Notes on Synthetic Geometry. *By* W. Esson, F.R.S.
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In a course of lectures on Synthetic Geometry, delivered at Oxford in 1894, I endeavoured to place the fundamental conceptions of the subject on a purely synthetic basis. In almost all the treatises results are assumed which have been proved by analytic processes. This is especially the case in the determination of Plücker's characteristics, and in the theory of united elements in correspondences. I hope to show in the following notes that these subjects admit of a purely synthetic treatment.

Definitions.

1. The degree of a plane curve is the number of points determined by the curve upon a straight line. This number remains the same whatever be the position of the straight line in the plane of the curve.

It is understood that the curve is described by a purely graphic method, so that the number of points in which the curve meets a straight line can be ascertained by the mode of description.

2. The class of a plane curve is the number of tangents determined by the curve at a point. This number remains the same whatever be the position of the point in the plane of the curve.

Plücker's Characteristics.

3. The class of a curve described in a given manner is m. By an alteration in the relative positions of the elements of description, a double point, not previously existing, is introduced. The class of the altered curve is $m-2$.

The tangent to each branch of the curve at the double point meets the branch to which it is a tangent in two consecutive points, and the other branch in one point ultimately coincident, but not consecutive, with the former points. This tangent counts as two at the consecutive points and one in addition as drawn from the point on the other branch. The number of remaining tangents determined by the curve at the double point is $m-6$. But in the altered curve the tangents at the double point are one for each branch, each counting as two. The class of the altered curve is therefore

$$m-6+4 = m-2.$$

It is seen that the original curve of class m degenerates into a curve of class $m-2$ and two points coinciding with the double point.

Cor.—If δ double points are introduced, the class of the altered curve is $m-2\delta$.

4. If the altered curve has a cusp, not previously existing, the class is $m-3$.

A cusp arises when by a further alteration in the elements of description the tangents at the double point become coincident and the loop at the point degenerates into a point.

A tangent to the loop from a point on one branch of the curve adjacent to the double point is lost in the altered curve, and the class of the curve is further diminished by 1.

Cor.—If κ cusps are introduced, the class of the altered curve is $m-3\kappa$.

4. If the original curve is of the n^{th} degree and has no double point or cusp, the class of the curve is $n(n-1)$.

Let u_n be the class of the original curve. Suppose that by an alteration in the elements of description the curve degenerates into a similar curve of the $(n-1)^{th}$ degree and a straight line. The altered curve has $n-1$ double points at the intersections of the curves of the $(n-1)^{th}$ degree and straight line, so that the class of the altered curve is, by § 3, diminished by $2(n-1)$, but the class is u_{n-1}, the number of tangents determined at a point by the similar curve of the $(n-1)^{th}$ degree. Hence

$$u_n - u_{n-1} = 2(n-1),$$

and $$u_1 = 0;$$

therefore $$u_n = n(n-1).$$

Cor.—The class of a curve of the n^{th} degree which has δ double points and κ cusps is $n(n-1)-2\delta-3\kappa$.

6. The degree of a curve described in a given manner is n. By an alteration in the relative positions of the elements of description a double tangent, not previously existing, is introduced. The degree of the altered curve is $n-2$.

The point of contact of the double tangent on each branch of the curve determines two consecutive tangents to that branch and one to the other branch ultimately coincident, but not consecutive, with the former tangents. This point of contact counts as two on the consecutive tangents and one in addition as determined by the tangent to the other branch. The number of remaining points determined by the curve on the double tangent is $n-6$. But in the altered curve the points of contact of the double tangent are one for each branch, each counting as two. The degree of the altered curve is therefore

$$n-6+4 = n-2.$$

It is seen that the original curve of degree n degenerates into a curve of degree $n-2$ and two straight lines coinciding with the double tangent.

Cor.—If τ double tangents are introduced, the degree of the altered curve is $n-2\tau$.

7. If the altered curve has an inflexion, not previously existing, the degree is $n-3$.

An inflexion arises when the points of contact of a double tangent become coincident. A tangent to one of the originally distinct portions of the curve at a point adjacent to the point of contact of the double tangent with this portion loses an intersection with the other portion of the curve which the double tangent touches, and the degree of the curve is further diminished by 1.

Cor.—If ι inflexions are introduced, the degree of the altered curve is $n-3\iota$.

