned} \right\} \quad (102)$$

where κ is a pos. imag. root of $\cosh \kappa h$, and $\frac{d^2 F_2}{dx'^2} + \frac{d^2 F_2}{dy'^2} + \kappa^2 F_2 = 0$.

The solutions are of the type $\psi = \sinh \kappa z' F_2(x', y')$.

(iii) Transitory modes, θ - ϕ or dilatational type.

Put $F_3 \equiv$ work difference from the system $\begin{cases} \phi = \frac{\sinh \kappa z' G_0 \kappa R}{8\pi\mu(\alpha + 1)\kappa^2 h(\cosh 2\kappa h - 1)} \\ \theta = -\cosh 2\kappa h \cdot \phi \end{cases}$

$$\left. \begin{aligned} \text{Then } u(x', y', z') &= \sum_{\kappa} \frac{dF_3}{dx'} \left\{ 2\kappa z' \cosh \kappa z' + (\alpha - \cosh 2\kappa h) \sinh \kappa z' \right\} \\ v(x', y', z') &= \sum_{\kappa} \frac{dF_3}{dy'} \left\{ 2\kappa z' \cosh \kappa z' + (\alpha - \cosh 2\kappa h) \sinh \kappa z' \right\} \\ w(x', y', z') &= \sum_{\kappa} \kappa F_3 \left\{ 2\kappa z' \sinh \kappa z' - (\alpha + \cosh 2\kappa h) \cosh \kappa z' \right\} \end{aligned} \right\} \quad (103)$$

where κ is a zero of $\sinh 2\kappa h - 2\kappa h$ with pos. imag. part, and $\frac{d^2 F_3}{dx'^2} + \frac{d^2 F_3}{dy'^2} + \kappa^2 F_3 = 0$.

The solutions are of the type $\psi = \sinh \kappa z' F_3(x', y')$, $\theta = -\cosh 2\kappa h \cdot \phi$.

34. Form of the solution for edge tractions deduced by another method.

We have thus shown that the most general deformation of a finite plate under edge tractions only is compounded of the types specified in (98) . . . (103). The deformation is of the same form as that given by our infinite plate solutions for any part of the solid free from body force or surface traction, and it may be of advantage to show in a direct manner why this should be so.

Suppose, then, that we have given a displacement (u, v, w) of a finite plate bounded by an external edge S and one or more internal edges S' , the only applied forces being tractions on the edges. Imagine the plate continued inwards and outwards so as to

form a complete infinite plate. By the general existence theorem of the subject, there exist values of u, v, w in the space within an internal edge, continuous at the edge with the values of the displacements of the original solid, and produced by edge tractions alone. Similarly, if we take any surface S'' , within the infinite plate, but completely enclosing the edge S , there exist values of u, v, w continuous with the original u, v, w at the external edge, and becoming zero on S'' ; these also being produced by edge tractions only, namely, on S and S'' .

If, then, we take u, v, w to be zero outside S'' , we obtain altogether a system of displacements continuous throughout the infinite solid. The forces required to maintain this system are given directly by the general equations of equilibrium. These forces form areal distributions on S, S', S'' , and are measured by the discontinuity of stress at these surfaces. Further, on the whole they make up an equilibrating system. But we have shown in the preceding pages how to find a solution for such a system of force, this solution giving displacements of order $\log R$ at most, and stresses of order R^{-2} at most, at a great distance. Only one solution fulfilling these conditions being possible, our solution is *the* solution.

Hence, finally, any displacement of a finite plate under edge tractions only is of the same form as that given by our infinite solid solutions for a certain system of areal force, distributed partly over the edges, and partly over an arbitrary external surface. This is what we proposed to prove.

35. *General solution for an infinite solid under any forces.*

It is now easy to determine the most general form of displacement of an infinite solid, under null body force and face traction, and free from singularity at a finite distance. For if u, v, w be any such displacement, then within any surface S , however distant, we have proved that u, v, w are given by the absolutely convergent series (98) (103).

If we take a right circular cylinder for the surface S , the functions F which satisfy equations of the form $\frac{d^2 F}{dx^2} + \frac{r^2 F}{dy^2} + \kappa^2 F = 0$ can be expressed in series of the form

$$\sum_m J_m \kappa \rho (A_m \cos m\omega + B_m \sin m\omega),$$

and the only restriction on the coefficients A_m, B_m is that they must make the double series in which the complete solution is thus expressed absolutely convergent for all values of ρ , however great.

The most general solution for any system of force applied at a finite distance is of course obtained by adding to this complete free solution the particular solution already investigated. It may be observed that this final result might have been obtained in one step by the process of § 33, if in that article we had taken for u, v, w any displacements under given body force and surface traction, instead of under edge traction only. The identity of the results of the two methods will be seen to depend essentially on the

fact that in the solution for a single force in any direction, the component displacement in that direction is *symmetrical in the accented and unaccented coordinates*, a theorem analogous to a well-known property of Green's function in Potential Theory.

It is interesting to observe that, in the process suggested in the last sentence but one, we only need to know the comparatively simple source solution for a single Z force in order to deduce the w displacement for any system of forces and face tractions whatever.

36. *Application of Betti's Theorem to the problem of given edge tractions.*

In the remaining pages, we shall be occupied almost exclusively with deformations of a finite plate under edge tractions only. For brevity we may refer to such deformations as *free*.

The formulæ (97) express the internal displacements in terms of the edge displacements and edge tractions. We may indicate here the general lines along which we naturally proceed in the attempt to reduce these formulæ to expressions in terms of displacements alone or of tractions alone.

Taking the first equation of (97), for example, if we wish a formula containing edge displacements only, we look for free displacements in the form of functions u_1', v_1', w_1' of x, y, z , such that $u_1 + u_1', v_1 + v_1', w_1 + w_1'$ shall be equal to zero at the edge.

If X_1', Y_1', Z_1' be the edge tractions in the system u_1', v_1', w_1' , then by Betti's Theorem

$$\iint (Xu_1' + Yv_1' + Zw_1' - X_1'u - Y_1'v - Z_1'w) dS = 0,$$

and by addition of this equation to (97),

$$u(x', y', z') = - \iint \left\{ u(X_1 + X_1') + v(Y_1 + Y_1') + w(Z_1 + Z_1') \right\} dS.$$

The problem of arbitrary edge displacements is thus reduced to a problem in which these displacements have a comparatively simple form.

When we attempt to find a formula in terms of edge tractions only, the procedure is not quite so simple, in consequence of the fact that the tractions X_1, Y_1, Z_1 are not equilibrating, but equivalent to a negative unit X force through (x', y', z') . From various methods of meeting this difficulty we select the following as the most convenient in the present case.

We have seen in § 30 that the system u_1, v_1, w_1 can be decomposed into four systems. The first system, say U_1, V_1, W_1 conveys no resultant stress; the second system conveys a stress equivalent to a unit X force through the origin, and the displacements are independent of x', y', z' ; the third system conveys a couple z' in the plane zOx , the displacements contain z' as a factor, but are otherwise independent of x', y', z' ; the fourth system conveys a couple $-y'$ in the plane xOy , the displacements involving x', y', z' only in the form of the factor y' .

The displacements u_2, v_2, w_2 and u_3, v_3, w_3 are similarly decomposable into equilibrating systems U_2, V_2, W_2 and U_3, V_3, W_3 with other systems conveying resultant forces and couples. The contributions to $u(x', y', z'), v(x', y', z'), w(x', y', z')$ in (97) from the various systems conveying forces and couples amount on the whole merely to a rigid body displacement of the plate. If this be neglected, then the value of $u(x', y', z')$, for instance, becomes simply the work difference from the system U_1, V_1, W_1 , the edge tractions due to which are equilibrating, and can be balanced by a free system U_1', V_1', W_1' . We then obtain from (97)

$$u(x', y', z') = \iint \left\{ X(U_1 + U_1') + Y(V_1 + V_1') + Z(W_1 + W_1') \right\} dS$$

and similarly for v, w .

37. *Exact solutions of special problems for a circular plate.*

As already stated, we are not at present in a position to complete the solution of the problem of arbitrary edge tractions, even for the simplest form of edge. The method just indicated may be used, however, whatever be the form of the edge, to obtain approximately the boundary conditions which define the permanent part of the solution. But before entering on this important application, we shall consider a few special problems which admit of exact solution. All of these have reference to a plate bounded by a right circular cylinder, with or without a concentric circular aperture, and to systems of displacement symmetrical about the axis.

The radius of the external edge is a , of the internal edge b ; and the axis of z coincides with the axis of the cylinder. u, v, w are the displacements in the directions in which the coordinates ρ, ω, z increase.

Problem 1. Symmetrical transverse displacement.

The displacement v , in the most general case, is given by a series involving cosines and sines of multiples of ω . We can determine the symmetrical term of the series. This constitutes the whole solution when the plate is subjected only to symmetrical torsional force.

For a transverse force Ω_1 applied at the point (ρ_1, ω_1, z_1) we have seen in article 29 that the solution is

$$\left. \begin{aligned} \psi &= \frac{d\psi'}{d\rho_1} \\ \theta &= -\frac{1}{\rho_1} \frac{d\theta'}{d\omega_1} \\ \phi &= -\frac{1}{\rho_1} \frac{d\phi'}{d\omega_1} \end{aligned} \right\} \cdot \frac{\Omega_1}{4\pi\mu(\alpha+1)}$$

The solution for a constant linear distribution of transverse force on the circle $\rho = \rho_1, z = z_1$, of intensity $\Omega_1/2\pi\rho_1$ per unit length, is found by integrating this with respect to ω_1 from 0 to 2π , and dividing by 2π . The result of the integration is simply to

eliminate all but the symmetrical part of ψ , and to eliminate θ, ϕ altogether. The solution is therefore,

when $\rho > \rho_1$,

$$v = \left. \begin{aligned} & \frac{1}{8\pi\mu h} \frac{\rho_1}{\rho} \\ & + \frac{1}{2\pi\mu h} \sum_{\kappa} \cosh \kappa z_1 \cosh \kappa z J_0' \kappa \rho_1 G_0' \kappa \rho, \quad (\kappa \text{ a pos. imag. root of } \sinh \kappa h) \\ & - \frac{1}{2\pi\mu h} \sum_{\kappa} \sinh \kappa z_1 \sinh \kappa z J_0' \kappa \rho_1 G_0' \kappa \rho, \quad (\kappa \text{ a pos. imag. root of } \cosh \kappa h) \end{aligned} \right\} \quad (i)$$

when $\rho < \rho_1$, ρ and ρ_1 have simply to be interchanged.

The only stress across a cylinder ρ is

$$\widehat{\rho\omega} = \mu \left(\frac{dv}{d\rho} - \frac{v}{\rho} \right).$$

Hence if Ω_a, Ω_b be the transverse components of the traction on $\rho = a, \rho = b$, and v_1 the transverse displacement at (ρ_1, ω_1, z_1) produced by this traction, Betti's Theorem gives, as in (97), for the permanent part of v_1 ,

$$\frac{1}{2\pi} \int_0^{2\pi} v_1 d\omega_1 = \frac{1}{8\pi\mu h} \iint \left(\Omega_a \frac{\rho_1}{a} + v_a \frac{2\mu\rho_1}{a^2} \right) dS_a + \frac{1}{8\pi\mu h} \iint \Omega_b \frac{b}{\rho_1} dS_b.$$

Also, since the distributions Ω_a, Ω_b have equal moments about Oz ,

$$a \iint \Omega_a dS_a + b \iint \Omega_b dS_b = 0.$$

In the case of a symmetrical deformation, we have therefore

$$v_1 = \frac{b^2}{4\mu h} \left(\frac{1}{\rho_1} - \frac{\rho_1}{a^2} \right) \int_{-h}^h \Omega_b dz + \frac{\rho_1}{a} v_a.$$

The displacement $v_1 = \rho_1$ being merely a rigid body rotation about Oz , the permanent solution is practically

$$v_1 = \frac{b^2}{4\mu h \rho_1} \int_{-h}^h \Omega_b dz = - \frac{a^2}{4\mu h \rho_1} \int_{-h}^h \Omega_a dz \quad \dots \quad (ii)$$

This might have been got at once by omitting the term $v = \rho_1/8\pi\mu h\rho$ from the source solution, in accordance with the method explained at the end of the last article.

For a uniform system of couple about the edge-normal $\int \Omega_b dz$ vanishes, and the permanent displacement is rigorously null.

For the transitory part of the solution we will, to simplify the algebra, suppose the cylinder solid.

Further, this not being a case where the separation into odd and even parts in z is of much consequence, we may shorten the formulæ by combining the two κ series into one.

