Preliminary Sketch of Biquaternions. *By* Prof. Clifford, M.A.

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I.

The *vectors* of Hamilton are quantities having magnitude and direction, but no particular position; the vector AB being regarded as identical with the vector CD when AB is equal and parallel to CD and in the same sense. The translation of a rigid body is an example of such a quantity; for since all particles of the body move through equal distances along parallel straight lines in the same sense, the motion is entirely specified by a straight line of the given length and direction drawn through any point whatever. A couple, again, may be adequately represented by a vector; since the axis of a couple is any line of length proportional to its moment drawn perpendicular from a given face of its plane.

For many purposes, however, it is necessary to consider quantities which have not only magnitude and direction, but *position* also. The rotational velocity of a rigid body is about a certain definite axis, and equal rotations about two parallel axes are not equivalent to one another. A force acting upon a solid has a definite line of action, and equal forces acting along parallel lines differ by a certain couple. The difference between the two kinds of quantities is clearly seen when we consider the geometric calculus which is used for the study of each. In studying the motions of a particle or the composition of couples, the only construction required is that of the "force-polygon," and the theory involved is that of the addition of vectors; but in the static or kinematic of solids we require in addition the construction of the "link-polygon," and there is involved the theory of the involution of lines in space, or of the linear complex.

The name *vector* may be conveniently associated with a velocity of *translation*, as the simplest type of the quantity denoted by it. In analogy with this, I propose to use the name *rotor* (short for *rotator*) to mean a quantity having magnitude, direction, and position, of which the simplest type is a velocity of *rotation* about a certain axis. A rotor will be geometrically represented by a length proportional to its magnitude measured upon its axis in a certain sense. The rotor AB will be identical with CD if they are in the same straight line, of the same length, and in the same sense; *i. e.*, a vector may move anywise parallel to itself, but a rotor *only* in its own line.

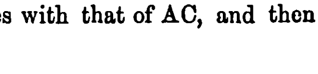
The *addition* of rotors will proceed by the rules which govern the composition of forces and rotations. Here, however, we come upon a very important break in the analogy between rotors and vectors. While the sum of any number of vectors is always a vector, it will only happen in special cases that the sum of a number of rotors is a rotor. In fact, the composition of two forces whose lines of action do not intersect, or of two rotation-velocities whose axes do not intersect, gives rise to a system of forces on the one hand, and the most general velocity of a rigid body on the other. These still more complex quantities have been studied, and the theory of their addition or composition completely worked out, by Dr. Ball.

A system of forces may be reduced in one way to a single force P, and a couple G whose plane is perpendicular to the line of action of the force, or *central axis.* Dr. Ball speaks of the system of forces as a *wrench* about a certain *screw;* the axis of the screw being the central axis, and the pitch being the ratio $\frac{G}{P}$ of the couple to the force. Similarly the velocity of a rigid body may be represented in one way only as a rotation-velocity ω about a certain axis combined with a translation-velocity v along that axis. Dr. Ball speaks of this velocity as a *twist*-velocity about a certain screw; the axis of the screw being the axis of rotation, and its pitch the ratio $\frac{v}{\omega}$ of the translation to the rotation. A *screw* is here a geometrical form resulting from the combination of an *axis* or straight line given in position with a *pitch* which is a linear magnitude. A *wrench* is the association with this geometrical form of a magnitude whose dimensions are those of a force; a *twist*-velocity the association of a magnitude whose dimensions are those of an angular velocity. The extreme convenience of this nomenclature is well exemplified in the remarkable memoir above referred to.

Just as a vector (translation-velocity or couple) is magnitude associated with direction, and as a rotor (rotation-velocity or force) is magnitude associated with an axis; so this new quantity, which is the sum of two or more rotors (twist-velocity or wrench) is magnitude associated with a screw. Following up the analogy thus indicated, I propose to call this quantity a *motor;* the simplest type of it being the general motion of a rigid body. And we shall say that in general the sum of rotors is a motor, but that in particular cases it may degenerate into a rotor or a vector.

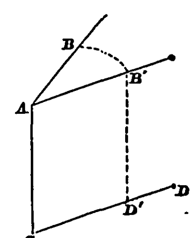
II.

A *quaternion* is the ratio of two vectors, or the operation necessary to make one into the other. Let the vectors be AB and AC, as they may both be made to start from any arbitrary point A. Then AB is made into AC by turning it round an axis through A perpendicular to the plane BAC until its direction coincides with that of AC, and then



magnifying or diminishing it until it is of the same length as AC. The ratio of two vectors then is the combination of an ordinary numerical ratio with a *rotation*; or, as Hamilton expresses it, a quaternion is the product of a tensor and a versor. Since the point A is perfectly arbitrary, this rotation is not about a definite axis; but is completely specified when its angular magnitude and the direction of its axis are given.

