then the order of the resultant is

$$\Pi p\left(\Sigma\frac{h}{p}+\mu\right).$$

For the equations may be written

$$a_h(u, v, \ldots)^p+\theta a_{h+\mu}(u, v, \ldots)^{-1}+\ldots+\theta^p a_{h+p\mu}=0,$$

where the h, p are to be affected successively with the suffixes $1, 2, \ldots k$, and θ may be considered $=1$. Now these equations may be regarded as having coefficients of the constant order h, but the weight of every coefficient of θ^r equal to $r\mu$. This being so, the degree of the resultant in the uneliminated variables will be the sum of its order and weight calculated on these suppositions. But its order is $h_1 p_2 p_3 \ldots p_k$, or $\frac{h_1}{p_1}\Pi p$, due to the coefficients of the first equation, $\frac{h_2}{p_2}\Pi p$ due to the coefficients of the second, and so on; while its weight is $\mu\Pi p$. Hence the entire order of the resultant is

$$\Pi p\left(\Sigma\frac{h}{p}+\mu\right),$$

as stated above.

March 10*th*, 1870.

Prof. CAYLEY, President, in the Chair.

Mr. E. A. L. Bradshaw Smith was elected a Member, and Messrs. A. and W. M. Ramsay were admitted into the Society. Visitor, Prof. Oppermann of Copenhagen.

Mr. Tucker read two communications by Mr. Clerk-Maxwell: the one on "Topographical Geometry" (on which paper the President and Mr. Archibald Smith made some remarks); the other

On the Displacement in a Case of Fluid Motion.

In most investigations of fluid motion, we consider the velocity at any point of the fluid as defined by its magnitude and direction, as a function of the coordinates of the point and of the time. We are supposed to be able to take a momentary glance at the system at any time, and to observe the velocities; but are not required to be able to keep our eye on a particular molecule during its motion. This method, therefore, properly belongs to the theory of a continuous fluid alike in all its parts, in which we measure the velocity by the volume which passes through unit of area rather than by the distance travelled by a molecule in unit of time. It is also the only method applicable to the case of a fluid, the motions of the individual molecules of which are not expressible as functions of their position, as in the motions due to heat and diffusion. When similar equations occur in the theory of the conduction of heat or electricity, we are constrained to use this method, for we cannot even define what is meant by the continued identity of a portion of heat or electricity.

The molecular theory, as it supposes each molecule to preserve its identity, requires for its perfection a determination of the position of each molecule at any assigned time. As it is only in certain cases that our present mathematical resources can effect this, I propose to point out a very simple case, with the results.

Let a cylinder of infinite length and of radius a move with its axis parallel to z, and always passing through the axis of x, with a velocity V, uniform or variable, in the direction of x, through an infinite, homogeneous, incompressible, perfect fluid. Let r be the distance of any point in the fluid from the axis of the cylinder; then it is easy to show that, if x_0 is the value of x for the axis of the cylinder, and x that of the point, and

$$\phi = \frac{a^2}{r^2}(x-x_0), \quad \text{and} \quad \psi = \left(1-\frac{a^2}{r^2}\right) y,$$

$V\phi$ will satisfy the conditions of the velocity-potential, and $V\psi$ that of the stream function;* and, since the expression for ψ does not contain the time, its value will remain constant for a molecule during the whole of its motion.

If we consider the position of a particle as determined by the values of z, r, and ψ, then z and ψ will remain constant during the motion, and we have only to find r in terms of the time. For this purpose we observe that, if we put ψ in polar coordinates, it becomes

$$\psi = \left(1-\frac{a^2}{r^2}\right) r \sin\theta,$$

and

$$\frac{dr}{dt} = \frac{V}{r}\frac{d\psi}{d\theta} = V\left(1-\frac{a^2}{r^2}\right)\cos\theta.$$

Expressing $\cos\theta$ in terms of r and ψ, this becomes

$$\frac{dr}{dt} = \frac{V}{r^2}\sqrt{\{r^4-(2a^2+\psi^2)\,r^2+a^4\}}.$$

If we make $\sqrt{(4a^2+\psi^2)}+\psi = 2\beta$, and $\dfrac{a^2}{\beta^2} = c$,

then β will be the value of y when the axis of the cylinder is abreast of the particle, and

$$\frac{dr}{dt} = \frac{V}{r^2}\sqrt{(r^2-\beta^2)}\,\sqrt{(r^2-c^2\beta^2)};$$

* The velocity-potential is a quantity such that its rate of variation along any line is equal to the velocity of the fluid resolved in the same direction. Whenever the motion of the fluid is irrotational, there is a velocity-potential.