Thus $\cosh \kappa (h+z) \cosh \kappa (h+z_1)$ being obviously equal to $\cosh \kappa z \cosh \kappa z_1$ when

$\sinh \kappa h = 0$, but equal to $-\sinh \kappa z \sinh \kappa z_1$ when $\cosh \kappa h = 0$, we may transform (i) if we put $h + z = \zeta$, $h + z_1 = \zeta_1$, into

$$v = \frac{1}{8\pi\mu h} \frac{\rho_1}{\rho} + \frac{1}{2\pi\mu h} \sum_{\kappa} \cosh \kappa \zeta_1 \cosh \kappa \zeta J_0' \kappa \rho_1 G_0' \kappa \rho, \quad (\kappa \text{ a pos. imag. root of } \sinh 2\kappa h). \quad (\text{iii})$$

For

$$\rho = G_0' \kappa \rho, \text{ we have when } \rho = \alpha, \quad \widehat{\rho\omega} = -\frac{\mu}{\alpha} (2G_0' \kappa \alpha + \kappa \alpha G_0 \kappa \alpha).$$

Hence, when the transitory part of v in (iii) is balanced, the solution becomes ($\rho > \rho_1$)

$$v = \frac{1}{8\pi\mu h} \frac{\rho_1}{\rho} + \frac{1}{2\pi\mu h} \sum_{\kappa} \cosh \kappa \zeta_1 \cosh \kappa \zeta J_0' \kappa \rho_1 \left(G_0' \kappa \rho - \frac{2(G_0' \kappa \alpha + \kappa \alpha G_0 \kappa \alpha)}{2J_0' \kappa \alpha + \kappa \alpha J_0 \kappa \alpha} J_0' \kappa \rho \right) \quad (\text{iv})$$

This gives at $\rho = \alpha$,

$$v = \frac{1}{8\pi\mu h} \frac{\rho_1}{\alpha} + \frac{1}{2\pi\mu h} \sum_{\kappa} \cosh \kappa \zeta_1 \cosh \kappa \zeta J_0' \kappa \rho_1 \left(-\frac{1}{2J_0' \kappa \alpha + \kappa \alpha J_0 \kappa \alpha} \right)$$

Hence, for the free displacement at (ρ_1, z_1) under symmetrical transverse traction Ω_a on $\rho = \alpha$, Betti's Theorem gives (omitting the rigid body rotation)

$$v_1 = \frac{\alpha}{\mu h} \sum_{\kappa} J_0' \kappa \rho_1 \cosh \kappa \zeta_1 \left(-\frac{1}{2J_0' \kappa \alpha + \kappa \alpha J_0 \kappa \alpha} \right) \int_0^{2h} \cosh \kappa \zeta \Omega_a d\zeta \quad (\text{v})$$

From this

$$(\rho\omega)_1 = \frac{1}{h} \sum_{\kappa} \frac{2J_0' \kappa \rho + \kappa \rho J_0 \kappa \rho}{2J_0' \kappa \alpha + \kappa \alpha J_0 \kappa \alpha} \cosh \kappa \zeta_1 \int_0^{2h} \cosh \kappa \zeta \Omega_a d\zeta$$

The series passes continuously, as ρ increases to α , into the limit

$$\frac{1}{h} \sum_{\kappa} \cosh \kappa \zeta_1 \int_0^{2h} \cosh \kappa \zeta \Omega_a d\zeta, \text{ provided this latter series converges.}$$

By Fourier's Theorem we know that it does, namely, to the value Ω_a , it being noted that $\int_0^{2h} \Omega_a d\zeta = 0$. The solution is thus verified. Of course, it could easily be obtained by the Fourier method *ab initio*.

The series (iv) converges very rapidly unless ρ and ρ_1 are nearly equal. By an application of the Residue Calculus, it may be transformed into a series in which the functions of ρ, ρ_1 are the fluctuating functions, and the functions of ζ, ζ_1 the convergence factors. For consider the function of κ ,

$$\frac{1}{\pi\mu} \frac{\cosh \kappa(2h - \zeta) \cosh \kappa \zeta_1}{\sinh 2\kappa h} J_0' \kappa \rho_1 \left(G_0' \kappa \rho - \frac{2G_0' \kappa \alpha + \kappa \alpha G_0 \kappa \alpha}{2J_0' \kappa \alpha + \kappa \alpha J_0 \kappa \alpha} J_0' \kappa \rho \right).$$

It is easy to see that $\log \kappa$ disappears from the last factor, and that the whole function is a *uniform, odd* function of κ ; also that if $\zeta > \zeta_1, \rho > \rho_1$, the function vanishes at infinity in such a way as to make the total sum of its residues equal to zero.

The poles of the function are $\kappa = 0$, the (pure imaginary) zeroes of $\sinh 2\kappa h$, and the (real) zeroes of $2J_0' \kappa \alpha + \kappa \alpha J_0 \kappa \alpha$. The function being odd, we have (series of residues at pos. imag. roots of $\sinh 2\kappa h$) + (series of residues at pos. roots of $2J_0' \kappa \alpha + \kappa \alpha J_0 \kappa \alpha$) + $\frac{1}{2}$ residue at $(\kappa = 0)$, equal to zero.

The first series of residues is the function v of (iv). Thus we obtain, ($\zeta > \zeta_1$)

$$v = \frac{4}{\pi\mu a} \sum_{\kappa} \frac{\cosh \kappa(2h - \zeta) \cosh \kappa\zeta_1}{\sinh 2\kappa h} \frac{1}{\kappa^3 a^3 (J_0 \kappa a)^2} J_0' \kappa \rho_1 J_0' \kappa \rho - \frac{2}{\pi\mu a^4} \rho_1 \rho \zeta + \frac{\rho \rho_1}{3\pi\mu a^4 h} (a^2 + 4h^2) - \frac{\rho_1}{8\pi\mu a^4 h} (\rho^3 - 4\rho\zeta^2) - \frac{\rho}{8\pi\mu a^4 h} (\rho_1^3 - 4\rho_1\zeta_1^2) \Bigg\} . \quad (\text{vi})$$

When $\zeta > \zeta_1$, we have merely to interchange ζ and ζ_1 in this formula.

We may verify in a moment from this, as from the perfectly equivalent form (iv), that the internal couple is balanced by the stress at the cylindrical boundary, and that there is no stress across the plane ends. But if we remove the last three terms from (vi), we make no change in the internal singularity, these terms being the same whether ζ or ζ_1 be the greater.

We thus obtain the displacement when the internal couple is balanced at the plane ends, namely, ($\zeta > \zeta_1$)

$$v' = -\frac{2}{\pi\mu a^4} \rho_1 \rho \zeta + \frac{4}{\pi\mu a} \sum_{\kappa} \frac{\cosh \kappa(2h - \zeta) \cosh \kappa\zeta_1}{\sinh 2\kappa h} \frac{1}{\kappa^3 a^3 (J_0 \kappa a)^2} J_0' \kappa \rho_1 J_0' \kappa \rho . \quad (\text{vii})$$

Here, as in (vi), the summation extends over the positive roots of $2J_0' \kappa a + \kappa a J_0 \kappa a$. The solution for symmetric transverse traction Ω_{2h} , Ω_0 on the ends, which might be obtained in an abnormal form from (iv) with the cognate formula for $\rho > \rho_1$, is given in normal form by a direct application of Betti's Theorem to (vii).

Thus

$$v(\rho_1, \zeta_1) = \frac{8}{\mu a} \sum_{\kappa} \frac{\cosh \kappa\zeta_1 J_0' \kappa \rho_1}{\sinh 2\kappa h} \frac{1}{\kappa^3 a^3 (J_0 \kappa a)^2} \int_0^a \Omega_{2h} \rho J_0' \kappa \rho d\rho + \frac{8}{\mu a} \sum_{\kappa} \frac{\cosh \kappa(2h - \zeta_1) J_0' \kappa \rho_1}{\sinh 2\kappa h} \frac{1}{\kappa^3 a^3 (J_0 \kappa a)^2} \int_0^a \Omega_0 \rho J_0' \kappa \rho d\rho - \frac{4}{\mu a^4} \rho_1 \zeta_1 \int_0^a \Omega_0 \rho^2 d\rho \Bigg\} . \quad (\text{viii})$$

The result belongs rather to the theory of a long rod than to that of a thin plate. The permanent term depends only on the integral couple, and coincides with that given by SAINT VENANT's theory of Torsion.*

38. *Problem 2. Boundary values of the normal displacement u , and the shearing stress normal to the plate $\widehat{x\rho}$, are given functions symmetrical about the axis; the displacement v , or the shearing stress $\widehat{\rho\omega}$, vanishes.*

We begin with the case of a solid cylinder.

(i) *Permanent extensional mode.*

Referring to § 33, I. (i), we see that under the conditions proposed the function E_1 must vanish, and the solution in cylinders is

$$u(\rho_1, z_1) = \frac{dE_2}{d\rho_1} + \frac{3-a}{a+1} \left(\frac{1}{2} z_1^2 - \frac{1}{3} h^2 \right) \frac{d}{d\rho_1} \nabla_1^2 E_2$$

$$w(\rho_1, z_1) = \frac{a-3}{a+1} z_1 \nabla_1^2 E_2$$

* The writer hopes to publish shortly a solution of the problem of equilibrium of an infinite circular cylinder, in which the celebrated solutions of SAINT VENANT will appear as the leading terms. It will be shown that in a finite cylinder the permanent modes are given *exactly* by SAINT VENANT's theory. In the theory of thin plates, the permanent modes can only, in general, be found *approximately*.

where $E_2 = \frac{1}{32\pi\mu h}$. work diffce. from $\theta = \phi = \chi(\rho) - \frac{1}{2}z^2 \log \rho + \frac{1}{4}\rho_1^2 \log \rho$,

or $E_2 = \frac{\rho_1^2}{128\pi\mu h}$. work diffce. from $\theta = \phi = \log \rho$, the part omitted being merely a constant.

Now in the system $\theta = \phi = \log \rho$

$$\left. \begin{aligned} u &= (\alpha + 1)/\rho \\ w &= 0 \end{aligned} \right\} \quad \left. \begin{aligned} \widehat{\rho\rho} &= -2\mu(\alpha + 1)/\rho^2 \\ \widehat{\rho z} &= 0 \end{aligned} \right\}$$

Hence the solution

$$\left. \begin{aligned} u &= -(\alpha + 1)\rho/\alpha^2 \\ w &= -2(\alpha - 3)z/\alpha^2 \end{aligned} \right\} \quad \left. \begin{aligned} \widehat{\rho\rho} &= 2\mu(\alpha - 7)/\alpha^2 \\ \widehat{\rho z} &= 0 \end{aligned} \right\}$$

will, taken along with $\theta = \phi = \log \rho$, give u and $\widehat{\rho z} = 0$ at $\rho = \alpha$. The balanced solution gives $\widehat{\rho\rho} = -16\mu/\alpha^2$, $w = 2(3 - \alpha)z/\alpha^2$, at the edge.

Hence

$$E_2 = \frac{\rho_1^2}{16\alpha h} \int_{-\alpha}^h \left\{ \frac{Z_\alpha}{2\mu} (3 - \alpha)z + 4u_\alpha \right\} dz$$

and

$$\left. \begin{aligned} u(\rho_1, z_1) &= \rho_1 \\ w(\rho_1, z_1) &= 2\frac{\alpha - 3}{\alpha + 1}z_1 \end{aligned} \right\} \cdot \frac{1}{8\alpha h} \int_{-\alpha}^h \left\{ \frac{Z_\alpha}{2\mu} (3 - \alpha)z + 4u_\alpha \right\} dz.$$

(ii) *Transitory extensional modes.*

The solution is given by (100) with

$$E_4 = \frac{J_0 \kappa \rho_1}{8\pi\mu(\alpha + 1)\kappa^2 h (\cosh 2\kappa h + 1)} \cdot \text{work diffce. from the system } \begin{cases} \phi = G_0 \kappa \rho \cosh \kappa z \\ \theta = \cosh 2\kappa h \cdot \phi \end{cases}$$

In the system mentioned

$$\begin{aligned} u &= \kappa G_0' \kappa \rho \{ (\cosh 2\kappa h + \alpha) \cosh \kappa z + 2\kappa z \sinh \kappa z \} \\ w &= \kappa G_0 \kappa \rho \{ (\cosh 2\kappa h - \alpha) \sinh \kappa z + 2\kappa z \cosh \kappa z \} \\ \frac{1}{\kappa^2} \frac{\widehat{z\rho}}{2\mu} &= G_0' \kappa \rho \{ (\cosh 2\kappa h + 1) \sinh \kappa z + 2\kappa z \cosh \kappa z \} \\ \frac{1}{\kappa^2} \frac{\widehat{\rho\rho}}{2\mu} &= -G_0 \kappa \rho \{ (\cosh 2\kappa h + 3) \cosh \kappa z + 2\kappa z \sinh \kappa z \} \\ &\quad - \frac{1}{\kappa \rho} G_0' \kappa \rho \{ (\cosh 2\kappa h + \alpha) \cosh \kappa z + 2\kappa z \sinh \kappa z \}. \end{aligned}$$

The balancing system for u and $\widehat{z\rho}$ at $\rho = \alpha$ is therefore

$$\phi = -\frac{G_0' \kappa \alpha}{J_0' \kappa \alpha} J_0 \kappa \rho \cosh \kappa z, \quad \theta = \cosh 2\kappa h \phi.$$

In the balanced system, at the edge

$$\begin{aligned} w &= \frac{1}{\alpha J_0' \kappa \alpha} \{ (\cosh 2\kappa h - \alpha) \sinh \kappa z + 2\kappa z \cosh \kappa z \} \\ \frac{\widehat{\rho\rho}}{2\mu} &= -\frac{\kappa}{\alpha J_0' \kappa \alpha} \{ (\cosh 2\kappa h + 3) \cosh \kappa z + 2\kappa z \sinh \kappa z \} \end{aligned}$$

Hence for the free solution with edge values $u = u_\alpha$, $\widehat{z\rho} = Z_\alpha$,

$$E_4 = \frac{1}{2(\alpha + 1)\kappa^2 h (\cosh 2\kappa h + 1)} \frac{J_0 \kappa \rho_1}{J_0' \kappa \alpha} \int_{-\alpha}^h \left\{ \frac{Z_\alpha}{2\mu} (\cosh 2\kappa h - \alpha) \sinh \kappa z + 2\kappa z \cosh \kappa z \right. \\ \left. + \kappa u_\alpha (\cosh 2\kappa h + 3) \cosh \kappa z + 2\kappa z \sinh \kappa z \right\} dz$$

(iii) *Permanent flexural mode.*

The solution is given by (101) with

$$F_1 = \frac{3}{32\pi\mu h^3} \cdot \frac{1}{4}\rho_1 z^2 \quad \text{work diffce. from} \quad \theta = -\phi = z \log \rho.$$