This quaternion* $\frac{AC}{AB} = q$, then, is an operation which, being performed on AB, converts it into AC, so that $q \, . \, AB = AC$. The axis of the quaternion is perpendicular to the plane BAC; and it is clear that the quaternion operating upon any other vector AD in this plane will convert it into a fourth vector AE in the same plane, the angle DAE being equal to BAC and the lengths of the four lines proportionals. But a quaternion can *only* operate upon a vector which is perpendicular to its axis. If AF be any vector not in the plane BAC, the expression $q \, . \, AF$ is absolutely unmeaning. A meaning is indeed subsequently given to an analogous expression *in which the signification of* AF *is different.* But it is very important to remark that so long as AF means a *vector* not perpendicular to the axis of q, the expression $q \, . AF$ has no meaning at all.



Let us now consider what is the operation necessary to convert one *rotor* into another. There is one straight line which meets at right angles the axes of any two rotors, and part of which constitutes the shortest distance between them. Let AC be the shortest distance between the rotors AB and CD. Then AB may be converted into CD by a process consisting of three steps. First, turn AB about the axis AC into the position AB′, parallel to CD. Then slide it along this axis into the position CD′. Lastly, magnify or diminish it in the ratio of CD′ to CD. The first two operations may be regarded as together forming a twist about a screw whose axis is AC and whose pitch is

$$\frac{AC}{\text{circ. meas. of } BAB'}.$$

The ratio of two rotors, then, is the combination of an ordinary numerical ratio with a *twist.* This twist is associated with a perfectly

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* Professor Cayley, by a very convenient notation, distinguishes $\frac{|AC}{AB|}$ and $\frac{AC|}{|AB}$; viz., $AB\frac{|AC}{AB|} = 1$, but $\frac{AC|}{|AB}AB = 1$. It should, I think, be a convention that $\frac{X}{Y}$ is *always* to mean $\frac{X|}{|Y}$, viz., the operation which converts Y into X, *or* which, coming after the operation Y, is equivalent to the operation X.

definite screw, and is only specified when its angular magnitude and the screw (involving direction, position, and pitch) are given. We may say also that just as the rotation (versor) involved in a quaternion is the ratio of two directions, so the twist involved in the ratio of two rotors is really the ratio of their axes.

Here again a remark must be made about the range of this operation. Using the expression *tensor-twist* to mean the ratio of two rotors (which is in fact a twist multiplied by a tensor), we may say that a tensor-twist can operate upon any rotor which meets its axis at right angles. Let t denote the operation which converts AB into CD, so that $t = \frac{CD}{AB}$, and $t.AB = CD$; then if EF be any other rotor which meets AC at right angles, the expression $t.EF$ will have a definite meaning, viz., it will mean a rotor obtained by sliding EF along a distance equal to AC, turning it about AC as axis through an angle equal to BAB′, and altering its length in the ratio AB : CD. But if EF be a rotor not meeting AC, or meeting it at any other than a right angle, the expression $t.EF$ will have no meaning whatever.

We have now defined the ratio of two rotors, and shown that like a quaternion it has a restricted range of operation. The question naturally arises, what now is the operation which converts one *motor* into another? We can answer this question very easily in the case in which the two motors have the same pitch; for in this case their ratio is a tensor-twist whose tensor is the ratio of their magnitudes and whose twist is the ratio of their axes. We are led to this by considering each motor as the sum of two rotors which do not intersect. Let α and β be two such rotors, t a tensor-twist whose axis meets them both at right angles; then $t\alpha$ is a rotor, say γ, and $t\beta$ is another rotor, say δ. If therefore we assume the distributive law, we have

$$t(m\alpha + n\beta) = m\gamma + n\delta,$$

or

$$t = \frac{m\gamma + n\delta}{m\alpha + n\beta}.$$

It is a mere translation of known theorems to say that the axis of t meets at right angles the axes of the motors $m\alpha + n\beta$ and $m\gamma + n\delta$, and that one of these axes is converted into the other by the same twist that makes α into γ or β into δ.

The solution of this problem in the general case in which the pitches are different, is not so easy. In the first place, we must remember that every motor consists of a rotor part and a vector part, and that its pitch is determined by the ratio of these two parts. By combining a suitable vector with a motor, therefore, we may make the pitch of it anything we like, without altering the rotor part. Now let it be required to find the operation which will convert a motor A into a motor

B. Let B′ be a motor having the same rotor part as B, and the same pitch as A; and let $B = B' + \beta$, where β is a vector parallel to the axis of B. Then the ratio $\frac{B}{A} = \frac{B'}{A} + \frac{\beta}{A}$; but $\frac{B'}{A}$ is a tensor-twist, say t, and we may write $$\frac{B}{A} = t + \frac{\beta}{A},$$ where it now only remains to find an operation which will convert a motor A into a vector β.