8. If the original curve is of the m^{th} class and has no double tangent or inflexion, the degree of the curve is $m(m-1)$.

Let u_m be the degree of the original curve. Suppose that by an alteration of the elements of description the curve degenerates into a similar curve of the $(m-1)^{\text{th}}$ class and a point. The altered curve has $m-1$ double tangents determined at the point by the curve of the $(m-1)^{\text{th}}$ class, so that the degree of the altered curve is diminished by $2(m-1)$, but the degree is u_{m-1}, the number of points determined on a straight line by the similar curve of the $(m-1)^{\text{th}}$ class.

Hence $\qquad u_m - u_{m-1} = 2(m-1), \quad \text{and} \quad u_1 = 0$

therefore $\qquad u_m = m(m-1).$

Cor.—The degree of a curve of the m^{th} class which has τ double tangents and ι inflexions is

$$m(m-1)-2\tau-3\iota.$$

9. The number of conditions which determine a curve described in a given manner is p. By an alteration of the relative positions of the elements of description δ double points and κ cusps, not previously existing, are introduced. The number of conditions which determine the altered curve is $p-\delta-2\kappa$.

As two branches of the altered curve pass through the same point, one condition is lost for each double point, and when the loop of the double point degenerates into a point an additional condition is lost for each cusp.

10. The number of conditions which determine a curve described in a given manner is q. By an alteration of the relative positions of the elements of description τ double tangents and ι inflexions are introduced. The number of conditions which determine the altered curve is $q-\tau-2\iota$.

As the same straight line is touched twice by the curve, one condition is lost for each double tangent, and when the points of contact of the double tangent become coincident an additional condition is lost for an inflexion.

11. The number of conditions which determine a curve of the n^{th} degree which has no double point and no cusp is $\frac{1}{2}n(n+3)$.

Let u_n be this number. Suppose the curve to degenerate into a similar curve of the $(n-1)^{\text{th}}$ degree and a straight line. The degenerate curve has $n-1$ double points, and the number of conditions which determine the altered curve is $u_n-(n-1)$; but the curve of the $(n-1)^{\text{th}}$ degree is determined by u_{n-1} conditions, and the straight line by 2, so that

$$u_n-(n-1)=u_{n-1}+2,$$

or $$u_n-u_{n-1}=n+1, \quad \text{and} \quad u_1=2;$$

therefore $$u_n=\tfrac{1}{2}n(n+3).$$

Cor.—The number of conditions which determine a curve of the n^{th} degree which has δ double points and κ cusps is

$$\tfrac{1}{2}n(n+3)-\delta-2\kappa.$$

12. The number of conditions which determine a curve of the m^{th} class which has no double tangent and no inflexion is $\frac{1}{2}m\,(m+3)$.

Let u_m be this number. Suppose the curve to degenerate into a similar curve of the $(m-1)^{th}$ class and a point. The degenerate curve has $m-1$ double tangents, and the number of conditions which determine the altered curve is $u_m-(m-1)$; but the curve of the $(m-1)^{th}$ class is determined by u_{m-1} conditions, and the point by 2, so that

$$u_m-(m-1)=u_{m-1}+2,$$

or

$$u_m-u_{m-1}=m+1, \quad \text{and} \quad u_1=2;$$

therefore

$$u_m=\tfrac{1}{2}m\,(m+3).$$

Cor. 1.—The number of conditions which determine a curve of the m^{th} class which has τ double tangents and ι inflexions is

$$\tfrac{1}{2}m\,(m+3)-\tau-2\iota.$$

13. The number of conditions which determine a curve is the same in whatever manner it is expressed. Hence for the same curve

$$\tfrac{1}{2}n\,(n+3)-\delta-2\kappa=\tfrac{1}{2}m\,(m+3)-\tau-2\iota.$$

14. The number of points common to two curves one of the n_1^{th} and the other of n_2^{th} degree is $n_1 n_2$.

Consider the curves taken together as a curve of the $(n_1+n_2)^{th}$ degree which has, in addition to the double points and cusps belonging to each curve, a number of double points equal to the number of points common to the two curves; this number is thus half the difference between

$$(n_1+n_2)(n_1+n_2-1)-2\,(\delta_1+\delta_2)-3\,(\kappa_1+\kappa_2)$$

and

$$m_1\,(n_1-1)-2\delta_1-3\kappa_1+n_2\,(n_2-1)-2\delta_2-3\kappa_2,$$

the class of the compound curve; *i.e.*, the number is $n_1 n_2$.