In the system

$$\theta = -\phi = z \log \rho, \quad u = -(\alpha+1)z/\rho \quad \left| \begin{array}{l} \widehat{z\rho} = 0 \\ \widehat{w} = (\alpha+1) \log \rho \end{array} \right| \quad \left| \begin{array}{l} \widehat{z\rho} = 0 \\ \widehat{\rho\rho} = 2\mu(\alpha+1)z/\rho^2 \end{array} \right|$$

Hence the solution balancing u and $\widehat{z\rho}$ at $\rho = \alpha$ is

$$u = (\alpha+1)z\rho/a^2 \\ w = -(\alpha+1)\rho^2/2a^2 + (\alpha-3)z^2/a^2 \quad \left| \begin{array}{l} \widehat{z\rho} = 0 \\ \widehat{\rho\rho} = 2\mu(7-\alpha)z/a^2 \end{array} \right|$$

In the balanced solution, at the edge

$$w = (\alpha+1)(\log \alpha - \frac{1}{2}) + (\alpha-3)z^2/a^2; \quad \widehat{\rho\rho} = 16\mu z/a^2$$

The constant term in the value of w will disappear since $\int_{-h}^h \widehat{z\rho} dz = 0$. Thus

$$F_1 = \frac{3}{32ah^3} \rho_1^2 \int_{-h}^h \left\{ \frac{Z_a}{2\mu} (\alpha-3)z^2 - 8zu_a \right\} dz$$

and

$$u(\rho_1, z_1) = -z_1\rho_1 \\ w(\rho_1, z_1) = \frac{1}{2}\rho_1^2 + \frac{3-\alpha}{\alpha+1}z_1^2 \quad \left\{ \begin{array}{l} \frac{3}{16ah^3} \int_{-h}^h \left\{ \frac{Z_a}{2\mu} (\alpha-3)z^2 - 8zu_a \right\} dz. \end{array} \right.$$

(iv) *Transitory flexural modes.*

The solution is given by (103) with

$$F_3 = \frac{J_0\kappa\rho_1}{8\pi\mu(\alpha+1)\kappa^2h(\cosh 2\kappa h - 1)} \quad \text{work diffce. from the system} \quad \left\{ \begin{array}{l} \phi = G_0\kappa\rho \sinh \kappa z \\ \theta = -\cosh 2\kappa h \cdot \phi \end{array} \right.$$

In this system

$$u = \kappa G_0' \kappa\rho \{ (\alpha - \cosh 2\kappa h) \sinh \kappa z + 2\kappa z \cosh \kappa z \} \\ w = \kappa G_0 \kappa\rho \{ -(\alpha + \cosh 2\kappa h) \cosh \kappa z + 2\kappa z \sinh \kappa z \} \\ \frac{1}{\kappa^2} \frac{\widehat{z\rho}}{2\mu} = G_0' \kappa\rho \{ (1 - \cosh 2\kappa h) \cosh \kappa z + 2\kappa z \sinh \kappa z \} \\ \frac{1}{\kappa^2} \frac{\widehat{\rho\rho}}{2\mu} = G_0 \kappa\rho \{ (\cosh 2\kappa h - 3) \sinh \kappa z - 2\kappa z \cosh \kappa z \} \\ - \frac{1}{\kappa\rho} G_0' \kappa\rho \{ (\alpha - \cosh 2\kappa h) \sinh \kappa z + 2\kappa z \cosh \kappa z \}$$

The system balancing u and $\widehat{z\rho}$ at $\rho = \alpha$ is

$$\phi = -\frac{G_0'\kappa\alpha}{J_0'\kappa\alpha} J_0\kappa\rho \sinh \kappa z, \quad \theta = -\cosh 2\kappa h \cdot \phi.$$

In the balanced system, at the edge

$$w = \frac{1}{\alpha J_0'\kappa\alpha} \left\{ -(\alpha + \cosh 2\kappa h) \cosh \kappa z + 2\kappa z \sinh \kappa z \right\} \\ \frac{\widehat{\rho\rho}}{2\mu} = \frac{\kappa}{\alpha J_0'\kappa\alpha} \left\{ (\cosh 2\kappa h - 3) \sinh \kappa z - 2\kappa z \cosh \kappa z \right\}$$

Hence for the free solution with edge values $u = u_a$, $\widehat{z\rho} = Z_a$,

$$F_3 = \frac{1}{2(\alpha+1)\kappa^2h(\cosh 2\kappa h - 1)} \frac{J_0\kappa\rho_1}{J_0'\kappa\alpha} \int_{-h}^h \left\{ \frac{Z_a}{2\mu} (-\alpha + \cosh 2\kappa h) \cosh \kappa z + 2\kappa z \sinh \kappa z \right\} dz.$$

If the given values of Z_a , u_a are the same as the edge values of $\widehat{z\rho}$, u , in one of the particular solutions, then clearly this particular solution by itself is *the* solution, and the

integrals which define the coefficients of the other particular solutions must vanish, while the integral corresponding to the solution left has its value determined. These results are easily verified by actual integration.

This remark may be used to find the solution for a hollow cylinder, which of course might also be obtained directly by the above process. We shall illustrate the method by finding the value of F_3 corresponding to any given root κ of $\sinh 2\kappa h - 2\kappa h$, when we have given u_a, u_b, Z_a, Z_b .

This value of F_3 we know is of the form $AJ_0\kappa\rho_1 + BG_0\kappa\rho_1$.

The complete values of u and of $\widehat{z}_\rho/2\mu$ for $\rho = \rho_1$ are given by series which manifestly converge uniformly so long as $b < \rho_1 < a$.

Multiply the series for $\widehat{z}_\rho/2\mu$ by $-(a + \cosh 2\kappa h) \cosh \kappa z + 2\kappa z \sinh \kappa z$,

the series for u by $-\kappa(3 - \cosh 2\kappa h \sinh \kappa z + 2\kappa z \cosh \kappa z)$,

add, and integrate with respect to z from $-h$ to h . All the terms disappear except that associated with the given root κ . We thus find

$$\begin{aligned} & (AJ_0'\kappa\rho_1 + BG_0'\kappa\rho_1)2(a+1)\kappa^2h(\cosh 2\kappa h - 1) \\ &= \int_{-h}^h \left\{ \frac{\widehat{z}_\rho}{2\mu}(\rho = \rho_1) \left(-a + \cosh 2\kappa h \right) \cosh \kappa z + 2\kappa z \sinh \kappa z \right. \\ & \quad \left. + \kappa u(\rho = \rho_1) (3 - \cosh 2\kappa h \sinh \kappa z + 2\kappa z \cosh \kappa z) \right\} dz. \end{aligned}$$

This is proved for the case $b < \rho_1 < a$. Now take the limits of both sides for $\rho_1 = a$. The limit of the integral is found simply by replacing \widehat{z}_ρ and $u(\rho = \rho_1)$ in the integrand by \widehat{z}_ρ and $u(\rho = a)$, provided the resulting integral has a meaning, which will be the case if \widehat{z}_ρ and $u(\rho = a)$ are integrable functions of z . Similarly we may take the limit for $\rho_1 = b$, and thus obtain two equations to determine A and B.

It will be observed that by this method we avoid two difficulties which in problems of this kind are often introduced unnecessarily by physical writers, namely, (i) the difficulty as to the convergence of the series for \widehat{z}_ρ and u , when the value $\rho_1 = a$ or b is substituted term by term, and (ii) the allied difficulty as to the continuity of the series right up to $\rho = a$ or b , even when it is known to converge. Judging from analogy, we may feel reasonably certain that the series will in fact converge at the limits, at least in the case of ordinary functions; but it is worth while noting that, whether they converge or not, the Fourier method of assuming the continuity and convergence, and determining the coefficients by integration, does give the correct values of these coefficients.

On the other hand, while our 'Green's function' method proves definitely that any possible solution has the form given above, it does not prove that a solution is possible for arbitrary edge values of \widehat{z}_ρ and u . The investigation might be completed by verifying that the solution obtained does actually satisfy the conditions, which would not be difficult in the present case. Alternatively, we may rely upon physical considerations, or upon a general analytical existence theorem. The proofs of theorems of this type in other branches of physical mathematics have been considerably improved within recent years by Poincaré and others, and their methods are equally applicable to the elastic equations.

39. *Problem 3.* To determine the permanent modes, having given the symmetrical edge tractions \widehat{z}_ρ and $\widehat{\rho\rho}$.

Supposing the cylinder solid, we have only to make a slight modification in the process of (i), (iii) in last article.

Extensional mode.

In (i) the balancing solution must now be taken as

$$u = \frac{(a+1)^2}{7-a} \frac{\rho}{a^2}, \quad w = -\frac{2(3-a)(a+1)}{7-a} \frac{z}{a^2}.$$

This, along with $\theta = \phi = \log \rho$, gives $\widehat{\rho\rho}$ and $\widehat{z\rho} = 0$ at $\rho = a$.

The balanced solution gives at the edge

$$u = \frac{8}{a} \frac{a+1}{7-a}, \quad w = -\frac{2(3-a)(a+1)}{7-a} \frac{z}{a}.$$

Hence

$$\left. \begin{aligned} u(\rho_1, z_1) &= \rho_1 \\ w(\rho_1, z_1) &= 2 \frac{a-3}{a+1} z_1 \end{aligned} \right\} \frac{a+1}{7-a} \frac{1}{16\mu ah} \int_{-h}^h (4aP_a + a-3 z Z_a) dz.$$

This gives the permanent mode *exactly*. The ordinary approximate theory omits the term in Z_a from the integral.

In CHREE'S solution of the problem of a rotating disc, the stress $\widehat{z\rho}$ vanishes identically at the edge, while $\int_{-h}^h P_a dz = 0$. His solution is therefore *exact*, so far as the fundamental mode is concerned.

As in last article, we infer from the form of the above solution that for any other than the permanent mode $\int_{-h}^h (4\rho \widehat{\rho\rho} + a-3 z \widehat{z\rho}) dz = 0$.

This may be verified by actual integration. Further, in the case of a hollow cylinder the solution is of the form

$$\left. \begin{aligned} u(\rho_1, z_1) &= A\rho_1 + B\rho_1 \\ w(\rho_1, z_1) &= 2A(a-3)z_1/(a+1) \end{aligned} \right\}$$

and the coefficients A, B are found from the conditions that

$$\int_{-h}^h (4\rho \widehat{\rho\rho} + a-3 z \widehat{z\rho}) dz \text{ must, for } \rho = a \text{ and } \rho = b$$

have the same value for the assumed form and for the given tractions.

Flexural mode.

Referring to (iii) of last article, the solution balancing

$\theta = \phi = z \log \rho$ at $\rho = a$ is now

$$u = -\frac{(a+1)^2}{7-a} \frac{z\rho}{a^2}; \quad w = \frac{(a+1)^2}{7-a} \frac{\rho^2}{2a^2} + \frac{(3-a)(a+1)}{7-a} \frac{z^2}{a^2}.$$

The balanced solution gives at the edge

$$u = -8 \frac{a+1}{7-a} \frac{z}{a}; \quad w = \text{const} + \frac{(3-a)(a+1)}{7-a} \frac{z^2}{a^2}.$$

Hence

$$\begin{aligned} u(\rho_1, z_1) &= -z_1 \rho_1 \\ v(\rho_1, z_1) &= \frac{1}{2} \rho_1^2 + \frac{3-a}{a+1} z_1^2 \left\{ \frac{3}{32\mu a h^3} \frac{a+1}{7-a} \int_{-h}^h (-8azP_a + 3 - a z^2 H_4) dz \right\}. \end{aligned}$$

The ordinary approximate theory takes account of the term in zP_a only.

The solution for a hollow cylinder can be obtained by this method, or by taking, in the notation of (94),

$$F = A\rho_1^2 + B \log \rho_1 + C \left(\frac{1}{4}\rho_1^2 \log \rho_1 - \frac{1}{4}\rho_1^2 \right),$$

determining C from the value of $\int_{-h}^h \widehat{z\rho} dz$ at either edge, and then determining A and B from the conditions that

$$\int_{-h}^h (-8\rho z \widehat{\rho\rho} + 3 - a z^2 \widehat{z\rho}) dz \text{ must for } \rho = a \text{ or } \rho = b$$

have the same value in the assumed form as in the actual displacement.

40. *Expansions of arbitrary functions.*

When we attempt to apply the method of last article to the determination of the modes corresponding to the various roots of $\sinh 2\kappa h \pm 2\kappa h$, we are at once confronted with an apparently insuperable difficulty. The determination of any one mode is reduced by the application of Betti's Theorem to the special problem of balancing the particular source solution involving a given root κ . Now in similar investigations connected with Laplace's equation, the equation of conduction of heat, and other partial differential equations of the second order which occur in physical mathematics, the analogous balancing problem can be solved without difficulty for certain simple forms of edge, and the balancing solution is of the same type as the particular source solution, that is, involves only the same root κ . In the present problem, however, the balancing solution will in general involve particular solutions of all types, as will be seen below.

Various theorems relating to the expansion of arbitrary functions may be found, similar to the theorem suggested at the end of § 38, but these do not help us, at all events immediately, to the general solution sought. One way of obtaining these expansion theorems may be indicated here; the method is of very wide application.

On a circular cylinder $\rho = a$, within the infinite plate, let areal force be distributed, the components of its intensity per unit area being $P \cos m\omega$, $\Omega \sin m\omega$, $Z \cos m\omega$, where P , Ω , Z are functions of z .

The infinite plate solution for this distribution of force can be written down, and the components of stress $\widehat{\rho\rho}$, $\widehat{\rho\omega}$, $\widehat{\rho z}$, calculated. These are given in different analytical forms according as ρ is greater or less than a . The expansion theorems are derived from the conditions of equilibrium

$$\lim_{\epsilon=0} \left\{ \widehat{\rho\rho}(\rho = a - \epsilon) - \widehat{\rho\rho}(\rho = a + \epsilon) \right\} = P$$

with two similar equations for Ω , Z .