In order to do this, we must introduce a symbol whose nature and operation will at first sight appear completely arbitrary, but will be justified in the sequel. *The symbol ω, applied to any motor, changes it into a vector parallel to its axis and proportional to the rotor part of it.* That is to say, it changes rotation about any axis into translation parallel to that axis, and a force into a couple in a plane perpendicular to its line of action. But if the rotation is accompanied by translation or the force by a couple, the symbol takes no account whatever of these accompaniments; and if made to operate directly on a vector, reduces it to zero. It follows from this that if it be made to operate twice upon a motor, it reduces it to zero; or $\omega^2 A = 0$ always. The portion of any expression which involves ω must therefore be treated as an infinitesimal of the first order; all higher orders being uniformly neglected.

Since then $\omega A = \alpha$, a vector, and the ratio $\frac{\beta}{\alpha}$ is a quaternion q so that $q\alpha = \beta$, we may write successively

$$\beta = q\alpha = q\omega A,$$

$$\frac{\beta}{A} = q\omega,$$

and then $$\frac{B}{A} = t + q\omega,$$

or *the ratio of two motors may be expressed as the sum of two parts, one of which is a tensor-twist, and the other is ω multiplied by a quaternion.*

The same ratio may be expressed in another form. Let an arbitrary point O be assumed as the origin; then every motor may be expressed in one way as the sum of a rotor passing through O and a vector. Now the theory of rotors passing through a fixed point is exactly the same as that of vectors in general, and the ratio of any two of them is a tensor-twist whose pitch is zero, or what is the same thing, a quaternion whose axis is constrained to pass through the fixed point. If we use cursive Greek letters (as α, β) in general to represent rotors through the origin, we may distinguish vectors from them by prefixing the symbol ω; thus $\omega\alpha$ denotes a vector parallel and proportional to the rotor α. The ratio $\frac{\beta}{\alpha}$ will then be a quaternion q, which is also the ratio $\frac{\omega\beta}{\omega\alpha}$.* The general expression for a motor is then $\alpha+\omega\beta$. Let it now be required to find the ratio of two motors $\alpha+\omega\beta$, $\gamma+\omega\delta$; or the value of the expression $\frac{\gamma+\omega\delta}{\alpha+\omega\beta}$.

First, let $\frac{\gamma}{\alpha}=q$; then $q(\alpha+\omega\beta)=\gamma+q\omega\beta=\gamma+\omega q\beta$.

The symbol $q\beta$ has at present no geometrical meaning; for in general the rotors α, β, γ will not be coplanar, and cannot therefore be operated on by the same quaternion q. If however (as in the Calculus of Quaternions) we consider all these quantities as expressed in terms of three rectangular unit rotors through the origin, $\frac{\delta-q\beta}{\alpha}$ will be a perfectly definite quaternion r. The equation

$$r\alpha=\delta-q\beta$$

is, like the equation $q(\alpha+\omega\beta)=\gamma+\omega q\beta$,

at present purely literal and devoid of meaning. Yet if (remembering the properties of the symbol ω) we add ω times the first equation to the second and assume the distributive law, we obtain

$$(q+\omega r)(\alpha+\omega\beta)=\gamma+\omega\delta.$$

In this way the ratio $\frac{\gamma+\omega\delta}{\alpha+\omega\beta}$ is expressed in the form $q+\omega r$, which expression may conveniently be called a *biquaternion.*† The final equation, however, is not susceptible of interpretation in the same sense as the equation $q\alpha=\gamma$. The expression $q+\omega r$ does not denote the sum of geometrical operations which can be applied to the motor $\alpha+\omega\beta$ as a whole; and the ratio of two motors is only expressed by a symbol as the sum of two parts, each of which separately has a definite meaning in certain other cases, but not in the case in point. In following sections this difficulty will be partly overcome by showing that the system here sketched is the limit of another in which it does not occur.

The preceding remarks may however explain, and be illustrated by, the following table:—

Geometrical Form	Quantity	Example	Ratio
Sense on st. line	Vector on st. line	Addition or Subtraction	Signed Ratio
Direction in plane	Vector in plane	Complex quantity	Complex Ratio
Direction in space	Vector in space	Translation, Couple	Quaternion
Axis	Rotor	Rotation-Velocity, Force	Twist
Screw	Motor	Twist-Velocity, System of Forces	Biquaternion

* It follows from this that $\omega q=q\omega$, or ω is commutative with quaternions.

