independent, and the illuminations due to them, and not the vibrations, are to be compounded. As it is commonly though not very correctly expressed, there is to be no interference. For this the sensible apparent magnitude of the sun, or other source of light, is an amply sufficient cause.

At page 106 of his excellent lectures, Verdet shows that the disturbance from the sun cannot be considered as a system of plane waves over a space greater than a circle of $\frac{1}{20}$ of a millimetre diameter. Between the vibrations at two points whose mutual distance is much greater than this, there is no permanent relation of phase. This shows that the vibrations corresponding to two holes in the imaginary screen cannot interfere regularly, but behave as if they were due to thoroughly independent sources of light.

Sir W. Thomson, Prof. Maxwell, and Mr. Strutt made some further remarks on the subject of the paper. Mr. Maxwell then gave a description of two singular solar halos which he had recently seen; and Prof. Adams, of King's College, gave some additional particulars in the case of one of the phenomena which had also been noticed by himself. Prof. Cayley, V.P., communicated an account of the following paper:—

On the Problem of finding the Circle which cuts three given Circles at three given Angles. By J. Griffiths, M.A.

I shall first show that a circle which cuts three given circles S_1, S_2, S_3 at the given angles $\theta_1, \theta_2, \theta_3$, is always touched by a variable circle $\lambda S_1+\mu S_2+\nu S_3$, where λ, μ, ν are connected by a certain relation of the second degree.

To simplify our investigation, let us take the centre of the required circle, U say, as the origin of coordinates, so that, if R be its radius, we have for its equation $\quad U = x^2+y^2-R^2 = 0$;

and for those of the given circles S_1, S_2, S_3, which are cut by U at angles $\theta_1, \theta_2, \theta_3$ respectively,

$$S_1 = x^2+y^2+2g'x+2f'y+R^2-2Rr_1\cos\theta_1 = 0,$$
$$S_2 = x^2+y^2+2g''x+2f''y+R^2-2Rr_2\cos\theta_2 = 0,$$
$$S_3 = x^2+y^2+2g'''x+2f'''y+R^2-2Rr_3\cos\theta_3 = 0,$$

r_1, r_2, r_3 being the respective radii of S_1, S_2, S_3.

Now let us consider the circle $\lambda S_1+\mu S_2+\nu S_3$. Its equation may be written in the form

$$(\lambda+\mu+\nu)(x^2+y^2+R^2)+2\Sigma\lambda g'x+2\Sigma\lambda f'y-2R\Sigma(\lambda r_1\cos\theta_1) = 0,$$

where $\quad \Sigma\lambda g' = \lambda g'+\mu g''+\nu g'''$, &c.;

and the condition that it shall *touch* U is

$$\{(R-r_1\cos\theta_1)\lambda+(R-r_2\cos\theta_2)\mu+(R-r_3\cos\theta_3)\nu\}^2$$
$$=(g'\lambda+g''\mu+g'''\nu)^2+(f'\lambda+f''\mu+f'''\nu)^2,$$

or $$(l^2-\delta'^2)\lambda^2+(m^2-\delta''^2)\mu^2+(n^2-\delta'''^2)\nu^2$$
$$+2(mn-g''g'''-f''f''')\mu\nu+\dots\dots+\dots\dots=0,$$

if we write l, m, n for $R-r_1\cos\theta_1$, $R-r_2\cos\theta_2$, $R-r_3\cos\theta_3$ respectively, and δ', δ'', δ''' for $\sqrt{g'^2+f'^2}$, $\sqrt{g''^2+f''^2}$, $\sqrt{g'''^2+f'''^2}$.

Observing, then, that δ', δ'', δ''' must be the distances between the centre of the required circle U and those of S_1, S_2, S_3, we have

$$\sqrt{\delta'^2-r_1^2\sin^2\theta_1}=R-r_1\cos\theta_1=l,$$
$$\sqrt{\delta''^2-r_2^2\sin^2\theta_2}=R-r_2\cos\theta_2=m,$$
$$\sqrt{\delta'''^2-r_3^2\sin^2\theta_3}=R-r_3\cos\theta_3=n;$$

since, evidently, by the conditions of the problem,

$$\delta'^2=R^2+r_1^2-2Rr_1\cos\theta_1,\quad \&c.$$

The coefficient of λ^2, for instance, in the above equation of condition, therefore becomes $\quad l^2-\delta'^2=-r_1^2\sin^2\theta_1$;