There are other expansions which we know must exist, but the coefficients of which we cannot determine. The following examples are of special importance, and will be found useful immediately. They refer to the case of a plate bounded by a single infinite edge, filling, say, the region $x > 0$, and the displacements considered are such that u, w are functions of x, z only, while v vanishes.

Extensional modes.

The permanent mode is of the form $u = x, w = \frac{a-3}{a+1}z$.

In a transitory mode with $\phi = e^{i\kappa x} \cosh \kappa z, \theta = \cosh 2\kappa h \cdot \phi$, we have

$$\begin{aligned}\frac{\widehat{u_x}}{2\mu i\kappa^2} &= ie^{i\kappa x}(\overline{\cosh 2\kappa h + 3} \cosh \kappa z + 2\kappa z \sinh \kappa z) \\ \frac{\widehat{w_z}}{2\mu i\kappa^2} &= e^{i\kappa x}(\cosh 2\kappa h + 1 \sinh \kappa z + 2\kappa z \cosh \kappa z)\end{aligned}$$

As a special case of the results of last article, it follows that the coefficient of the permanent mode is determined from the given value of $\int_{-h}^h \widehat{u_x} dz$ at the edge; and this integral normal stress is zero for each of the transitory modes.

Hence if $P(z)$ be any even function of z , with $\int_{-h}^h P(z) dz = 0$, and $Z(z)$ be any odd function of z , coefficients C_κ exist such that at the same time

$$\text{and} \quad \left. \begin{aligned} \text{Lt}_{x=0} \sum_{\kappa} C_{\kappa} ie^{i\kappa x} (\overline{\cosh 2\kappa h + 3} \cosh \kappa z + 2\kappa z \sinh \kappa z) &= P(z) \\ \text{Lt}_{x=0} \sum_{\kappa} C_{\kappa} e^{i\kappa x} (\cosh 2\kappa h + 1 \sinh \kappa z + 2\kappa z \cosh \kappa z) &= Z(z) \end{aligned} \right\} \quad (i)$$

Flexural modes.

In the permanent mode F of equation (95) is of the form $Ax^2 + Bx^3$. A and B are found from the edge values of $\int_{-h}^h z \cdot \widehat{u_x} dz$ and $\int_{-h}^h \widehat{w_z} dz$. These integrals vanish for a transitory mode, in which, with $\phi = e^{i\kappa x} \sinh \kappa z, \theta = -\cosh 2\kappa h \cdot \phi$,

$$\begin{aligned}\frac{\widehat{u_x}}{2\mu i\kappa^2} &= ie^{i\kappa x}(3 - \cosh 2\kappa h \sinh \kappa z + 2\kappa z \cosh \kappa z) \\ \frac{\widehat{w_z}}{2\mu i\kappa^2} &= e^{i\kappa x}(1 - \cosh 2\kappa h \cosh \kappa z + 2\kappa z \sinh \kappa z).\end{aligned}$$

We infer that if $P(z)$ be any odd function of z , with $\int_{-h}^h z P(z) dz = 0$, and $Z(z)$ any even function of z , with $\int_{-h}^h Z(z) dz = 0$, values of C_κ exist such that simultaneously

$$\left. \begin{aligned} \text{Lt}_{x=0} \sum_{\kappa} C_{\kappa} ie^{i\kappa x} (3 - \cosh 2\kappa h \sinh \kappa z + 2\kappa z \cosh \kappa z) &= P(z) \\ \text{Lt}_{x=0} \sum_{\kappa} C_{\kappa} e^{i\kappa x} (1 - \cosh 2\kappa h \cosh \kappa z + 2\kappa z \sinh \kappa z) &= Z(z) \end{aligned} \right\} \quad (ii)$$

The limit for $x=0$ may be taken term by term, provided the resulting series converge. In the following analysis we shall assume that they do so, but this is merely in order to avoid lengthy forms of statement; the argument could be put, if necessary, in a form independent of this assumption.

41. *The problem of given edge tractions for a thin plate.*

The *form* of the complete solution is exactly known, and the three boundary conditions in their exact forms could, therefore, at once be written down. The whole strain is compounded of an infinite number of modes of equilibrium of known types, and it is obviously suggested as the method of attack that we should try to disentangle from the general boundary conditions those special conditions by which each mode is separately defined. When the plate is thin we find that within certain limits this can be done, and, in particular, the conditions defining the permanent modes, which in the case supposed are incomparably the most important, can be found with considerable exactness.

We shall understand that the edge traction, or any component of the edge traction, is given as a function of $x, y, z/h$ or of $s, z/h$, where s is the arc of the edge line, so that if ζ be put for z/h the form of this function is completely independent of h . The theory may be applied to cases in which the proviso is not fulfilled, but before such application the given traction is to be separated into parts of ascending order in h , say, for example, $f_0(x, y, \zeta) + h f_1(x, y, \zeta) + h^2 f_2(x, y, \zeta) + \text{etc.}$; then for a first approximation we deal only with $f_0(x, y, \zeta)$. The theory does *not* contemplate such a distribution of traction, as, for example, $\sin(ms/h)$, m being a number, where the rate at which the traction varies along the arc is of a lower order in h than the traction itself.

The trace of the cylindrical edge on the middle plane of the plate is the *edge line*; the outward normal, and the tangent, to the edge line will be referred to as the *normal*, and *tangent* simply; the generator of the cylindrical edge at right angles to these at their point of intersection may be called the *perpendicular*.

Let l, m be the direction cosines of the normal,
then $-m, l$ are those of the tangent.

The normal displacement is $p = lu + mv$
and the tangential displacement $q = -mu + lv$.

The tractions on the edge in the directions of normal, tangent, and perpendicular, are $\widehat{nn}, \widehat{ns}, \widehat{nz}$ or N, S, Z .

42. *Extensional strain.*

In this case N, S are even functions, and Z an odd function of z .

It will be advantageous to express as far as possible the displacements and tractions at an edge in the various types of solution in terms of derivatives along the tangent and normal.

Alongside the symbol α we shall use the more familiar σ , the relation between the two being given by $\alpha + 1 = 4(1 - \sigma)$; $3 - \alpha = 4\sigma$.

(i) *Permanent mode.*

$$\left. \begin{aligned} u &= U + \frac{\sigma}{1-\sigma} \frac{z}{2} \frac{d}{dx} \left(\frac{dU}{dx} + \frac{dV}{dy} \right) \\ v &= V + \frac{\sigma}{1-\sigma} \frac{z}{2} \frac{d}{dy} \left(\frac{dU}{dx} + \frac{dV}{dy} \right) \\ w &= -\frac{\sigma}{1-\sigma} \frac{z}{2} \left(\frac{dU}{dx} + \frac{dV}{dy} \right) \end{aligned} \right\}$$

where

$$\left. \begin{aligned} 2 \frac{d}{dx} \left(\frac{dU}{dx} + \frac{dV}{dy} \right) + (1-\sigma) \frac{d}{dy} \left(\frac{dU}{dy} - \frac{dV}{dx} \right) &= 0 \\ 2 \frac{d}{dy} \left(\frac{dU}{dx} + \frac{dV}{dy} \right) - (1-\sigma) \frac{d}{dx} \left(\frac{dU}{dy} - \frac{dV}{dx} \right) &= 0 \end{aligned} \right\}$$

Put

$$\left. \begin{aligned} \Delta &= \frac{1}{1-\sigma} \left(\frac{dU}{dx} + \frac{dV}{dy} \right) = \frac{1}{1-\sigma} \left(\frac{du}{dx} + \frac{dv}{dy} \right) \\ \Pi &= \frac{1}{2} \left(\frac{dU}{dy} - \frac{dV}{dx} \right) = \frac{1}{2} \left(\frac{du}{dy} - \frac{dv}{dx} \right) \end{aligned} \right\}$$

Then

$$\left. \begin{aligned} \frac{d\Delta}{dx} + \frac{d\Pi}{dy} &= 0 \\ \frac{d\Delta}{dy} - \frac{d\Pi}{dx} &= 0 \end{aligned} \right\} \text{ and at an edge } \left. \begin{aligned} \frac{d\Delta}{dn} + \frac{d\Pi}{ds} &= 0 \\ \frac{d\Delta}{ds} - \frac{d\Pi}{dn} &= 0 \end{aligned} \right\}$$

Also

$$\left. \begin{aligned} \widehat{xx} &= \frac{2\mu}{1-\sigma} \left(\frac{du}{dx} + \sigma \frac{dv}{dy} \right) \\ \widehat{yy} &= \frac{2\mu}{1-\sigma} \left(\sigma \frac{du}{dx} + \frac{dv}{dy} \right) \\ \widehat{xy} &= \frac{2\mu}{1-\sigma} \cdot \frac{1-\sigma}{2} \left(\frac{du}{dy} + \frac{dv}{dx} \right) \end{aligned} \right\}$$

The components of traction parallel to the axes are

$$X = l\widehat{xx} + m\widehat{xy} : Y = l\widehat{xy} + m\widehat{yy}.$$

These are easily transformed into

$$X = 2\mu \left(l\Delta + m\Pi - \frac{dv}{ds} \right); \quad Y = 2\mu \left(-l\Pi + m\Delta + \frac{du}{ds} \right).$$

Hence

$$\left. \begin{aligned} \widehat{nn} &= lX + mY = 2\mu \left(\Delta + m \frac{du}{ds} - l \frac{dv}{ds} \right) \\ \widehat{ns} &= -mX + lY = 2\mu \left(-\Pi + l \frac{du}{ds} + m \frac{dv}{ds} \right) \\ \widehat{nz} &= 0 \end{aligned} \right\}$$

also

or

$$\left. \begin{aligned} \widehat{nn} &= 2\mu \left(\Delta - \frac{dq}{ds} - \frac{p}{\rho} \right) \\ \widehat{ns} &= 2\mu \left(-\Pi + \frac{dp}{ds} - \frac{q}{\rho} \right) \end{aligned} \right\}$$

If P, Q be the values of p, q when $z=0$, then

$$p = P - \frac{1}{2} \sigma z \frac{d\Pi}{ds}; \quad q = Q + \frac{1}{2} \sigma z \frac{d\Delta}{ds}$$

$$\left. \begin{aligned} \frac{\widehat{nn}}{2\mu} &= \Delta - \frac{dQ}{ds} - \frac{P}{\rho} - \frac{1}{2} \sigma z \left(\frac{d^2\Delta}{ds^2} - \frac{1}{\rho} \frac{d\Pi}{ds} \right) \\ \frac{\widehat{ns}}{2\mu} &= -\Pi + \frac{dP}{ds} - \frac{Q}{\rho} - \frac{1}{2} \sigma z \left(\frac{d^2\Pi}{ds^2} + \frac{1}{\rho} \frac{d\Delta}{ds} \right) \end{aligned} \right\}$$

(ii) *Rotational Transitory Modes.*

$$\left. \begin{aligned} u &= 2 \frac{d\psi}{dy} \\ v &= -2 \frac{d\psi}{dx} \end{aligned} \right\} \quad \left. \begin{aligned} \frac{\widehat{xx}}{2\mu} &= 2 \frac{d^2\psi}{dx dy} \\ \frac{\widehat{xy}}{2\mu} &= \frac{d^2\psi}{dy^2} - \frac{d^2\psi}{dx^2} \\ \frac{\widehat{xz}}{2\mu} &= \frac{d^2\psi}{dy dz} \end{aligned} \right\}$$

Taking the axes of x and y for a moment along the normal and tangent, these give at once by means of (k)

$$\left. \begin{aligned} p &= 2 \frac{d\psi}{ds} \\ q &= -2 \frac{d\psi}{dn} \end{aligned} \right\} \quad \left. \begin{aligned} \frac{\widehat{nn}}{2\mu} &= 2 \frac{d}{ds} \frac{d\psi}{dn} - \frac{2}{\rho} \frac{d\psi}{ds} \\ \frac{\widehat{ns}}{2\mu} &= \frac{d^2\psi}{dz^2} + \frac{2}{\rho} \frac{d\psi}{dn} + 2 \frac{d^2\psi}{ds^2} \\ \frac{\widehat{nz}}{2\mu} &= \frac{d^2\psi}{ds dz} \end{aligned} \right\}$$

The function ψ can be expressed as a series of terms of the form

$$\psi_n(x, y) \cos \frac{n\pi z}{h}, \quad \text{where } \nabla^2 \psi_n - \frac{n^2 \pi^2}{h^2} \psi_n = 0.$$

Hence in cases where the values of ψ along an edge are given independently of h , or generally, when the rate of variation of ψ along an edge is of the same order in h as ψ itself, say order zero, terms of various orders occur in the expressions for the displacements and tractions.

Thus

$$\left. \begin{aligned} \frac{d\psi}{ds}, \frac{d^2\psi}{ds^2} & \text{ are of order } \dots 0 \\ \frac{d\psi}{dn}, \frac{d}{ds} \frac{d\psi}{dn}, \frac{d}{ds} \frac{d\psi}{dz} & \dots \dots \dots -1 \\ \frac{d^2\psi}{dz^2} & \text{ is } \dots \dots -2 \end{aligned} \right\}$$

It follows that in such cases this type of strain contributes mainly to the *tangential* displacement and traction at an edge.

We see also that the principal part of the displacement is of one order higher in h than the principal part of the traction.