† Hamilton's *biquaternion* is a quaternion with complex coefficients; but it is convenient (as Prof. Peirce remarks) to suppose from the beginning that all scalars may be complex. As the word is thus no longer wanted in its old meaning, I have made bold to use it in a new one.

III.

That geometry of three-dimensional space which assumes the Euclidian postulates has been called by Dr. Klein the *parabolic* geometry of space, to distinguish it from two other varieties, which assume uniform positive and negative curvature respectively, and which he calls the *elliptic* and *hyperbolic* geometry of space. The investigations which follow involve the postulates of elliptic geometry. As, however, the postulate of uniform positive curvature is not sufficient to define this, it may be worth while to devote a short space to an explanation of its nature.

Space of three dimensions is that the points of which may be associated with systems of values of three variables x, y, z. It is not in general possible, however, so to make this association that to every system of values there shall correspond in general one point, and to every point in general one system of values. When this is the case, the space is called *unicursal.* An *algebraic* space is one in which the position of a point may be uniquely defined by a set of values of periodic algebraic integrals, without exceptions which form a part of the space. Thus, unicursal spaces are a particular case of algebraic. Attending now to unicursal spaces only, we must observe that there are in general exceptions to the unique correspondence of points and value-systems; namely, there are certain points to each of which correspond an infinite number of values of the coordinates satisfying a certain equation or equations; and there are certain value-systems to which correspond, not points, but loci in the space. The assignment of these point-equations and loci-values and of their relations with one another serves to determine the *projective-connection* of the space; and when once these are known, the whole of its projective geometry may be worked out. The point-equations and loci-values may or may not involve imaginary values of the variables or their coefficients; but in all cases they must be taken into account. The points which correspond to real systems of values are called real points; those which correspond to imaginary systems, imaginary points: the study of these latter, which does not strictly belong to that of three-dimensional space, is undertaken only for the sake of the former.

Loci which correspond to linear equations between the coordinates may at present be called *planes*, and their intersections *lines;* this is a purely projective definition, and these loci are not necessarily *flat* planes and *straight* lines in the metrical sense. Points, lines, and planes are included in the name *elements.*

The *metric* geometry of space* is the theory of the projective relations of certain fixed geometrical forms with all other geometrical forms, or of the invariant relations of certain fixed algebraic forms with all other

* This theory of metric geometry is due to Prof. Cayley: Sixth Memoir on Quantics, Phil. Trans., 1859.

algebraic forms. The word *power* will be explained as much as is wanted in the sequel; meanwhile it may be said that these fixed forms (called all together *the absolute*) are given when we know the points, the lines, and the planes of the absolute, or say the elements of the absolute; and that the power of an element of the absolute in regard to any arbitrary element is infinite. In other words, we *require* in general equations of the absolute in point-, line-, and plane-coordinates respectively.

A unicursal space the points of which may be represented uniquely by value-systems of the coordinates x, y, z, without the exception of any point-equations or loci-values, is called a *linear* space. This is merely a projective definition, and leaves the absolute, therefore the whole of metric geometry, undetermined.

There is a particular determination of the absolute in a linear space which is of the utmost importance. It is that in which the points of the absolute are those of a certain quadric surface, while the lines and planes of the absolute are those which touch this surface; or in which the three equations of the absolute are of the second degree. There are three cases* to be considered, as being the only ones of which observed space can form a part:—

(1) *Elliptic* geometry; all the elements of the absolute are imaginary.

(2) *Hyperbolic* geometry; the absolute contains no real straight lines, and surrounds us. In this case, real points situate on the other side of the surface are called *ideal.*

(3) *Parabolic* geometry; the surface degenerates into an imaginary conic in a real plane. The points of the absolute are points in the (real) plane of this conic; the lines and planes are the imaginary lines and planes which meet and touch the conic respectively.

The *first* of these suppositions will be made in what follows. It may be well here to set down in what it consists.

(1) The space to be considered is such that there is one point of it for every set of values of the coordinates x, y, z, and one set of values for every point, without any exception whatever.

(2) There is a certain quadric surface, called the absolute, all whose points and tangent planes are imaginary. If the line joining two points a, b meet the absolute in i, j, the quantity

$$\frac{ab \,.\, ij}{\sqrt{(ai \,.\, aj \,.\, bi \,.\, bj)}} \equiv \overline{ab},$$

(which is a function of anharmonic ratios, and therefore an invariant,) is called the *power* of the points a, b in regard to one another, or of

* On this division see Dr. Klein, "Ueber die so-genannte Nicht-Euklidische Geometrie," Math. Annalen, Bd. 4. The second case is the geometry of Lobatschewsky and Bolyai.

either in regard to the other. The *distance* of these two points is an angle θ such that $\sin\theta = \overline{ab}$.