and that of $\mu\nu$,

$$2(mn-g''g'''-f''f''')$$
$$=-(m-n)^2+(g''-g''')^2+(f''-f''')^2+m^2+n^2-g''^2-g'''^2-f''^2-f'''^2$$
$$=-(r_2\cos\theta_2-r_3\cos\theta_3)^2+\delta_1^2+\delta''^2-r_2^2\sin^2\theta_2+\delta'''^2-r_3^2\sin^2\theta_3-\delta''^2-\delta'''^2$$
$$=\delta_1^2-r_2^2-r_3^2+2r_2r_3\cos\theta_2\cos\theta_3,$$

where $\delta_1^2=(g''-g''')^2+(f''-f''')^2=$ square of the distance between the centres of S_2 and S_3.

Hence, finally, if r_1, r_2, r_3 be the radii of the given circles S_1, S_2, S_3, and δ_1, δ_2, δ_3 the lengths of the sides of the triangle formed by joining their centres, we see that a circle which cuts S_1, S_2, S_3 at the respective angles θ_1, θ_2, θ_3 will always be touched by a variable circle $\lambda S_1+\mu S_2+\nu S_3$, provided λ, μ, ν satisfy the relation

$$A\lambda^2+B\mu^2+C\nu^2+2F\mu\nu+2G\nu\lambda+2H\nu\lambda=0,$$

where $$A=r_1^2\sin^2\theta_1,\quad B=r_2^2\sin^2\theta_2,\quad C=r_3^2\sin^2\theta_3,$$
$$2F=-(\delta_1^2-r_2^2-r_3^2+2r_2r_3\cos\theta_2\cos\theta_3),$$
$$2G=-(\delta_2^2-r_3^2-r_1^2+2r_3r_1\cos\theta_3\cos\theta_1),$$
and $$2H=-(\delta_3^2-r_1^2-r_2^2+2r_1r_2\cos\theta_1\cos\theta_2).$$

Again, if the angle of intersection of two circles be taken to be the definite angle subtended by the line joining the centres of the two circles at one of their common points of intersection, it is easily seen that the relation between λ, μ, ν, just obtained, is also precisely the condition that $\lambda S_1+\mu S_2+\nu S_3$ shall touch V, a circle cutting S_1, S_2, S_3 at the supplementary angles $\pi-\theta_1$, $\pi-\theta_2$, $\pi-\theta_3$.

It follows, then, that two circles U and V which cut the given circles at angles θ_1, θ_2, θ_3, and their supplements $\pi-\theta_1$, $\pi-\theta_2$, $\pi-\theta_3$ respectively, will *both be touched* by a variable circle $\lambda S_1+\mu S_2+\nu S_3$, when

$$\Sigma A\lambda^2+2\Sigma F\mu\nu=0.$$

Conversely, by seeking the envelope of $\lambda S_1+\mu S_2+\nu S_3$ under this condition, we find that the quartic

$$\Sigma(BC-F^2)S_1^2+2\Sigma(GH-AF)S_2S_3=0,$$

where $S_1=x^2+y^2+2g_1x+2f_1y+c_1$, $S_2=\ldots\ldots$, $S_3=\ldots\ldots$,

$$A=r_1^2\sin^2\theta_1,\quad B=\ldots\ldots,\quad C=\ldots\ldots,$$

$$2F=-(\delta_1^2-r_2^2-r_3^2+2r_2r_3\cos\theta_2\cos\theta_3),\quad 2G=\ldots\ldots,\quad 2H=\ldots\ldots,$$

must represent *a pair of circles* U and V, cutting S_1, S_2, S_3 at the given angles $\theta_1, \theta_2, \theta_3$; $\pi-\theta_1$, $\pi-\theta_2$, $\pi-\theta_3$ respectively.

Again, if we change θ_1, for instance, into $\pi-\theta_1$, it is clear that the values of G and H will be altered; so that we shall get another quartic of the above form, giving the pair of circles which cut S_1, S_2, S_3 at the angles $\pi-\theta_1$, θ_2, θ_3, and θ_1, $\pi-\theta_2$, $\pi-\theta_3$, respectively.

Hence, evidently, there will be in all *four* quartics of the above form; or, in other words, there are in general eight solutions to the problem, "To describe a circle to cut three given circles at given angles."