(iii) *Dilatational transitory modes.*

There would be some advantage in working with the functional symbols θ , ϕ , as with ψ in the last case, but on the whole it seems clearer to deal with a typical solution corresponding to a single root κ of $\sinh 2\kappa h + 2\kappa h$.

$$\begin{aligned} \phi &= \cosh \kappa z f(x, y); \quad \theta = \cosh 2\kappa h \cdot \phi \\ u &= \frac{df}{dx} \left\{ (\cosh 2\kappa h + \alpha) \cosh \kappa z + 2\kappa z \sinh \kappa z \right\} \\ v &= \frac{df}{dy} \left\{ (\cosh 2\kappa h + \alpha) \cosh \kappa z + 2\kappa z \sinh \kappa z \right\} \\ w &= \kappa f \left\{ (\cosh 2\kappa h - \alpha) \sinh \kappa z + 2\kappa z \cosh \kappa z \right\} \\ \frac{xx}{2\mu} &= -\kappa^2 f \left\{ (\cosh 2\kappa h + 3) \cosh \kappa z + 2\kappa z \sinh \kappa z \right\} \\ &\quad - \frac{d^2 f}{dy^2} \left\{ (\cosh 2\kappa h + \alpha) \cosh \kappa z + 2\kappa z \sinh \kappa z \right\} \\ \frac{xy}{2\mu} &= \frac{d^2 f}{dx dy} \left\{ (\cosh 2\kappa h + \alpha) \cosh \kappa z + 2\kappa z \sinh \kappa z \right\} \\ \frac{xz}{2\mu} &= \kappa \frac{df}{dx} \left\{ (1 + \cosh 2\kappa h) \sinh \kappa z + 2\kappa z \cosh \kappa z \right\} \end{aligned}$$

Hence

$$\begin{aligned} \frac{\widehat{nn}}{2\mu} &= -\kappa^2 f \left\{ (\cosh 2\kappa h + 3) \cosh \kappa z + 2\kappa z \sinh \kappa z \right\} \\ &\quad - \left(\frac{1}{\rho} \frac{df}{dn} + \frac{d^2 f}{ds^2} \right) \left\{ (\cosh 2\kappa h + \alpha) \cosh \kappa z + 2\kappa z \sinh \kappa z \right\} \\ \frac{\widehat{ns}}{2\mu} &= \left(\frac{d}{ds} \frac{df}{dn} - \frac{1}{\rho} \frac{df}{ds} \right) \left\{ (\cosh 2\kappa h + \alpha) \cosh \kappa z + 2\kappa z \sinh \kappa z \right\} \\ \frac{\widehat{nz}}{2\mu} &= \kappa \frac{df}{dn} \left\{ (1 + \cosh 2\kappa h) \sinh \kappa z + 2\kappa z \cosh \kappa z \right\} \end{aligned}$$

Thus at an edge where the rate of variation of f is of the same order in h as f itself, say order zero, the *normal* and *perpendicular* displacements are of order -1 , while the tangential displacement is of an order one higher; the *normal* and *perpendicular* tractions are of order -2 , the tangential traction being of order -1 , or again one higher.

Hence this type of strain contributes most to those components of displacement and traction to which the ψ type contributes least, at an edge.

(iv) It is now possible to assign approximately to each of the three types of strain the portion which it carries of any given distribution of edge traction. Let this distribution be N, S, Z , functions of z, s , of order zero in h . We can satisfy the conditions to the first order by a solution in which the *principal* part of the traction

is of order zero for each type of strain. For, taking account only of these principal parts, the equations to be satisfied at the edge are on this supposition

$$\left. \begin{aligned} N &= N_p + N_d \\ S &= S_p + S_r \\ Z &= Z_d \end{aligned} \right\}$$

the suffixes referring to the permanent, rotational, and dilatational types respectively. Now

$$S_r = 2\mu \frac{d^2\psi}{dz^2} \quad \text{and} \quad \int_{-h}^h S_r dz = 2\mu \left[\frac{d\psi}{dz} \right]_{-h}^h = 0.$$

Also

$$N_d = -2\mu \sum_{\kappa} \kappa^2 f_{\kappa} \left\{ (\cosh 2\kappa h + 3) \cosh \kappa z + 2\kappa z \sinh \kappa z \right\},$$

and, as in art. 40,

$$\int_{-h}^h N_d dz = 0.$$

Hence the above equations give

$$\left. \begin{aligned} N_p &= \frac{1}{2h} \int_{-h}^h N dz \\ S_p &= \frac{1}{2h} \int_{-h}^h S dz \end{aligned} \right\}$$

and these conditions determine the permanent mode.

ψ can now be found from the boundary condition

$$2\mu \frac{d^2\psi}{dz^2} = S - \frac{1}{2h} \int_{-h}^h S dz.$$

For, taking

$$\psi = \sum_{n=1} \psi_n(x, y) \cos \frac{n\pi z}{h},$$

the condition is

$$\sum_{n=1} \psi_n \cdot n^2 \cos \frac{n\pi z}{h} = -\frac{h^2}{2\mu\pi^2} \left\{ S - \frac{1}{2h} \int_{-h}^h S dz \right\}.$$

Now the right-hand member here is a function of s, z , even in z , the z -integral of which from $-h$ to h is zero for all values of s . It can therefore be expanded by Fourier's Theorem in the form, valid from $z = -h$ to $z = h$,

$$\sum_{n=1} A_n \cos \frac{n\pi z}{h}.$$

ψ_n is then determined as satisfying $\nabla^2 \psi_n + \frac{n^2 \pi^2}{h^2} \psi_n = 0$ throughout the plate, and taking the value A_n/μ^2 at the boundary.

Lastly, the equations to determine the dilatational mode are (since at the edge $\frac{df}{dn} = -i\kappa f$ to the first order),

$$\left. \begin{aligned} \sum_{\kappa} f_{\kappa} \cdot \kappa^2 \left\{ (\cosh 2\kappa h + 3) \cosh \kappa z + 2\kappa z \sinh \kappa z \right\} &= -\frac{1}{2\mu} \left\{ N - \frac{1}{2h} \int_{-h}^h N dz \right\} \\ \sum_{\kappa} f_{\kappa} \cdot i\kappa^2 \left\{ (1 + \cosh 2\kappa h) \sinh \kappa z + 2\kappa z \cosh \kappa z \right\} &= -\frac{1}{2\mu} Z \end{aligned} \right\}$$

By the same method as in the case of ψ , but using the theorem (i) of § 40, instead of Fourier's Theorem, we see that functions $f_{\kappa}(x, y)$ exist, solutions of $(\nabla^2 + \kappa^2)f = 0$ and satisfying the above boundary equations.

Thus the apportionment proposed for the edge tractions does actually satisfy the conditions to a first approximation. The solution found gives tractions of which the principal parts are the tractions actually assigned in the problem. The residual traction given by the solution is of the first and higher orders; and a second approximation to the problem will be obtained by subtracting a solution giving the residual tractions of the first order, such solution being found by the method used in the first approximation.

This process would be tedious, and the way would be blocked at an early stage by our ignorance of the coefficients of the expansion (i) of § 40.

We therefore pass at once to the consideration of the powerful method furnished by Betti's Theorem for the determination of the permanent mode.

43. *Extensional strain. The Green's Function method for the permanent mode.*

The method has already been explained (§ 36). If we wish the permanent displacement at (x', y', z') in any direction (say the displacement u), we take the permanent part of the solution for a unit force in that direction (a unit X force), modify it by removing the terms which convey resultant stress, and then try to *balance* it at the edge by adding a solution, without internal singularity, which shall neutralise its edge tractions.

The displacements at the edge in the balanced solution, *i.e.* in the solution obtained as the sum of the source and balancing solutions, being u', v', w' , or p', q', w' , and the given tractions X, Y, Z, or N, S, Z, we have

$$\begin{aligned} u(x', y', z') &= \iint (Xu' + Yv' + Zw') ds dz \\ &= \iint (Np' + Sq' + Zw') ds dz \end{aligned}$$

the integral being taken over the cylindrical boundary.

The thickness $2h$ being supposed infinitesimal, the object of the method is to determine a few of the terms of p', q', w' of lowest order in h .

An alternative method would be to determine the functions E_1, E_2 of § 33, I. (i), in terms of edge tractions; but the least confusing method of all is perhaps to determine

$$U \equiv \frac{\partial E_1}{\partial y'} + \frac{\partial E_2}{\partial x'} \quad \text{and} \quad V \equiv -\frac{\partial E_1}{\partial x'} + \frac{\partial E_2}{\partial y'}$$

These do not contain z' , but when they are known the complete solution can obviously be written down. We begin with U' , and in fact it will not be necessary to determine V' separately.

Now U' is the work difference from the system

$$\left. \begin{aligned} \psi &= \frac{1}{8\pi\mu h} \frac{d}{dy} (\chi - \tfrac{1}{2} z^2 \nabla^2 \chi) \\ \theta = \phi &= \frac{1}{32\pi\mu h} \frac{d}{dx} (\chi - \tfrac{1}{2} z^2 \nabla^2 \chi) \end{aligned} \right\}.$$

From this system let those terms be removed which transmit the resultant stress (equivalent to a unit X force through x', y', z'). Then the remaining displacements have still the forms discussed in § 42 (i), and we shall use for them, and for the various quantities related to them, the notation given there, modified by the addition of a suffix 0 in each case.

The problem is now to balance the edge tractions due to the system u_0, v_0, w_0 .

The principal parts of these tractions, which in this case are simply the terms independent of z , are balanced by a solution of the permanent type (which we shall distinguish by the suffix 1) such that at the edge

$$\left. \begin{aligned} \left(\Delta_0 - \frac{dQ_0}{ds} - \frac{P_0}{\rho} \right) + \left(\Delta_1 - \frac{dQ_1}{ds} - \frac{P_1}{\rho} \right) &= 0 \\ \left(-\Pi_0 + \frac{dP_0}{ds} - \frac{Q_0}{\rho} \right) + \left(-\Pi_1 + \frac{dP_1}{ds} - \frac{Q_1}{\rho} \right) &= 0 \end{aligned} \right\} \quad (1)$$

These conditions define the solution with suffix 1.

The residual tractions from the compound solution $u_0 + u_1, v_0 + v_1, w_0 + w_1$ contain the factor z^2 and are of order h^2 as compared with those already balanced. To balance these residual tractions, solutions of all types are required, but, as in § 42 (iv), the permanent solution (which will be marked with suffix 2) is determined from the integral residual tractions; the permanent displacements are of order h , while those from the transitory solutions are of order h^2 , those from the source solution being of order h^{-1} .

The displacements of the balanced solution are therefore to terms of order h inclusive in the notation of § 42 (i),

$$\left. \begin{aligned} p' &= P_0 + P_1 & + P_2 - \tfrac{1}{2} \sigma z^2 \frac{d}{ds} (\Pi_0 + \Pi_1) \\ q' &= Q_0 + Q_1 & + Q_2 + \tfrac{1}{2} \sigma z^2 \frac{d}{ds} (\Delta_0 + \Delta_1) \\ w' &= & - \sigma z (\Delta_0 + \Delta_1) \end{aligned} \right\} \quad (2)$$

and with these values of p', q', w'

$$U'(x', y', z') = \iint (Np' + Sq' + Zw') \rho ds dz \quad (3)$$

All the steps of the above process can actually be carried out in the case of a circular plate, and the final formula gives a perfectly definite solution provided merely that N, S, Z are functions integrable over the edge. It should be specially noted that, in this form of the solution, discontinuity of the applied traction gives rise to no difficulty whatever.

On the other hand, the formula does not give a ready answer to such an important question as "What are the relations between the tractions actually applied, and the

tractions required to maintain the permanent solution alone?" or the practically equivalent question, "What conditions must the edge stress satisfy in order that the permanent mode may be absent from the resulting strain?"

The expression found for U' may be transformed so as to supply answers to these questions.

In the first place, we note that the values of p', q', w' will still be correct to the order stated (but will contain superfluous terms) if in the expressions just given for them we write

$$\Pi_0 + \Pi_1 + \Pi_2 \quad \text{for} \quad \Pi_0 + \Pi_1$$

and

$$\Delta_0 + \Delta_1 + \Delta_2 \quad \text{for} \quad \Delta_0 + \Delta_1.$$

Write also

$$\Pi \quad \text{for} \quad \Pi_0 + \Pi_1 + \Pi_2, \quad \Delta \quad \text{for} \quad \Delta_0 + \Delta_1 + \Delta_2,$$

and similarly for the other quantities.

Hence

$$\left. \begin{aligned} p' &= P - \frac{1}{2}\sigma z \frac{d\Pi}{ds} \\ q' &= Q + \frac{1}{2}\sigma z \frac{d\Delta}{ds} \\ w' &= -\sigma z \Delta \end{aligned} \right\} \begin{array}{l} \text{the error in each case} \\ \text{being of order } h^2 \end{array}$$

and

$$U'(x', y', z') = \iint \left\{ N \left(P - \frac{1}{2}\sigma z \frac{d\Pi}{ds} \right) + S \left(Q + \frac{1}{2}\sigma z \frac{d\Delta}{ds} \right) - Z \sigma z \Delta \right\} ds dz \quad . \quad . \quad (4)$$

Also from (1) $\Delta - \frac{dQ}{ds} - \frac{P}{\rho}$, $-\Pi + \frac{dP}{ds} - \frac{Q}{\rho}$ are of order h , and the formula for U' will therefore be correct to the same order as before if we substitute $\frac{dQ}{ds} + \frac{P}{\rho}$ for Δ , and $\frac{dP}{ds} - \frac{Q}{\rho}$ for Π . We shall also write

$$\begin{aligned} \int_{-h}^h N dz &= N_0, & \int_{-h}^h z^2 N dz &= N_2 \\ \int_{-h}^h S dz &= S_0, & \int_{-h}^h z^2 S dz &= S_2 \\ \int_{-h}^h z Z dz &= Z_1. \end{aligned}$$

Thus

$$U'(x', y', z') = \int ds \left\{ N_0 P - \frac{1}{2}\sigma N_2 \frac{d}{ds} \left(\frac{dP}{ds} - \frac{Q}{\rho} \right) + S_0 Q + \frac{1}{2}\sigma S_2 \frac{d}{ds} \left(\frac{dQ}{ds} + \frac{P}{\rho} \right) - \sigma Z_1 \left(\frac{dQ}{ds} + \frac{P}{\rho} \right) \right\} \quad . \quad (5)$$

If $N_2, S_2, Z_1, \frac{dN_2}{ds}, \frac{dS_2}{ds}$ are continuous over each edge line, integration by parts transforms this into

$$\int ds \left\{ P \left(N_0 - \frac{\sigma}{\rho} Z_1 - \frac{1}{2}\sigma \frac{d^2 N_2}{ds^2} - \frac{1}{2}\frac{\sigma}{\rho} \frac{dS_2}{ds} \right) + Q \left(S_0 + \sigma \frac{dZ_1}{ds} - \frac{1}{2}\frac{\sigma}{\rho} \frac{dN_2}{ds} + \frac{1}{2}\sigma \frac{d^2 S_2}{ds^2} \right) \right\} \quad . \quad (6)$$

Hence, in order that the permanent mode should be absent from the strain due to N, S, Z the following two conditions are *sufficient* :—

$$\left. \begin{aligned} N_0 - \frac{\sigma}{\rho} Z_1 - \frac{1}{2}\sigma \frac{d^2 N_2}{ds^2} - \frac{1}{2} \frac{\sigma}{\rho} \frac{dS_2}{ds} &= 0 \\ S_0 + \sigma \frac{dZ_1}{ds} - \frac{1}{2} \frac{\sigma}{\rho} \frac{dN_2}{ds} + \frac{1}{2}\sigma \frac{d^2 S_2}{ds^2} &= 0 \end{aligned} \right\} \cdot \cdot \cdot \cdot \cdot \cdot \quad (7)$$

at every point of the edge line or lines.