Similarly, if through the line of intersection of the planes A, B there be drawn the tangent planes I, J to the absolute, the power of the planes A, B in regard to one another is the quantity

$$\frac{\mathrm{AB}\,.\,\mathrm{IJ}}{\sqrt{(\mathrm{AI}\,.\,\mathrm{AJ}\,.\,\mathrm{BI}\,.\,\mathrm{BJ})}} = \overline{\mathrm{AB}},$$

and the angle between them is an angle ϕ such that

$$\sin\phi = \overline{\mathrm{AB}}.$$

(3) If two points are conjugate in regard to the absolute, they are distant a *quadrant* from one another; if two lines or planes are conjugate in regard to the absolute, they are at right angles. Thus all the points at a quadrant distance from a given point are situate on its polar plane in regard to the absolute, and every plane through it cuts this polar plane at right angles. Every line has a polar line in regard to the absolute, such that every point on the polar line is distant a quadrant from every point on the line; and every line which is at right angles to either meets the other. Through an arbitrary point can in general be drawn *one* line perpendicular to a given plane; namely, the line joining the point to the pole of the plane. If, however, the point *is* the pole of the plane, every line through it is perpendicular to the plane. Similarly, from a point not on the polar of a given line can be drawn one and only one perpendicular to the line; namely, the line through the point which meets the given line and its polar.

(4) *In general, two lines can be drawn so that each meets two given lines at right angles, and these are polars of one another.* One line may therefore be converted into another by rotation about two polar axes. These axes are determined as the lines which meet the two given lines and their polars. If we travel continuously along one of these lines and draw perpendiculars on the other, one of these axes determines the shortest distance between the lines, and the other the longest. If then these two are equal, the lines are equidistant along their whole length. Thus *there is a case of exception in which two lines and their polars belong to the same set of generators of a hyperboloid; the lines are then equidistant along their whole length, and meet the same two generators of one system of the absolute.* I shall use the word *parallel* to denote two lines so situated; and they shall be called *right* parallel or *left* parallel according as one is converted into the other by a right-handed or left-handed twist. Through an arbitrary point can be drawn one right parallel and one left parallel to a given line; the angle between them is twice the distance of the point from the line. There are many points of analogy between the *parallels* here defined and those of parabolic geometry. Thus, if a line meet two parallel lines, it makes equal

angles with them; and a series of parallel lines meeting a given line constitute a ruled surface of zero curvature. The geometry of this surface is the same as that of a finite parallelogram whose opposite sides are regarded as identical.

(5) A twist-velocity of a rigid body must be regarded as having *two* axes. For a motion of translation along any axis is the same thing as a rotation about the polar axis, and *vice versâ*. Hence a twist-velocity is compounded of rotation-velocities about two polar axes; say these are θ, ϕ. Then the motion may be regarded either as a twist-velocity about a screw whose pitch is $\frac{\phi}{\theta}$ and whose axis is the first axis, or about a screw whose pitch is $\frac{\theta}{\phi}$ and whose axis is the polar axis. In general, then, a motor has two axes, and is expressible in one way only as the sum of two polar rotors. There is, however, one case of exception in which the axes of a motor are indeterminate; that, namely, in which the magnitudes of the two polar rotors are equal.* If a rigid body receive at the same time a rotation about an axis and an equal translation along it, all the points of the body will describe parallel straight lines; and the motion of the body is at the same time a rotation about any one of these lines combined with an equal translation along it. Such a motion may be adequately represented by a line of given length drawn through any point whatever parallel to a given line. A motor of pitch unity, or which is its own polar, may therefore be regarded as having the nature of a *vector*, and shall in future be denoted by that name. For we may define a vector as a motor whose axes are indeterminate; and the case we are now considering is the only case of such indetermination which occurs in elliptic geometry. Vectors will be called *right* or *left* according as the twist of them is right- or left-handed.

Prop.: *Every motor is the sum of a right and a left vector.* For let A be a motor, and A′ the polar motor; then we have $A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A')$. Now $A+A'$ and $A-A'$ are both motors of pitch unity, but one right-handed and the other left-handed.

IV.