The stream function exists in every case of the motion of an incompressible fluid in two dimensions, and is such that the total instantaneous flow across any curve, referred to unit of time, is equal to the difference of the values of the stream function at the extremities of the curve.

and if we now use instead of r a new angular variable χ such that

$$\sin\chi = \frac{\beta}{r} = \frac{\sqrt{(4a^2+\psi^2)}+\psi}{2r},$$

then we can express $\int V dt$ or x_0 in terms of elliptic functions of the first and second kinds,

$$\int V dt = x_0 = \beta \cot\chi \sqrt{(1-c^2\sin^2\chi)} + \beta \{E_c(\chi) - F_c(\chi)\},$$

where the position of the axis of the cylinder is expressed in terms of the position of a molecule with respect to it.

Now let us take a molecule originally on the axis of y, at a distance η from the origin, and let the cylinder begin to move from an infinite distance on the negative side of the axis of x; then

$$\psi = \eta, \text{ and } 2\beta = \sqrt{(4a^2+\eta^2)}+\eta, \text{ and } \frac{a^2}{\beta^2} = c;$$

and when the cylinder has passed from negative infinity to positive infinity in the direction of x, then the coordinates of the molecule will be

$$x = \frac{2a}{\sqrt{c}}(F_c - E_c), \text{ and } y = \frac{a(1-c)}{\sqrt{c}}.$$

It appears from this expression, that after the passage of the cylinder every particle is at the same distance as at first from the plane of xz, but that it is carried forward in the direction of the motion of the cylinder by a quantity which is infinite when $y=0$, but finite for all other values of y.

The motion of a particle at any instant is always inclined to the axis of x at double the inclination of the line drawn to the axis of the cylinder. Hence it is in the forward direction till the inclination of this line is 45°, backward from 45° to 135°, and forward again afterwards. The forward motion is slower than the backward motion, but lasts for a longer time, and it appears that the final displacement of every particle is in the forward direction. It follows from this that the condition fulfilled by the fluid at an infinite distance is not that of being contained in a fixed vessel; for in that case there would have been, on the whole, a displacement backwards equal to that of the cylinder forwards. The problem actually solved differs from this only by the application of an infinitely small forward velocity to the infinite mass of fluid such as to generate a finite momentum.

In drawing the accompanying figures, I began by tracing the stream-lines in Fig. 1, p. 86, by means of the intersections of a system of straight lines equi-distant and parallel to the axis, with a system of circles touching the axis at the origin and having their radii as the reciprocals of the natural numbers. (See Prof. Rankine's Papers on Stream-Lines in the "Phil. Trans.")

The cylinder is $\frac{2}{3}$ inch radius, and the stream-lines are originally $\frac{4}{15}$ inch apart.

I then calculated the coordinates, x and y, of the final form of a transverse straight line from the values of the complete elliptic functions for values of c corresponding to every 5°. The result is given in the continuous curve on the left of Fig. 2, p. 87.

I then traced the path of a particle in contact with the cylinder from the equation

$$\tan \tfrac{1}{2}\theta = e^{-\frac{2x_0}{a}},$$

where $x = x_0 + a\cos\theta$ and $y = a\sin\theta$.

The form of the path is the curve nearest the axis in Fig. 3. The dots indicate the positions at equal intervals of time.

The paths of particles not in contact with the cylinder might be calculated from Legendre's tables for incomplete functions, which I have not got.