Further, all systems of traction for which the left-hand members of (7) have given values at every point of an edge will produce the same permanent mode. *Now as one such system of traction we may take the traction due to, or producing, the permanent mode by itself*, as given in § 42 (i). This gives at once the boundary conditions satisfied by the functions U, V of that section, and these boundary conditions, with the internal equations

$$\frac{d\Delta}{dx} + \frac{d\Pi}{dy} = 0; \quad \frac{d\Delta}{dy} - \frac{d\Pi}{dx} = 0,$$

completely define U, V which are thus determined, to a third approximation in general. The defining differential equations and surface conditions being practically of the same form as in the familiar first approximation, we need not detail the proof that U, V are actually determinate from the conditions, but pass at once to the important conclusion, an immediate consequence of this determinateness, that the permanent strain will *not* be absent unless (7) are satisfied, or, in other words, that these conditions are *necessary*, as well as *sufficient*. From this again it follows that these conditions are fulfilled, to the order stated, by each of the transitory modes; and this remark is valuable, because, once it has been verified by direct integration, it obviously leads, by an extension of the process of § 42 (iv), to a completely independent method of dealing with the whole problem. The method is noticeable for its simplicity and directness, but a somewhat serious defect is the difficulty of adapting it to the case when the edge stress is discontinuous.

This leads us to consider the correction that must be applied to the integral (6) when the conditions of continuity stated in connection with it are not fulfilled. It will be sufficient to take a case in which breach of continuity occurs at only one point E of the edge line.

We have defined the positive direction of an edge line in (k); let the excess of the value of $f(s)$ just on the negative side of E over its value just on the positive side be denoted by $[f(s)]$. Then if $\phi(s)$ be continuous

$$\int f(s) \frac{d}{ds} \phi(s) ds = \phi(s)[f(s)] - \int \phi(s) \frac{d}{ds} f(s) ds,$$

the integrals, we may suppose, being taken round the edge line from the positive to the negative side of E , and the value of $\phi(s)$ in the integrated term being taken as at E .

Then to (6) we have in general to add

$$\left. \begin{aligned} & -\frac{1}{2}\sigma \left(\frac{dP}{ds} - \frac{Q}{\rho} \right) [N_2] + \frac{1}{2}\sigma \left(\frac{dQ}{ds} + \frac{P}{\rho} \right) [S_2] - \sigma Q[Z_1] \\ & + \frac{1}{2}\sigma P \left[\frac{dN_2}{ds} \right] - \frac{1}{2}\sigma Q \left[\frac{dS_2}{ds} \right] \end{aligned} \right\}$$

or, otherwise arranged,

$$\left. \begin{aligned} & P \left\{ \frac{1}{2}\frac{\sigma}{\rho} [S_2] + \frac{1}{2}\sigma \left[\frac{dN_2}{ds} \right] \right\} + Q \left\{ -\sigma[Z_1] + \frac{1}{2}\frac{\sigma}{\rho} [N_2] - \frac{1}{2}\sigma \left[\frac{dS_2}{ds} \right] \right\} \\ & - \frac{dP}{ds} \frac{1}{2}\sigma [N_2] + \frac{dQ}{ds} \frac{1}{2}\sigma [S_2] \end{aligned} \right\}$$

The various terms of this expression may be interpreted with the help of the conceptions of sources and doublets. Thus, to go back to (3), we see that the part of U' arising from an element $N_0 ds$ of normal traction at E has $(P_0 + P_1)N_0 ds$ for principal term. $P_0 + P_1$ is therefore (principal term of the) value of U' due to a unit element of normal force at E . (Since this unit element can only exist in any actual deformation as part of an equilibrating system, the phrase *due to* in the last sentence must be taken under reservation. The solution of which $P_0 + P_1$ is the x -displacement at $(x', y', 0)$ is in fact maintained by a unit element of normal traction at E' , acting along with a continuous system of force in equilibrium with this element, and distributed over the edge in a manner depending only on the statical value of the element, and not at all on the position of E . For any equilibrating combination of elements, the aggregate of these continuous systems will disappear.)

Now the first of the above integrated terms is equivalent to $(P_0 + P_1)\frac{1}{2}\frac{\sigma}{\rho}[S_2]$.

Hence the discontinuity in S_2 at E has the same effect at a distance from the edge as would have an element of normal traction distributed over the perpendicular at E so as to give a resultant $\frac{1}{2}\frac{\sigma}{\rho}[S_2]$.

Again an element $-A$ of normal traction at E , combined with an element A at E' , where $EE' = ds$, will give

$$\begin{aligned} U' &= A \frac{d}{ds} (P_0 + P_1) ds \\ &= \frac{d}{ds} (P_0 + P_1), \text{ if we take } A ds = 1. \end{aligned}$$

$\frac{d}{ds} (P_0 + P_1)$ is therefore due to a unit doublet of normal force at E , and from the term $-\frac{dP}{ds} \frac{1}{2}\sigma[N_2]$ we conclude that the discontinuity in N_2 at E has the same interior effect as a doublet of normal force at E of strength $-\frac{1}{2}\sigma[N_2]$.

The other terms may be interpreted similarly. It does not seem possible to account on physical grounds for any except the principal terms of the solution given above. The principal terms are of course the same as those deduced in the ordinary theory from the 'Principle of the elastic equivalence of statically equipollent systems of load.'

With reference to the equivalence of mere discontinuities to line elements and doublets one or two remarks may be made. Discontinuity in the applied force will not produce infinite displacement at a line where it takes place, but a line element of load, and, *a fortiori*, a doublet, will do so. The permanent mode may therefore contain infinities at the edge which do not exist in the exact solution. There is really no difficulty in this, since the permanent mode does not purport to represent the strain, even approximately, in the immediate vicinity of the edge. The point may be illustrated by the permanent part of the infinite solid solution for a single force. This becomes infinite on the perpendicular through the source in a totally different way from the exact solution. A good deal of discussion took place at one time over a similar point in the flexural solution. This will be referred to again, but the considerations we have adduced seem to remove the chief part of the difficulty.

44. *Flexural strain.*

In this case N , S are odd functions, and Z an even function of z .

(i) *Permanent mode.*

This mode is defined in terms of one function F of (x, y) satisfying $\nabla^4 F = 0$, and may be referred to simply as an F strain.

$$\begin{aligned}\phi &= -(zF - \frac{1}{6}z^3\nabla^2 F); \quad \theta = zF - \frac{1}{6}z^3\nabla^2 F - 2h^2z\nabla^2 F \\ p &= \frac{d}{dn} \left\{ -4(1-\sigma)(zF - \frac{1}{6}z^3\nabla^2 F) + 2(\frac{1}{3}z^3 - h^2z)\nabla^2 F \right\} \\ q &= \frac{d}{ds} \left\{ -4(1-\sigma)(zF - \frac{1}{6}z^3\nabla^2 F) + 2(\frac{1}{3}z^3 - h^2z)\nabla^2 F \right\} \\ w &= 4(1-\sigma)(F - \frac{1}{2}z^2\nabla^2 F) + 2(z^2 - h^2)\nabla^2 F\end{aligned}$$

For shortness in writing out the stresses, we shall work with symbols $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3$, denoting operations of differentiation applied to F , and defined by the equations

$$\left. \begin{aligned}\mathfrak{I}_1 &= 4\nabla^2 + 4(\sigma-1)\left(\frac{1}{\rho}\frac{d}{dn} + \frac{d^2}{ds^2}\right) \\ \mathfrak{I}_2 &= 4(1-\sigma)\left(\frac{d}{ds}\frac{d}{dn} - \frac{1}{\rho}\frac{d}{ds}\right) \\ \mathfrak{I}_3 &= 4\frac{d}{dn}\nabla^2\end{aligned}\right\}$$

Then

$$\begin{aligned}\frac{\widehat{nn}}{2\mu} &= -\mathfrak{I}_1 F + \left\{ \frac{1}{6}z^3 + \frac{1}{2}\frac{1}{1-\sigma}\left(\frac{1}{3}z^3 - h^2z\right) \right\} \mathfrak{I}_1 \nabla^2 F \\ \frac{ns}{2\mu} &= -\mathfrak{I}_2 F + \left\{ \frac{1}{6}z^3 + \frac{1}{2}\frac{1}{1-\sigma}\left(\frac{1}{3}z^3 - h^2z\right) \right\} \mathfrak{I}_2 \nabla^2 F \\ \frac{\widehat{nz}}{\mu 2} &= \frac{1}{2}(z^2 - h^2)\mathfrak{I}_3 F\end{aligned}$$

(ii) *Rotational transitory modes.*

These are as in § 42 (ii), except that now

$$\psi = \sum_{n=0} \psi_n(x, y) \sin \frac{2n+1}{2} \frac{\pi z}{h}$$

where $\nabla^2 \psi_n - (2n+1)^2 \frac{\pi^2}{4h^2} \psi_n = 0$.

(iii) *Dilatational transitory modes.*

$$\phi = \sinh \kappa z f(x, y); \quad \theta = -\cosh 2\kappa h \cdot \phi$$

As in § 42 (iii),

$$\left. \begin{aligned} p &= \frac{df}{dn} \left\{ (\alpha - \cosh 2\kappa h) \sinh \kappa z + 2\kappa z \cosh \kappa z \right\} \\ q &= \frac{df}{ds} \left\{ (\alpha - \cosh 2\kappa h) \sinh \kappa z + 2\kappa z \cosh \kappa z \right\} \\ w &= \kappa f \left\{ -(\alpha + \cosh 2\kappa h) \cosh \kappa z + 2\kappa z \sinh \kappa z \right\} \\ \frac{\widehat{nn}}{2\mu} &= -\kappa^2 f \left\{ (3 - \cosh 2\kappa h) \sinh \kappa z + 2\kappa z \cosh \kappa z \right\} \\ &\quad - \left(\frac{1}{\rho} \frac{df}{dn} + \frac{d^2 f}{ds^2} \right) \left\{ (\alpha - \cosh 2\kappa h) \sinh \kappa z + 2\kappa z \cosh \kappa z \right\} \\ \frac{\widehat{ns}}{2\mu} &= \left(\frac{d}{ds} \frac{df}{dn} - \frac{1}{\rho} \frac{df}{ds} \right) \left\{ (\alpha - \cosh 2\kappa h) \sinh \kappa z + 2\kappa z \cosh \kappa z \right\} \\ \frac{\widehat{nz}}{2\mu} &= \kappa \frac{df}{dn} \left\{ (1 - \cosh 2\kappa h) \cosh \kappa z + 2\kappa z \sinh \kappa z \right\} \end{aligned} \right\}$$

We have $\frac{df}{dn} = -i\kappa f$ to the lowest order,

$$\text{or } f = \frac{i}{\kappa} \frac{df}{dn}.$$

Hence if we put

$$\left. \begin{aligned} \kappa \frac{df}{dn} &= g_\kappa \\ i \{ (\cosh 2\kappa h - 3) \sinh \kappa z - 2\kappa z \cosh \kappa z \} &= N(\kappa z) \\ (1 - \cosh 2\kappa h) \cosh \kappa z + 2\kappa z \sinh \kappa z &= Z(\kappa z) \end{aligned} \right\}$$

the above strain gives

$$\left. \begin{aligned} \frac{\widehat{nn}}{2\mu} &= g_\kappa N(\kappa z) \\ \frac{\widehat{ns}}{2\mu} &= 0 \\ \frac{\widehat{nz}}{2\mu} &= g_\kappa Z(\kappa z) \end{aligned} \right\} \quad \text{with an error of relative order } h,$$

exactly,

and we may note that $\int_{-h}^h z N(\kappa z) dz = 0$ and $\int_{-h}^h Z(\kappa z) dz = 0$. (§ 41.)

The same remarks as in the extensional case might be made here about the complementary character of the types (ii), (iii) in regard to their contributions to edge displacement or traction, when h is small.

(iv) If we follow the lines of the discussion of the extensional case, we have now to consider the approximate allocation of a given system of edge traction among the three types of strain.