It may be observed that the radii of any one of these four pairs of circles may be obtained by writing down the relation (investigated by Prof. Cayley; see Dr. Salmon's "Conic Sections," p. 129, 5th ed.) which connects the mutual distances of any four points in a plane; and substituting therein $R^2+r_1^2-2Rr_1\cos\theta_1$ for $(14)^2$, &c. The result is a quadratic in R; and similarly for the radii of any of the other pairs.

Remarks on the above Quartics.

Putting $\quad BC-F^2=a$, &c., $\quad GH-AF=f$, &c.,

I propose to verify the fact that the quartic

$$\Omega=(a, b, c, 2f, 2g, 2h)(S_1, S_2, S_3)^2=0$$

breaks up into two circles.

It may be remarked, in the first place, that a curve of the form Ω is one of the fourth degree having a node at each of the two circular points at infinity, and that if it has two additional nodes, it will represent a pair of circles. (See a paper by Prof. Cayley, entitled "Investigations in Connexion with Casey's Equation," published in the "Quarterly Journal of Mathematics," No. 32, vol. viii.) Hence we have to show that the special quartic under consideration has two additional nodes.

1°. If the discriminant $\begin{vmatrix} a, h, g \\ h, b, f \\ g, f, c \end{vmatrix}$ be supposed to vanish, Ω breaks up into linear factors, and so, obviously, represents a pair of circles.

2°. If the discriminant in question does not vanish, and Ω has two additional nodes, it is easy to show that they both lie on the Jacobian, J, of the given circles S_1, S_2, S_3; and in this case, therefore, we must prove that Ω meets J in two pairs of coincident points. Taking, then, the centre of the Jacobian as the origin of coordinates, we may write for the equations of the several circles J, S_1, S_2, S_3,

$$J = x^2+y^2-\rho^2 = 0$$

$$\left.\begin{aligned} S_1 &= x^2+y^2+2g_1x+2f_1y+\rho^2 = 0 \\ S_2 &= \dots\dots\dots\dots\dots\dots\dots +\rho^2 = 0 \\ S_3 &= \dots\dots\dots\dots\dots\dots\dots +\rho^2 = 0 \end{aligned}\right\} \text{ where } \rho = \text{radius of Jacobian,}$$

so that the points common to Ω and J (exclusive of the two circular points at infinity) coincide with the intersections of J and the conic whose equation is $(a, b, c, 2f, 2g, 2h)(x', y', z')^2 = 0$; where

$$\left.\begin{aligned} x' &= g_1x+f_1y+\rho^2 \\ y' &= g_2x+f_2y+\rho^2 \\ z' &= g_3x+f_3y+\rho^2 \end{aligned}\right\}.$$

Now, if we put $r_1^2 = A'$, $r_2^2 = B'$, $r_3^2 = C'$, $\delta_1^2 - r_2^2 - r_3^2 = -2F'$, &c., (where $\delta_1, \delta_2, \delta_3$; r_1, r_2, r_3 denote the same quantities as before,) it may be shown that the equation of J in terms of x', y', z' is

$$(a', b', c', 2f', 2g', 2h')(x', y', z')^2 = 0;$$

where $a' = B'C'-F'^2$, &c., as usual.

[This equation may be derived from the ordinary trilinear equation of a circle whose radius is r, and centre the point $(\alpha', \beta', \gamma')$; viz.,

$$\Sigma\{\beta'^2+\gamma'^2+2\beta'\gamma'\cos A-r^2\sin^2 A\}\,\alpha^2$$
$$+2\Sigma\{(\alpha'^2-r^2)\cos A-\beta'\gamma'-\gamma'\alpha'\cos C-\alpha'\beta'\cos B-r^2\cos B\cos C\}\,\beta\gamma = 0;$$

by writing $\frac{x'}{\delta'}, \frac{y'}{\delta''}, \frac{z'}{\delta'''}$ for α, β, γ, and expressing the several coefficients of x'^2, &c., in terms of given quantities.]

Hence we have to verify the fact that the conics given by the equations

$$(a', b', c', 2f', 2g', 2h')(x, y, z)^2 = 0,$$
$$(a, b, c, 2f, 2g, 2h)(x, y, z)^2 = 0,$$

have *double contact*; or, what is the same thing, that their reciprocals,

$$(A', B', C', 2F', 2G', 2H')(\lambda, \mu, \nu)^2 = 0,$$
$$(A, B, C, 2F, 2G, 2H)(\lambda, \mu, \nu)^2 = 0,$$

have double contact.