I have therefore drawn them by eye so as to fulfil the following conditions:—

The radius of curvature is $\dfrac{1}{2}\,\dfrac{a^2 y}{a^2\sin^2\theta + y^2}$, which, when y is large compared with a, becomes nearly $\dfrac{a^2}{2y}$.

The paths of particles at a great distance from the axis are therefore very nearly circles.

To draw the paths of intermediate particles, I observed that their two extremities must lie at the same distance from the axis of x as the asymptote of a certain stream-line, and the middle point of the path at a distance equal to that of the same stream-line when abreast of the cylinder; and, finally, that the distance between the extremities is the same as that given in Fig. 2.

In this way I drew the paths of different particles in Fig. 3. I then transferred these to Fig. 2, to show the paths of a series of particles, originally in a straight line, and finally in the curve already described.

I then laid Fig. 1 on Fig. 2, and drew, through the intersections of the stream-lines and the paths of the corresponding particles in the fluid originally at rest, the lines which show the form taken by a line of particles originally straight as it flows past the cylinder. This method, however, does not give the point where the line crosses the axis of x. I therefore calculated this from the equation

$$x = r + \tfrac{1}{2}a\log\frac{r-a}{r+a},$$

calculating r for values of x differing by $\tfrac{1}{3}$ inch.

The curves thus drawn appear to be as near the truth as I could get without a much greater amount of labour.

If a maker of "marbled" paper were to rule the surface of his bath with straight lines of paint at right angles, and then to draw a cylindrical ruler through the bath up to the middle, and apply the painted lines to his paper, he would produce the design of Fig. 1, p. 86.