The investigation is this time more complex, chiefly in consequence of the presence of the stress \widehat{nz} in the permanent mode. Since, moreover, flexure is much more important physically than extension, we shall give a fairly detailed discussion in the next article, but in the meantime we may examine what could be done with a solution in which, as in § 42 (iv), the principal part of the edge traction is of order zero for each type of strain, and the parts of higher order are neglected.

Such a solution would give

$$\left. \begin{aligned} N &= N_p & + N_a \\ S &= S_p + S_r \\ Z &= & Z_a \end{aligned} \right\}$$

But we see at once that we do not in this way get a perfectly general distribution of N, S, Z , since the last equation gives $\int_{-h}^h Z dz = 0$. A closer examination is therefore necessary, and it will perhaps conduce to lucidity if we consider separately the three cases of normal, tangential, and perpendicular traction.

45. *Flexural forces.*

(i) *Normal traction.*

N being of order zero, we can satisfy the conditions by taking zF , $\frac{d^2\psi}{dz^2}$, and g_κ all of this order, but besides the terms of order zero in the stresses, it will be necessary to take account of the terms of \widehat{nz} which come from F and ψ , albeit these are of an order one higher. Then

$$\left. \begin{aligned} \frac{N}{2\mu} &= -z\mathfrak{F}_1 F & + & \Sigma g_\kappa N(\kappa z) & . & . & . & (1) \\ 0 &= -z\mathfrak{F}_2 F & + & \frac{d^2\psi}{dz^2} & . & . & . & (2) \\ 0 &= \frac{1}{2}(z^2 - h^2)\mathfrak{F}_3 F & + & \frac{d^2\psi}{ds dz} & + & \Sigma g_\kappa Z(\kappa z) & . & . & (3) \end{aligned} \right\}$$

Assuming these provisionally, multiply (1) by z and integrate from $-h$ to h .

$$\text{Thus} \quad \mathfrak{F}_1 F = -\frac{3}{4\mu h^3} \int_{-h}^h zN dz \quad . \quad . \quad . \quad (4)$$

$$\text{and} \quad \Sigma g_\kappa N(\kappa z) = \frac{N}{2\mu} - \frac{3z}{4\mu h^3} \int_{-h}^h zN dz \quad . \quad . \quad . \quad (5)$$

From (2), since ψ is odd in z , and $\frac{d\psi}{dz} = 0$ for $z = \pm h$, we get

$$\psi = (\frac{1}{6}z^3 - \frac{1}{2}h^2z)\mathfrak{F}_2 F \quad . \quad . \quad . \quad (6)$$

In (3) the terms are of different orders; thus, with the help of (6),

$$z\mathfrak{F}_3 F + \frac{d}{ds}\mathfrak{F}_2 F = 0 \quad . \quad . \quad . \quad (7)$$

$$\Sigma g_\kappa Z(\kappa z) = 0 \quad . \quad . \quad . \quad (8)$$

(4) and (7) define F , (6) then gives the edge value of ψ , and (5), (8) determine g_κ , the functions to be expanded obviously satisfying the conditions of § 40 (ii).

Further, with the values of F , ψ , g_κ so determined, the equations (1), (2), (3) are all satisfied, the traction \widehat{nz} vanishes exactly, and the residual tractions \widehat{nn} , \widehat{ns} are of order 1.

(ii) *Tangential traction.*

S being of order 0, zF and $\frac{d^2\psi}{dz^2}$ will again be of this order, but it will be seen that g_κ is of order 1. For, making these suppositions, we have

$$0 = -z\mathfrak{I}_1 F \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{S}{2\mu} = -z\mathfrak{I}_2 F + \frac{d^2\psi}{dz^2} \quad . \quad . \quad . \quad . \quad (2)$$

$$0 = \frac{1}{2}(z^2 - h^2)\mathfrak{I}_3 F + \frac{d^2\psi}{ds dz} + \Sigma g_\kappa Z(\kappa z) \quad . \quad . \quad . \quad . \quad (3)$$

$$\text{From (1),} \quad \mathfrak{I}_1 F = 0 \quad . \quad . \quad . \quad . \quad (4)$$

$$\text{From (2),} \quad \frac{1}{2\mu} \int_{-h}^z S dz = -\frac{1}{2}(z^2 - h^2)\mathfrak{I}_2 F + \frac{d\psi}{dz} \quad . \quad . \quad . \quad . \quad (5)$$

Differentiate this with respect to s , and subtract from (3). Thus

$$-\frac{1}{2\mu} \frac{d}{ds} \int_{-h}^z S dz = \frac{1}{2}(z^2 - h^2)(\mathfrak{I}_3 F + \frac{d}{ds}\mathfrak{I}_2 F) + \Sigma g_\kappa Z(\kappa z) \quad . \quad (6)$$

and, integrating the last from $-h$ to h ,

$$\mathfrak{I}_3 F + \frac{d}{ds}\mathfrak{I}_2 F = -\frac{3}{4\mu h^3} \frac{d}{ds} \int_{-h}^h z S dz \quad . \quad . \quad . \quad . \quad (7)$$

(4) and (7) define F , and (5) integrated from 0 to z gives ψ . With the values so determined, and with (6) satisfied by g_κ (and this is possible in virtue of (7)), the equations (1), (2), (3) are all satisfied, the traction \widehat{nz} vanishes exactly, and the residual tractions \widehat{nn} , \widehat{ns} are of order 1.

By combining this with the preceding case, we see that the results do in fact give a first approximation to the solution, since the residual stresses N , S are each of an order higher than their original given values.

The additional equation required to define g_κ might be found by carrying out the process of (i) with the residual normal traction $4\mu \frac{d}{ds} \frac{d\psi}{dn}$. The analysis would be practically the same as will be given in connection with the next case.

(iii) *Perpendicular traction.*

Z being of order zero, we shall have to take zF and $\frac{d^2\psi}{dz^2}$ of order -1 , g_κ of order 0.

On this hypothesis, we shall write down the exact expression for \widehat{nz} , and the terms of order -1 and 0 in \widehat{nn} and \widehat{ns} . We are then to have

$$0 = -z\mathfrak{I}_1 F \quad \left\{ \begin{array}{l} + 2 \frac{d}{ds} \frac{d\psi}{dn} + \Sigma g_\kappa N(\kappa z) \end{array} \right. \quad . \quad . \quad . \quad (1)$$

$$0 = -z\mathfrak{I}_2 F + \frac{d^2\psi}{dz^2} \quad \left\{ \begin{array}{l} + \frac{2}{\rho} \frac{d\psi}{dn} \end{array} \right. \quad . \quad . \quad . \quad (2)$$

$$\frac{Z}{2\mu} = \frac{1}{2}(z^2 - h^2)\mathfrak{I}_3 F + \frac{d^2\psi}{ds dz} + \Sigma g_\kappa Z(\kappa z) \quad . \quad . \quad . \quad (3)$$

where

$$\nabla^2 \psi_m - \frac{(2m+1)^2 \pi^2}{4h^2} \psi_m = 0,$$

and ψ_m has the edge value $\mathfrak{J}_2 F$.

Thus

$$\frac{d\psi}{dn} = -\frac{32}{\pi^4} h^2 \left(\frac{d\psi_0}{dn} \sin \frac{\pi z}{2h} - \frac{1}{3^4} \frac{d\psi_1}{dn} \sin \frac{3\pi z}{2h} + \dots \right)$$

the principal value of which is

$$-\frac{16}{\pi^5} h^2 \mathfrak{J}_2 F \left(\sin \frac{\pi z}{2h} - \frac{1}{3^3} \sin \frac{3\pi z}{2h} + \frac{1}{5^3} \sin \frac{5\pi z}{2h} - \dots \right)$$

Let

$$\beta(z/h) \equiv \frac{16}{\pi^3} \left(\sin \frac{\pi z}{2h} - \frac{1}{3^3} \sin \frac{3\pi z}{2h} + \frac{1}{5^3} \sin \frac{5\pi z}{2h} - \dots \right) \quad (11)$$

then

$$\frac{d\psi}{dn} = -h^2 \beta(z/h) \mathfrak{J}_2 F \quad (12)$$

and

$$\left. \begin{aligned} \int_{-h}^h z \frac{d\psi}{dn} dz &= -\frac{128}{\pi^5} h^4 \left(1 + \frac{1}{3^5} + \frac{1}{5^5} + \dots \right) \mathfrak{J}_2 F \\ &= -\frac{128}{\pi^5} \gamma_5 h^4 \mathfrak{J}_2 F \end{aligned} \right\} \quad (13)$$

if

$$\gamma_5 \equiv 1 + \frac{1}{3^5} + \frac{1}{5^5} + \dots$$

Now multiply (8) by z , and integrate from $-h$ to h .

Hence

$$\mathfrak{J}_1 F' = -\frac{384}{\pi^5} \gamma_5 h \frac{d}{ds} \mathfrak{J}_2 F \quad (14)$$

and

$$\sum g_\kappa N(\kappa z) = \left\{ 2h^2 \beta(z/h) - 384 \gamma_5 h z / \pi^5 \right\} \frac{d}{ds} \mathfrak{J}_2 F \quad (15)$$

This, and equation (7), define g_κ .

As we do not require g'_κ , we will eliminate it from (10) at once by integrating.

Thus

$$0 = -2 \frac{h^2}{3} \mathfrak{J}_3 F' + \frac{d}{ds} \left[\psi' \right]_{-h}^h.$$

Multiply (9) by z and integrate. Then

$$-\frac{2}{\rho} \int_{-h}^h z \frac{d\psi}{dn} dz = -\frac{2h^3}{3} \mathfrak{J}_2 F' - \left[\psi' \right]_{-h}^h$$

and from this with the last

$$\frac{d}{ds} \left\{ \frac{2}{\rho} \int_{-h}^h z \frac{d\psi}{dn} dz \right\} = 2 \frac{h^3}{3} \left(\mathfrak{J}_2 F' + \frac{d}{ds} \mathfrak{J}_2 F' \right)$$

or, from (13),

$$\mathfrak{J}_3 F' + \frac{d}{ds} \mathfrak{J}_2 F' = -\frac{384}{\pi^5} \gamma_5 h \frac{d}{ds} \left(\frac{1}{\rho} \mathfrak{J}_2 F \right) \quad (16)$$

(14) and (16) give F' , and ψ' may be found from (9).

If we write \bar{F} for $F + F'$, we get from (4), (14)

$$\begin{aligned}\mathfrak{I}_1 \bar{F} &= -\frac{384}{\pi^5} \gamma_s h \frac{d}{ds} \mathfrak{I}_2 F \\ &= -\frac{384}{\pi^5} \gamma_s h \frac{d}{ds} \mathfrak{I}_2 \bar{F}\end{aligned}$$

neglecting the term in hF' , which is of order h^2 relative to \bar{F} .

This may be written

$$\left(\mathfrak{I}_1 + \frac{384}{\pi^5} \gamma_s h \frac{d}{ds} \mathfrak{I}_2 \right) \bar{F} = 0 \quad . \quad . \quad (17)$$

Similarly from (6), (16) we obtain

$$\left(\mathfrak{I}_3 + \frac{d}{ds} \mathfrak{I}_2 + \frac{384}{\pi^5} \gamma_s h \frac{d}{ds} \frac{1}{\rho} \mathfrak{I}_2 \right) \bar{F} = -\frac{3}{4h^3} \int_{-h}^h Z dz \quad . \quad . \quad (18)$$

We may regard (17) and (18) as the equations giving F to a second approximation. If we combine the results of the three cases of this article, we obtain

$$\left. \begin{aligned} \mathfrak{I}_1 F &= -\frac{3}{4\mu h^3} \int_{-h}^h z N dz \\ \mathfrak{I}_2 F + \frac{d}{ds} \mathfrak{I}_2 F &= -\frac{3}{4\mu h^3} \left\{ \int_{-h}^h Z dz + \frac{d}{ds} \int_{-h}^h z S dz \right\} \end{aligned} \right] \quad (19)$$

These are the equations usually referred to as Kirchhoff's boundary conditions. The extension of the more approximate conditions (17), (18) to the general case will be given in the next article.

46. *Flexural strain under given edge tractions. The Green's function method for the permanent mode.*

The displacement at (x', y', z') due to tractions N, S, Z is defined in § 33, II. (i), in terms of the work difference from the system

$$\begin{aligned}\phi &= -(3/32\pi\mu h^3) (z\chi - \frac{1}{6}z^3\nabla^2\chi) \\ \psi &= (3/32\pi\mu h^3) (z\chi - \frac{1}{6}z^3\nabla^2\chi - 2h^2z\nabla^2\chi).\end{aligned}$$

From this system let the terms conveying resultant stress be removed; the residue is still an F strain with $F = F_0$ say, and F_0 is of order h^{-3} .

We have to balance F_0 at the edge, and the edge displacements in the balanced solution being p', q', w' , the work difference required (F_1 of § 33) is

$$\iint (Np' + Sq' + Zw') ds dz.$$

The problem is to determine p', q', w' as closely as is practicable.

The tractions to be balanced are

$$\begin{aligned} N/2\mu &= -z\mathfrak{I}_1 F_0 \quad \text{with terms in } z^3 \\ S/2\mu &= -z\mathfrak{I}_2 F_0 \\ Z/2\mu &= \frac{1}{2}(z^2 - h^2)\mathfrak{I}_3 F_0.\end{aligned}$$

With tractions of these orders, of quite general form in z , the analysis of last article would lead us to expect that in the balancing solution \mathbf{F} would be of order -3 , ψ of order 0, and g_κ of order -2 . But in consequence of \mathbf{N} being simply proportional to z , it will be noticed that § 45 (i) (5) gives $\Sigma g_\kappa \mathbf{N}(\kappa z) = 0$, and it follows that in the present case g_κ will be of order -1 at lowest. The displacements p' , w' derived from the strain defined by the functions g_κ will therefore be of order 0, and q' of order 1. Thus we may anticipate that the first *two* terms of p' , and the first *three* terms of q' and w' will be obtained in practicable forms, *i.e.* independently of series associated with the zeroes of $\sinh 2\kappa h - 2\kappa h$.