Now this is clearly the case; for we have

$$(A', B', C', 2F', 2G', 2H')(\lambda, \mu, \nu)^2-(A, B, C, 2F, 2G, 2H)(\lambda, \mu, \nu)^2$$
$$= (r_1\cos\theta_1\,\lambda+r_2\cos\theta_2\,\mu+r_3\cos\theta_3\,\nu)^2.$$

Hence Ω meets J in a pair of two-fold points (exclusive of the circular points at infinity), or Ω has two additional nodes, and so breaks up into a pair of circles.*

The result just obtained enables us to find the radical axis of the pair of circles Ω; for this must evidently be the polar with respect to J, represented by $x^2+y^2-\rho^2=0$, of the point whose coordinates are given by the equations

$$\frac{g_1x+f_1y+\rho^2}{r_1\cos\theta_1}=\frac{g_2x+f_2y+\rho^2}{r_2\cos\theta_2}=\frac{g_3x+f_3y+\rho^2}{r_3\cos\theta_3}.$$

Hence we have the following theorem,—proved, I believe, by Plücker in his solution of the problem under consideration,—viz., that the

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* With reference to the above investigation, Prof. Cayley has remarked that "if $\Omega=(a,\ldots)(S_1,S_2,S_3)^2=0$ has a double point, then $J=J(S_1,S_2,S_3)=0$ passes through this double point;" but, he asks, "is it always true conversely, that if Ω, J have a twofold intersection, this is a double point of Ω?"

It may be replied that this is generally so, as I shall thus endeavour to show.

In the first place, there exists a quadric relation between

$$J=x^2+y^2-\rho^2,\ \text{and}\ S_1=x^2+y^2+2g_1x+2f_1y+\rho^2,\quad S_2=\ldots\ldots,\quad S_3=\ldots\ldots,$$

of the form $$(a',\ldots)(S_1,S_2,S_3)^2=kJ^2.$$

For if $g_1x+f_1y+\rho^2$, $g_2x+f_2y+\rho^2$, $g_3x+f_3y+\rho^2$ be written, for shortness, x', y', z', the quartic $(a',\ldots)(S_1,S_2,S_3)^2$ becomes

$$(a'+b'+c'+2f'+2g'+2h')\,J^2+4\left\{(a'+h'+g')\,x'+\ \ldots\ldots\right\}J+4\,(a',\ldots)(x',y',z')^2,$$

which will reduce to $(a'+\ldots)J^2$, provided $a', b', \ldots$ satisfy certain linear equations. In fact, it will be found that when $(a',\ldots)(x',y',z')^2$ reduces to $(a'+b'+\ldots)\,\rho^2(\rho^2-x^2-y^2)$, the quartic function $(a',\ldots)(S_1,S_2,S_3)^2$ becomes at the same time $(a'+b'+\ldots)\times(x^2+y^2-\rho^2)^2$.

It follows from this, as in the analogous case of conics, that we can always determine θ so that the function $\Omega+\theta J^2$ breaks up into linear factors of S_1, S_2, S_3, *i. e.*, so that $$\Omega+\theta J^2=(\alpha S_1+\beta S_2+\gamma S_3)\,(\alpha' S_1+\beta' S_2+\gamma' S_3).$$

Hence, if J be supposed to have a twofold intersection with Ω, it is clear that θ may be so determined that one of the circles represented by the factors on the right-hand side of this equation, say $\alpha' S_1+\beta' S_2+\gamma' S_3$, shall *touch* J; but when a circle of the form $\alpha' S_1+\beta' S_2+\gamma' S_3$ touches J, it is easy to see that it reduces to a *point*.

If, therefore, the twofold intersection of Ω and J be taken as origin of coordinates, Ω becomes of the form

$$\Omega=(\alpha S_1+\beta S_2+\gamma S_3)\,(x^2+y^2)-\theta\,(x^2+y^2+2gx+2fy)^2;$$

where $x^2+y^2+2gx+2fy$ represents the Jacobian $J(S_1,S_2,S_3)$; or, in other words, a twofold intersection of Ω and J is a node of Ω.