A fixed point being chosen as origin, let three lines perpendicular to one another be drawn through it, and let three unit-rotors having these lines as axes be denoted by the symbols i, j, k. Then every rotor through the origin will be denoted by an expression of the form $ix+jy+kz$, where x, y, z are scalar quantities, or the ratios of magnitudes. The symbols i, j, k shall have also another meaning; viz., each

* This motion is described in another connection by Drs. Klein and Lie, Math. Annalen, Bd. 4; it is a transformation of the absolute into itself in which two generators remain unaltered.

shall signify the rotation through a right angle about its axis of any rotor which meets that axis at right angles. When they are performed on rotors passing through the origin, these operations satisfy the equations $i^2 = j^2 = k^2 = ijk = -1$, by the ordinary rules of quaternions; and it is easy to see that the same equations hold good when the operations are performed on rotors not passing through the origin. The compound symbol $ix+jy+kz$ is also to have an analogous secondary meaning; viz., a rectangular rotation about the axis of the rotor which it previously denoted, combined with a tensor $\sqrt{(x^2+y^2+z^2)}$. It can operate only on a rotor which meets its axis at right angles. This being so, the ratio of any two rotors through the origin is a *quaternion* of the form $q \equiv w+ix+jy+kz \equiv w+\rho$, say. The axis ρ of this quaternion is perpendicular to the plane of the two rotors. If a be a rotor through the origin and q a quaternion, the product qa can be formed according to the Hamiltonian rules of multiplication, and is in general a quaternion r. In this general case the equation $qa = r$ can only be interpreted by giving to a its *secondary* meaning; and the translation of this statement into words is as follows:—If a rotor be capable of being successively operated upon by the rectangular versor a and the quaternion q, the final result will be the same as if it had been originally operated upon by the quaternion r. If, however, the axes of q and a are at right angles, the scalar part of r will be wanting, and we may write the equation $qa = \rho$. This equation is now susceptible of a *primary* interpretation; viz., the quaternion q operating on the rotor a produces the rotor ρ; although the *secondary* interpretation does not cease to be true.

With such conventions, the two sides of the equation

$$(q+r)\,s = qs+rs$$

(in which q, r, s are quaternions) have always the same meaning when both are interpretable; which is what is meant by saying that the distributive law holds good for these symbols.

The ratio of two rotors which do not meet is a twist which in general has perfectly definite axes. But when the rotors are polars of one another, the axes of the twist are indeterminate; for any line meeting both meets them at right angles, and will serve for an axis. It is therefore always possible to find a twist which shall simultaneously convert two given rotors into their polars; and any two rectangular twists with pitch 1 or -1 have a pair of common rotors on which they can operate, and which they convert into one another. Hence we may consider that

All rectangular twists of pitch 1 *are equivalent to one another; and all rectangular twists of pitch* -1 *are equivalent to one another.*

The rectangular twist of pitch 1 shall be denoted by the symbol ω; the expression ωa will denote the rotor polar to a and equal to it in magnitude, obtained from it by a left-handed twist. During the operation of this twist, every point of the rotor describes a straight line; if therefore the twist be continued through two right angles, the rotor will be replaced in its original position, *not* reversed; we have therefore $\omega^2 = 1.$

Every motor can be expressed as the sum of two rotors, one passing through the origin and the other being polar to a rotor through the origin. The general expression for a motor is therefore

$$\alpha + \omega\beta.$$

This will represent a *rotor* if the two rotor constituents intersect, or if each is perpendicular to the polar of the other; or if $S\alpha\beta = 0$.

Let now $$\xi = \frac{1+\omega}{2}, \quad \eta = \frac{1-\omega}{2};$$

then $$\xi^2 = \frac{1+2\omega+\omega^2}{4} = \frac{2+2\omega}{4} = \xi,$$

$$\eta^2 = \frac{1-2\omega+\omega^2}{4} = \frac{2-2\omega}{4} = \eta,$$

$$\xi\eta = \frac{1-\omega^2}{4} = 0.$$

Any motor $\alpha+\omega\beta$ can also be expressed in the form $\xi\gamma+\eta\delta$. It is clear that $\xi\gamma$ is the right vector part of this motor, and that $\eta\delta$ is the left vector part. If we multiply $\xi\gamma+\eta\delta$ by ξ, the result is merely $\xi\gamma$; so the effect of multiplying a motor by ξ is merely to pick out the right vector part of it. The symbols ξ, η are thus in a certain sense *selective* symbols, and are analogous to the S and V of quaternions.