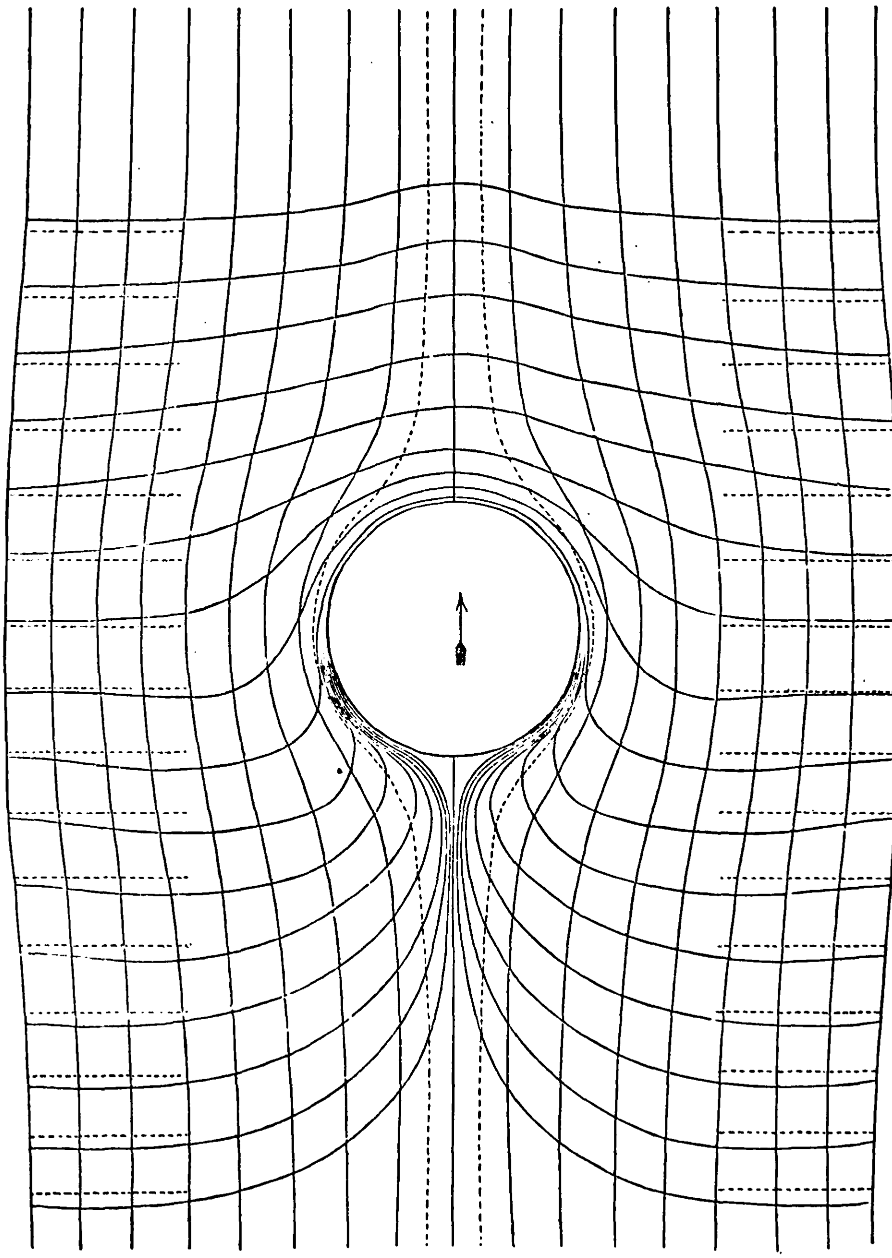

FIG. 1.
Fluid flowing past a fixed cylinder.

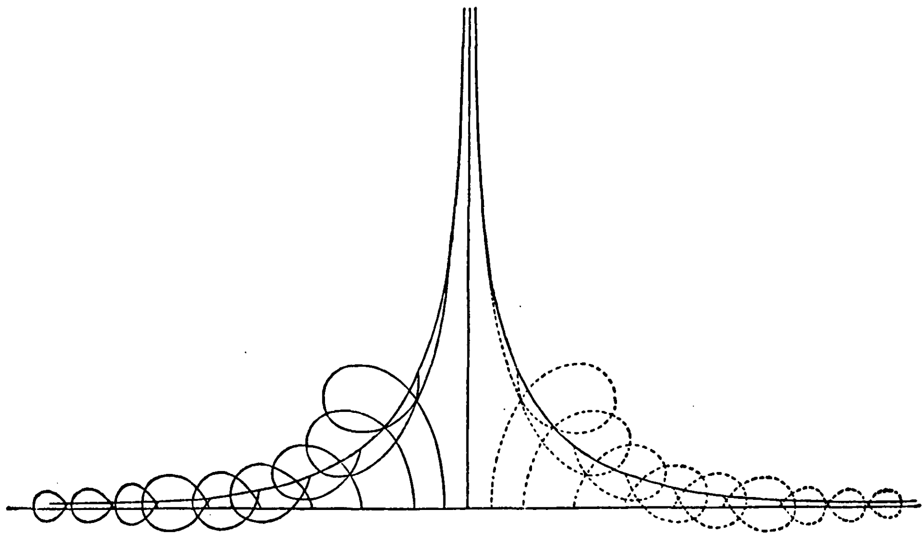

FIG. 2.

Paths of particles of the fluid when a cylinder moves through it.

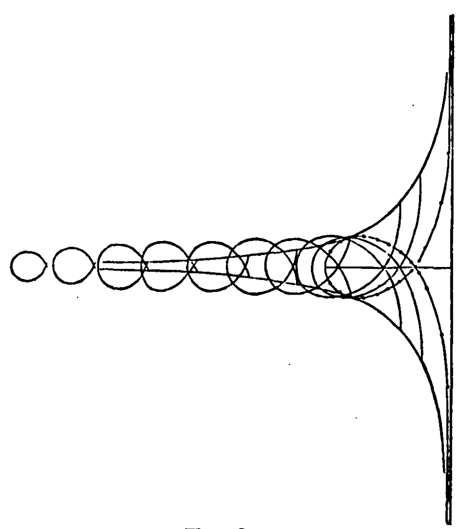

FIG. 3.

Paths of particles at different distances from the cylinder: radius of cylinder, ⅜ inch. At great distances (β) the path is a circle of radius $\frac{a^2}{2\beta}$, and in this circle $\tan\frac{\theta}{2} = \frac{Vt}{\beta}$.