15. The number of tangents common to two curves one of the m_1^{th} class and the other of the m_2^{th} class is $m_1 m_2$.

Consider the curves taken together as a curve of the $(m_1+m_2)^{th}$ class, which has, in addition to the double tangents and inflexions common to each, a number of double tangents equal to the number of tangents common to the two curves; this number is thus half the difference between

$$(m_1+m_2)(m_1+m_2-1)-2(r_1+r_2)-3(\iota_1+\iota_2)$$

and $$m_1(m_1-1)-2r_1-3\iota_1+m_2(m_2-1)-2r_2-3\iota_2,$$

the degree of the compound curve; *i.e.*, the number is m_1m_2.

16. The maximum number of double points which a proper curve of the n^{th} degree can have is $\frac{1}{2}(n-1)(n-2)$.

If a curve has one more than the maximum number, it degenerates into two curves of lower degree. Let these two curves be a curve of the $(n-1)^{\text{th}}$ degree with the maximum number of double points and a straight line. Let u_n be the maximum number for a curve of the n^{th} degree. The straight line and curve of the $(n-1)^{\text{th}}$ degree have together $u_{n-1}+n-1$ double points, so that

$$u_n+1 = u_{n-1}+n-1,$$

or $$u_n-u_{n-1} = n-2 \quad \text{and} \quad u_2 = 0;$$

therefore $$u_n = \tfrac{1}{2}(n-1)(n-2).$$

17. The maximum number of double tangents which a proper curve of the m^{th} class can have is $\frac{1}{2}(m-1)(m-2)$.

If a curve has one more than the maximum number, it degenerates into two curves of lower class. Let the two curves be a curve of the $(m-1)^{\text{th}}$ class with the maximum number of double tangents and a point. Let u_m be the maximum number for a curve of the m^{th} class. The point and curve of the $(m-1)^{\text{th}}$ class have together $u_{m-1}+m-1$ double tangents, so that

$$u_m+1 = u_{m-1}+m-1,$$

or $$u_m-u_{m-1} = m-2 \quad \text{and} \quad u_2 = 0;$$

therefore $$u_m = \tfrac{1}{2}(m-1)(m-2).$$

18. It is convenient to express Plücker's characteristics in terms of the degree n, class m, and deficiency D.

We have

$$n(n-1)-m-2\delta-3\kappa = m(m-1)-n-2\tau-3\iota = 0,$$

$$\tfrac{1}{2}n(n+3)-\delta-2\kappa = \tfrac{1}{2}m(m+3)-\tau-2\iota,$$

whence, by subtraction,

$$\tfrac{1}{2}(n-1)(n-2)-\delta-\kappa = \tfrac{1}{2}(m-1)(m-2)-\tau-\iota = D,$$

and
$$\kappa = 2(n-1) - m + 2D, \qquad \text{(i.)}$$
$$\iota = 2(m-1) - n + 2D, \qquad \text{(ii.)}$$
$$\delta = \tfrac{1}{2}(n-1)(n-6) + m - 3D, \qquad \text{(iii.)}$$
$$\tau = \tfrac{1}{2}(m-1)(m-6) + n - 3D. \qquad \text{(iv.)}$$

Correspondences.

19. *Definition.*—If two sets of elements of the same kind are such that to one element of one set correspond n elements of the other set, the groups of n elements of the second set are said to be in involution.

20. There are $2(n-1)$ double elements of an involution of the n^{th} order.

Let the involution be represented by rays of a linear pencil each group of n rays of which corresponds to a single ray of another linear pencil.

To the join of the centres of the pencil correspond, (1) n rays of the first pencil which are tangents at the centre of this pencil to the locus of intersection of corresponding rays of the two pencils, (2) one ray of the second pencil which is tangent at the centre of this pencil to the same locus. There is no double ray of the second pencil, and therefore no tangent to the locus from the centre of the first pencil except those at the centre, each of which counts as two. The locus is therefore of the $2n^{\text{th}}$ class. Of the $2n$ tangents determined by the locus at the centre of the second pencil, the tangent at the centre counts as two; the remaining $2(n-1)$ tangents are those rays of the second pencil which correspond to the double rays of the first pencil, of which the number is therefore $2(n-1)$.