The tractions written down above may be balanced approximately by strains \mathbf{F}' , ψ' . We require

$$\left. \begin{aligned} -z\mathfrak{F}_1(\mathbf{F}_0 + \mathbf{F}') &= 0 \\ -z\mathfrak{F}_2(\mathbf{F}_0 + \mathbf{F}') + \frac{d^2\psi'}{dz^2} &= 0 \\ \frac{1}{2}(z^2 - h^2)\mathfrak{F}_3(\mathbf{F}_0 + \mathbf{F}') + \frac{d^2\psi'}{ds dz} &= 0 \end{aligned} \right\}$$

These are equivalent to

$$\left. \begin{aligned} \mathfrak{F}_1(\mathbf{F}_0 + \mathbf{F}') &= 0 \\ \psi' &= \left(\frac{1}{6}z^3 - \frac{1}{2}h^2z\right)\mathfrak{F}_2(\mathbf{F}_0 + \mathbf{F}') \\ \left(\mathfrak{F}_3 + \frac{d}{ds}\mathfrak{F}_2\right)(\mathbf{F}_0 + \mathbf{F}') &= 0 \end{aligned} \right\} \quad (1)$$

which determine \mathbf{F}' , ψ' .

The principal terms of the residual tractions are

$$\left. \begin{aligned} \frac{\widehat{nn}}{2\mu} &= 2 \frac{d}{ds} \frac{d\psi'}{dn} = -2h^2\beta(z/h) \frac{d}{ds} \mathfrak{F}_2(\mathbf{F}_0 + \mathbf{F}') \\ \frac{\widehat{ns}}{2\mu} &= \frac{2}{\rho} \frac{d\psi'}{dn} = -\frac{2h^2}{\rho} \beta(z/h) \mathfrak{F}_2(\mathbf{F}_0 + \mathbf{F}') \end{aligned} \right\} \quad (2)$$

as in (iii) (12) of last article, and they may be dealt with in the manner illustrated there.

To balance them take strains defined by \mathbf{F}'' , ψ'' , g_κ'' with $z\mathbf{F}''$, $\frac{d^2\psi''}{dz^2}$, g_κ'' of order -1 .

We must therefore have

$$\left. \begin{aligned} 2h^2\beta(z/h) \frac{d}{ds} \mathfrak{F}_2(\mathbf{F}_0 + \mathbf{F}') &= -z\mathfrak{F}_1\mathbf{F}'' + \Sigma g_\kappa'' \mathbf{N}(\kappa z) \\ (2h^2/\rho)\beta(z/h)\mathfrak{F}_2(\mathbf{F}_0 + \mathbf{F}') &= -z\mathfrak{F}_2\mathbf{F}'' + \frac{d^2\psi''}{dz^2} \\ 0 &= \frac{1}{2}(z^2 - h^2)\mathfrak{F}_3\mathbf{F}'' + \frac{d^2\psi''}{ds dz} + \Sigma g_\kappa'' \mathbf{Z}(\kappa z) \end{aligned} \right\} \quad (3)$$

From these, as in last article,

$$\left. \begin{aligned} \mathfrak{F}_1\mathbf{F}'' &= -\frac{384}{\pi^3}\gamma_3h \frac{d}{ds} \mathfrak{F}_2(\mathbf{F}_0 + \mathbf{F}') \\ \left(\mathfrak{F}_3 + \frac{d}{ds}\mathfrak{F}_2\right)\mathbf{F}'' &= -\frac{384}{\pi^3}\gamma_3h \frac{d}{ds} \left\{ \frac{1}{\rho} \mathfrak{F}_2(\mathbf{F}_0 + \mathbf{F}') \right\} \\ \psi'' &= \left(\frac{1}{6}z^3 - \frac{1}{2}h^2z \right) \mathfrak{F}_2\mathbf{F}'' - \frac{128}{\pi^3\rho} h^4 \mathfrak{F}_2(\mathbf{F}_0 + \mathbf{F}') \left(\sin \frac{\pi z}{2h} - \frac{1}{3^5} \sin \frac{3\pi z}{2h} + \dots \right) \end{aligned} \right\} \quad (4)$$

whence

$$\frac{d\psi''}{dn} = -h^2\beta(z/h)\mathfrak{I}_2F'' + \frac{2}{\rho}\left(\frac{1}{6}z^3 - \frac{1}{2}h^2z\right)\mathfrak{I}_2(F_0 + F') \quad (5)$$

We are now left with tractions \widehat{nn} , \widehat{ns} of order zero. In the solution balancing these, F''' will be of order -1 , ψ''' of order 2 , g_κ''' of order 0 . We do not think it worth while to write down the equations defining F''' , but it should be noticed that they can be found explicitly. In fact, although the residual tractions \widehat{nn} , \widehat{ns} with which we are now dealing are partly defined by κ series, the integrals $\int_{-h}^h z \widehat{nn} dz$ and $\int_{-h}^h z \widehat{ns} dz$ will be found to vanish to the order concerned so far as they come from these series, in virtue of the relation $\sum g_\kappa'' Z(\kappa z) = 0$, which follows from (iii) (7) of last article.

The functions g_κ'' give terms of order zero in p' , w' .

Hence, including in p' terms of orders $-2, -1$

$$\begin{array}{lll} \dots & q' & \dots \dots \dots -2, -1, 0 \\ \dots & w' & \dots \dots \dots 3, -2, -1 \end{array}$$

we have

$$\left. \begin{aligned} p' &= -4(1-\sigma)z \frac{d}{dn}(F_0 + F' + F'') \\ q' &= -4(1-\sigma)z \frac{d}{ds}(F_0 + F' + F'' + F''') + \left(\frac{4-2\sigma z^3}{3} - 2h^2z\right) \frac{d}{ds} \nabla^2(F_0 + F') \\ &\quad - 2 \frac{d}{dn}(\psi' + \psi'') \\ w' &= 4(1-\sigma)(F_0 + F' + F'' + F''') + 2(\sigma z^2 - h^2) \nabla^2(F_0 + F') \end{aligned} \right\} \quad (6)$$

The value of $\frac{d\psi'}{dn}$ to a second approximation is

$$\frac{d\psi'}{dn} = -h^2\beta(z/h)\mathfrak{I}_2(F_0 + F') - \frac{1}{2\rho}\left(\frac{1}{6}z^3 - \frac{1}{2}h^2z\right)\mathfrak{I}_2(F_0 + F') \quad (7)$$

and $\frac{d\psi''}{dn}$ is given by (5).

The function F_1 which (§ 33) defines the permanent solution is

$$\iint (Np' + Sq' + Zw') ds dz \quad (8)$$

For the case of a solid or hollow circular plate all the quantities in the right-hand members of (6) can actually be calculated, and we thus obtain the solution for normal traction to a *second* approximation, and for tangential or perpendicular traction to a *third* approximation, in a form, moreover, applicable without modification or addition even when the given edge stresses are discontinuous.

We conclude by deducing the equations corresponding to Kirchhoff's boundary conditions to a second approximation. (They might be found to one order higher in the case of vanishing normal traction.)

We suppose that the given tractions N , S are of the same order in h , and that Z is of an order one higher. Any case may be reduced to combinations of cases satisfying this condition.

If we write F for $F_0 + F' + F''$, then it follows from (6), (7), (8) that the terms of two lowest orders in F_1 are given by

$$\frac{F_1}{4(1-\sigma)} = \int \left[-\frac{dF}{dn} \int_{-h}^h zNdz - \frac{dF}{ds} \int_{-h}^h zSdz + F \int_{-h}^h Zdz \right] ds \quad (9)$$

$$+ \left(\frac{d}{ds} \frac{dF}{dn} - \frac{1}{\rho} \frac{dF}{ds} \right) \int_{-h}^h 2h^2 \beta(z/h) Sdz$$

As in the extensional case, § 43, the integral with respect to s may be modified by means of integration by parts so that only F and $\frac{dF}{dn}$ appear under the integral sign.

Write

$$\left. \begin{aligned} \int_{-h}^h zNdz &= N_1, \quad \int_{-h}^h zSdz = S_1, \quad \int_{-h}^h Zdz = Z_0 \\ \int_{-h}^h 2h^2 \beta(z/h) Sdz &= S_2. \end{aligned} \right\} \quad (10)$$

Then if N_1 , S_1 , Z_0 , S_2 are continuous functions of s ,

$$\frac{F_1}{4(1-\sigma)} = \int \left\{ -\frac{dF}{dn} \left(N_1 + \frac{dS_1}{ds} \right) + F \left(\frac{dS_1}{ds} + Z_0 + \frac{d}{ds} \frac{S_2}{\rho} \right) \right\} ds \quad (11)$$

Hence any two systems of traction will give the same permanent mode to a second approximation, provided the values of $N_1 + \frac{dS_1}{ds}$ and $\frac{dS_1}{ds} + Z_0 + \frac{d}{ds} \frac{S_2}{\rho}$ are the same for the two systems.

Now for the system F_1 , § 44 (i),

$$\left. \begin{aligned} N_1 &= -\frac{4}{3} \mu h^3 \mathfrak{g}_1 F_1, \quad S_1 = -\frac{4}{3} \mu h^3 \mathfrak{g}_2 F_1, \quad Z_0 = -\frac{4}{3} \mu h^3 \mathfrak{g}_3 F_1, \\ S_2 &= -\frac{4}{3} \mu h^3 \frac{384}{\pi^3} \gamma_3 h \mathfrak{g}_2 F_1, \end{aligned} \right\} \quad (12)$$

§ 45, (iii) (13).

Thus, with $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$ defined as in § 44 (i), the boundary conditions are

$$\left. \begin{aligned} -\frac{4\mu h^3}{3} \left(\mathfrak{g}_1 F_1 + \frac{384}{\pi^3} \gamma_3 h \frac{d}{ds} \mathfrak{g}_2 F_1 \right) &= N_1 + \frac{dS_1}{ds} \\ -\frac{4\mu h^3}{3} \left(\frac{d}{ds} \mathfrak{g}_2 F_1 + \mathfrak{g}_3 F_1 + \frac{384}{\pi^3} \gamma_3 h \frac{d}{ds} \frac{\mathfrak{g}_2 F_1}{\rho} \right) &= \frac{dS_1}{ds} + Z_0 + \frac{d}{ds} \frac{S_2}{\rho} \end{aligned} \right\} \quad (13)$$

When only the principal terms are retained, these reduce to Kirchhoff's conditions.

If S_1 or S_2 is discontinuous at any point of the edge, integrated terms will appear in equation (11), as in the extensional case, § 43. Thus, if the normal couple S_1 be discontinuous at a point P ($s = s'$), there will appear on the right of (11) a term

$$F(s') \mid S_1 \mid_{s=s'-}^{s=s'+}$$

The method of dealing with such a discontinuity in any actual problem is obvious, for by (9) its effect is the same as that of an *element* $\mid S_1 \mid$ of shearing traction applied at P , a result which on the 'elastic equivalence' theory may easily be obtained by a trifling modification of the process by which THOMSON and TAIT reconciled the conditions of KIRCHHOFF and POISSON.

ADDITION TO PAPER BY J. DOUGALL ON—

“AN ANALYTICAL THEORY OF THE EQUILIBRIUM OF AN ISOTROPIC ELASTIC PLATE.”

(Note added May 20, 1904.)

The kindness of one of the referees enables me to supply the following references to recent work bearing on the subject of the paper.

(a) J. H. MICHELL, in a paper “On the direct determination of Stress in an Elastic Solid, with application to the Theory of Plates,” *Proc. Lond. Math. Soc.*, vol. 31, 1899, shows how the stresses might be found without previous determination of the displacements. In the case of the stress Ξ or R , he finds that $\nabla^4 R$ is a given function in the body of the plate, while R and dR/dz are given on the faces. If we neglect the conditions at the edge, which have practically no influence on the result, a value of R satisfying these conditions can be found, in terms of Fourier integrals for instance. Mr MICHELL does not determine R —this has been done in the present paper—but proceeds to deduce the forms of the remaining stresses, and the differential equation for the normal displacement of a point on the mid plane. One special case of normal force is worked out to a first approximation, and Lagrange’s equation for this case deduced.

For the conditions at the edge, reference is made to the ordinary Thomson-Boussinesq theory, which uses the principle of equipollent loads.

(b) L. N. G. FILON, “On an approximate solution for the bending of a Beam of rectangular cross section under any system of Load, with special reference to points of concentrated or discontinuous Loading,” *Phil. Trans. R. Soc. Lond. (Sec. A)*, vol. 201 (1903).

Dr FILON’s solution applies to a beam in which the ratios of breadth to depth, and of depth to length, are both small. The axis of z being taken in the direction of the breadth, the stress Ξ is taken as negligible, and equations are deduced for the mean values, across the breadth, of the displacements u, v . These equations are the same as equations (90), page 182 of the present paper, with the body force null. In order to see the reason of this from our standpoint, we may notice that the assumption that Ξ vanishes eliminates all the solutions of what we have called the dilatational transitory type, and that taking the mean of the displacements eliminates all the flexural solutions, as well as the rotational transitory solutions.

As regards the conditions at the ends, the beam is treated as a long rod.

It may be of interest to remark that the results of § 43 above furnish the data for a more approximate treatment of the problem on the lines followed by Dr FILON.

(c) A note appended to a paper by Professor LAMB in *Proc. Lond. Math. Soc.*, vol. 34 (1902), pp. 283, 284, contains a solution of a special case of the problem of face traction.

(d) In connection with existence theorems relating to the elastic equations, reference should be made to the work of Italian elasticians, as SOMIGLIANA, LAURICELLA, and TEDONE.