Ratio of two motors.—We can find immediately now the operation which converts a motor $\xi\gamma+\eta\delta$ into a motor $\xi\alpha+\eta\beta$. For if we perform the operation $$\left(\xi\frac{\alpha}{\gamma} + \eta\frac{\beta}{\delta}\right)(\xi\gamma+\eta\delta),$$

remembering the laws of multiplication of ξ, η, we obtain the result $\xi\alpha+\eta\beta$. If then $\frac{\alpha}{\gamma} = q$, $\frac{\beta}{\delta} = r$, we may write

$$\frac{\xi\alpha+\eta\beta}{\xi\gamma+\eta\delta} = \xi\frac{\alpha}{\gamma} + \eta\frac{\beta}{\delta} = \xi q + \eta r,$$

and the latter may be written in the form

$$\frac{q+r}{2} + \omega\,.\,\frac{q-r}{2} = s + \omega t,$$

showing that *the ratio of two motors is a biquaternion.*

The motor $\xi\alpha+\eta\beta$ will be a *rotor* if

$$S(\alpha+\beta)(\alpha-\beta) = 0,$$

or if $$T\alpha = T\beta;$$

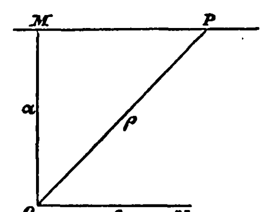
and it is easy to see from this that the biquaternion $\xi q+\eta r$ will be a *twist*, or the ratio of two rotors, if $Tq = Tr$.

V.

1. *Position-Rotor of a Point.*—The coordinates of a point in regard to a quadrantal tetrahedron 1234 being x_1, x_2, x_3, x_4, the equation to the absolute is $\Sigma x^2 = 0$. The rotor from the origin (the point 4) to the point x is represented by $i_1\frac{x_1}{x_4} + i_2\frac{x_2}{x_4} + i_3\frac{x_3}{x_4}$, or $\Sigma i_k\frac{x_k}{x_4}$ $(k = 1, 2, 3)$, where i_1, i_2, i_3 are rotors along the edges of the tetrahedron from the origin to the middle points of the edges. The tensor of this rotor is the tangent of the angular distance from the origin to the point it represents. For if
$$\rho = i_1\frac{x_1}{x_4} + i_2\frac{x_2}{x_4} + i_3\frac{x_3}{x_4},$$
$$[T\rho]^2 = \frac{x_1^2 + x_2^2 + x_3^2}{x_4^2} = \tan^2 \widehat{ox}, \text{ where } o \text{ is the origin.}$$

The angular distance from the origin to a point has an infinite number of values, which differ by multiples of π. If therefore a rotor be considered to have this angular distance as its length, the rotor of a point can only be defined by such an equation as $\breve{\rho} \equiv \breve{\alpha}$ (mod. $\breve{\pi}_\cdot$). To obviate this indetermination, there is required a one-valued unicursal function having the period π; the tangent of the angular distance is hereby completely singled out.

2. *Equation of a Straight Line.*—Let OM be the perpendicular from the origin O upon the straight line MP; and let ON be a line perpendicular to OM in the plane MOP. Then from the triangle MOP we have
$$\frac{\tan \text{OM}}{\tan \text{OP}} = \cos \text{MOP};$$
or if $\text{OM} = \alpha,\quad \text{OP} = \rho,\quad \text{ON} = \beta,$
$$T\alpha = T\rho \cos \text{MOP};$$



so that α is the component of ρ in the direction OM, and we have $\rho = \alpha + \beta x$, where x is some scalar.

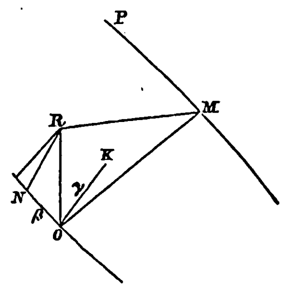
By varying x, then, we get all the points in the line MP. But if α_1 is *any* particular value of ρ, the equation may just as well be written
$$\rho = \alpha_1 + \beta x,$$
where now α_1 is not necessarily perpendicular to β.

This form may be reduced to the preceding as follows:

To find the perpendicular from O, put $\delta T\rho = 0$; this gives
$$S\alpha_1\beta + \beta^2 x = 0,$$
and the equation becomes $\rho = \alpha_1 - \beta S\frac{\alpha_1}{\beta} - \beta x$,

where $\alpha_1 - \beta S\frac{\alpha_1}{\beta} = \alpha$ of the former equation.

3. *Rotor along Straight Line whose Equation is given.*



Let OR be the rotor through the origin which has right parallelism with MP. Then $\angle$ NOR = OM. Let OK be perpendicular to ON and OM, and of such length that $\frac{\tan OK}{\tan ON} = \tan NOR$. Then, if $\gamma = OK$, $OR = \beta + \gamma$. Now $\frac{T\gamma}{T\beta} = T\alpha$, and $U\gamma = U\alpha\beta$, since γ is perpendicular to α and β. Hence $\gamma = \alpha\beta$; and if R be a rotor along MP, m a scalar,

$$\text{right vector of } R = \xi R = m\xi(\beta+\gamma) = m\xi(\beta+\alpha\beta),$$

so $$\text{left vector of } R = \eta R = m\eta(\beta-\gamma) = m\eta(\beta-\alpha\beta);$$