21. Two involutions whose corresponding elements admit of coincidence, and which have a (m, n) correspondence, have $m+n$ united corresponding elements.

Let the involutions be projective with two linear pencils. The locus of intersection of their corresponding rays is of the $(m+n)^{\text{th}}$ degree, for the join of their centres meets the locus in $m+n$ points, viz., m at one centre and n at the other, determined by the rays which correspond to the join of the centre.

Any other straight line meets the locus in $m+n$ points, viz., the united corresponding points of the involutions determined on the straight line by the pencils. The number of united corresponding elements of the original involutions is therefore $m+n$.

The locus employed in the preceding proof has interesting characteristics.

22. The class of the locus when it has no cusps and no multiple points except at the centres of the pencils is $2mn$.

At the centre of one pencil the locus determines $2mn$ tangents, viz., $2m$ at the centre and $2m(n-1)$ determined as the rays corresponding to the $2(n-1)$ double rays of the other pencil. The class of the locus is therefore $2mn$.

23. The deficiency of such a locus is $(m-1)(n-1)$.

The multiple points at the centres are equivalent to

$$\tfrac{1}{2}m(m-1)+\tfrac{1}{2}n(n-1)$$

double points, and the deficiency is therefore the excess of

$$\tfrac{1}{2}(m+n-1)(m+n-2)$$

over this number of double points, viz., $(m-1)(n-1)$.

24. The Plücker's characteristics of such a locus are, by § 18,

$$\begin{aligned}\kappa &= 0,\\ \iota &= 3(2mn-m-n),\\ \delta &= \tfrac{1}{2}m(m-1)+\tfrac{1}{2}n(n-1),\\ \tau &= 2(mn-1)(mn-2)-4(m-1)(n-1).\end{aligned}$$

25. When $m=1$, the locus has no deficiency, and in this case (the locus in § 20, of degree $n+1$ and class $2n$),

$$\begin{aligned}\kappa &= 0,\\ \iota &= 3(n-1),\\ \delta &= \tfrac{1}{2}n(n-1),\\ \tau &= 2(n-1)(n-2).\end{aligned}$$

26. The number of conditions which determine the locus of the $(m+n)^{\text{th}}$ degree and class $2mn$ which has one given multiple point of the m^{th} and one of the n^{th} order is $(m+1)(n+1)+1$.

Each multiple point of the m^{th} order is determined by $\tfrac{1}{2}m(m+1)$ conditions, so that the number of conditions including the points which are assigned to be of multiplicity m and n respectively, the number of conditions is

$$\tfrac{1}{2}(m+n)(m+n+3)-\tfrac{1}{2}m(m+1)-\tfrac{1}{2}n(n+1)+2 = (m+1)(n+1)+1.$$

27. A pencil of such curves which has two given multiple centres, one of the m^{th} and another of the n^{th} order, and $(m+1)(n+1)-2$ other given centres, has $(m-1)(n-1)$ other centres determined by the given centres.

By § 26 the pencil is determined by $(m+1)(n+1)$ conditions, *i.e.*, by $(m+1)(n+1)-2$ other points in addition to the two multiple centres; but two curves of the pencil intersect in

$$m^2+n^2+(m+1)(n+1)-2 = (m+n)^2-(m-1)(n-1)$$

points at the given centres, and therefore in $(m-1)(n-1)$ other points which are also centres of the pencil of curves.

28. When $m=1$, the number of conditions which determine the locus of the $(n+1)^{th}$ degree and $2n^{th}$ class which has one given multiple point of the n^{th} order is $2n+3$.

29. When $m=1$, a pencil of such curves is completely determined by one given multiple centre of the n^{th} order and $2n+1$ other centres.

30. *Example.*—A pencil of cubics of the fourth class is determined by five single centres and one double centre.

Construct the conic a determined by the five single centres, and the conic b determined by the double centre and any four of the single centres. Take any fixed point O on a. A ray through O determines a point P on a and two points Q on b; P and Q determine at the remaining single centre and at the double centre respectively rays which have a (1, 2) correspondence and whose intersections therefore lie on a cubic of the fourth class having a double point at the given double centre. Each position of O determines one such cubic passing through all the given centres.