therefore $$R = m(\beta+\omega\alpha\beta).$$

Now if R have the same length as β, we have

$$\beta^2 = R^2 = m^2(\beta^2+\overline{\alpha\beta}^2) = m^2\beta^2(1-\alpha^2);$$

therefore $$R = \frac{\beta+\omega\alpha\beta}{\sqrt{(1-\alpha^2)}}.$$

Conversely, equation to axis of rotor $\gamma+\omega\delta$ is

$$\rho = \frac{\delta}{\gamma}+\gamma x.$$

This finds the rotor in the case in which $\rho = \alpha+\beta x$, where $S\alpha\beta = 0$. But in the general case we have only to write the equation in the form

$$\rho = \alpha-\beta S\frac{\alpha}{\beta}+\beta x,$$

whence $$R = \frac{\beta+\omega\left(\alpha-\beta S\frac{\alpha}{\beta}\right)\beta}{\sqrt{\left(1-\alpha^2-\beta^2S^2\frac{\alpha}{\beta}+2S\alpha\beta S\frac{\alpha}{\beta}\right)}}$$

$$= \frac{\beta+\omega V\alpha\beta}{\sqrt{\left(1+S\alpha\beta S\frac{\alpha}{\beta}-\alpha^2\right)}}.$$

4. *Rotor ab joining Points whose Position-Rotors are α, β.*

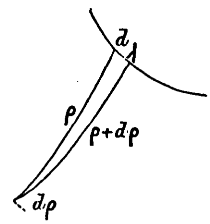
The equation of this rotor is

$$\rho = \alpha+(\beta-\alpha)x,$$

whence $$mR = \beta-\alpha+\omega V\alpha\beta.$$

Now if a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4 are the coordinates of the points, we have $$[TR]^2 = \tan^2 ab = \frac{\Sigma(a_h b_k - a_k b_h)^2}{(\Sigma a_h b_h)^2} = -\frac{(\alpha-\beta)^2+(V\alpha\beta)^2}{(1-S\alpha\beta)^2},$$

therefore $$R = \frac{\beta - \alpha + \omega V\alpha\beta}{1 - S\alpha\beta}.$$



COR.—If ρ be the rotor of a variable point on a curve, $d\lambda$ a rotor along the tangent of length equal to the arc of the curve between ρ and $\rho + d\rho$, we have

$$d\lambda = \frac{d\rho + \omega V\rho d\rho}{1 - \rho^2}.$$

5. *Rotor parallel to β through Point whose Position-Rotor is α.*

The general equation to a line through the point α is $\rho = \alpha + \lambda x$, where λ is any rotor through the origin. A rotor along this line is $\lambda + \omega V\alpha\lambda$; if this is right parallel to β, we have

$$\xi(\lambda + V\alpha\lambda) = \xi\beta, \qquad [\xi\omega = \xi]$$

or $$\lambda + V\alpha\lambda = \beta.$$

Operating by $S\alpha$, we have, since $S.\alpha V\alpha\lambda = 0$,

$$S\alpha\lambda = S\alpha\beta,$$

whence, by addition, $$\lambda + \alpha\lambda = \beta + S\alpha\beta,$$

and $$\lambda = (1+\alpha)^{-1}(\beta + S\alpha\beta) = \beta - (1+\alpha)^{-1} V\alpha\beta.$$

The rotor required is $\lambda + \omega V\alpha\lambda$, or $\lambda + \omega(\beta - \lambda)$. This becomes, then,

$$\beta - (1+\alpha)^{-1} V\alpha\beta + \omega(1+\alpha)^{-1} V\alpha\beta = \beta - 2\eta(1+\alpha)^{-1} V\alpha\beta.$$

Instead of operating by $S\alpha$ on the equation

$$\lambda + V\alpha\lambda = \beta,$$

we might have operated with $V\alpha$, and got

$$V\alpha\lambda + \alpha V\alpha\lambda = V\alpha\beta, \quad \text{since} \quad V.\alpha V\alpha\lambda = \alpha V\alpha\lambda,$$

therefore $$V\alpha\lambda = (1+\alpha)^{-1} V\alpha\beta,$$

and $$\lambda = \beta - V\alpha\lambda = \beta - (1+\alpha)^{-1} V\alpha\beta.$$

Similarly, we have for the rotor *left* parallel to β,

$$\lambda = \beta + (1-\alpha)^{-1} V\alpha\beta,$$

and the rotor is

$$\lambda + \omega(\lambda - \beta) = \beta + (1-\alpha)^{-1} V\alpha\beta + \omega(1-\alpha)^{-1} V\alpha\beta$$
$$= \beta + 2\xi(1-\alpha)^{-1} V\alpha\beta.$$