Again, if θ can be so determined (as in the case of the quartic we have been considering) that $\Omega+\theta^2J^2$ is the square of a factor $\alpha S_1+\beta S_2+\gamma S_3$, it is evident that Ω then represents the two circles $\alpha S_1+\beta S_2+\gamma S_3+\theta J$, and $\alpha S_1+\beta S_2+\gamma S_3-\theta J$.

It may be observed that the nodes of Ω in the case before us coincide with the envelope-points $\lambda S_1+\mu S_2+\mu S_3=0$, where $\Sigma A'\lambda^2+2\Sigma F'\mu\nu=0$ and $\Sigma r_1\cos\theta_1\lambda=0$;

i. e., if $\quad S_1=(x-g_1)^2+(y-f_1)^2-r_1^2,\quad S_1=(x-g_2)^2+(y-f_2)^2-r_2^2,$

and $$S_3=(x-g_3)^2+(y-f_3)^2-r_3^2,$$

the nodes of Ω are the points

$$x=\frac{\lambda g_1+\mu g_2+\nu g_3}{\lambda+\mu+\nu},\quad y=\frac{\lambda f_1+\mu f_2+\nu f_3}{\lambda+\mu+\nu};$$

where $\Sigma r_1^2\lambda^2-\Sigma\left\{(g_2-g_3)^2+(f_2-f_3)^2-r_2^2-r_3^2\right\}\mu\nu=0$, and $\Sigma r_1\cos\theta_1\lambda=0$.

radical axis of the pair of circles Ω coincides with an axis of similitude of three circles concentric respectively with S_1, S_2, S_3, and whose radii have to each other the ratios $r_1 \cos\theta_1 : r_2 \cos\theta_2 : r_3 \cos\theta_3$.

Again, by considering that a quartic of the form Ω is the envelope of a circle $\lambda S_1 + \mu S_2 + \nu S_3$ moving subject to a certain condition, we are readily led to another property of the four pairs of circles which cut three given circles at given angles; viz.,

Any two of these pairs form four circles, which are all touched by each of four other circles orthogonal to the Jacobian J.

For instance, the two pairs of circles which cut S_1, S_2, S_3 at the angles (1) $\theta_1, \theta_2, \theta_3$; $\pi-\theta_1, \pi-\theta_2, \pi-\theta_3$; (2) $\pi-\theta_1, \theta_2, \theta_3$; $\theta_1, \pi-\theta_2, \pi-\theta_3$, are all touched by each of the circles $\lambda S_1 + \mu S_2 + \nu S_3$; where the ratios $\lambda : \mu : \nu$ are given by the two simultaneous equations

$$\begin{aligned} r_1^2 \sin^2\theta_1 \lambda^2 + r_2^2 \sin^2\theta_2 \mu^2 + r_3^2 \sin^2\theta_3 \nu^2 &- (\delta_1^2 - r_2^2 - r_3^2 + 2r_2 r_3 \cos\theta_2 \cos\theta_3)\,\mu\nu \\ &- (\delta_2^2 - r_3^2 - r_1^2 + 2r_3 r_1 \cos\theta_3 \cos\theta_1)\,\nu\lambda \\ &- (\delta_3^2 - r_1^2 - r_2^2 + 2r_1 r_2 \cos\theta_1 \cos\theta_2)\,\lambda\mu = 0, \end{aligned}$$

$$\begin{aligned} r_1^2 \sin^2\theta_1 \lambda^2 + r_2^2 \sin^2\theta_2 \mu^2 + r_3^2 \sin^2\theta_3 \nu^2 &- (\delta_1^2 - r_2^2 - r_3^2 + 2r_2 r_3 \cos\theta_2 \cos\theta_3)\,\mu\nu \\ &- (\delta_2^2 - r_3^2 - r_1^2 - 2r_3 r_1 \cos\theta_3 \cos\theta_1)\,\nu\lambda \\ &- (\delta_3^2 - r_1^2 - r_2^2 - 2r_1 r_2 \cos\theta_1 \cos\theta_2)\,\lambda\mu = 0. \end{aligned}$$

The bicircular quartic $\Omega = (a, b, c, 2f, 2g, 2h)(S_1, S_2, S_3)^2 = 0$ has hitherto been regarded as an *envelope*; *i.e.*, the envelope of a variable circle $\lambda S_1 + \mu S_2 + \nu S_3$, where λ, μ, ν satisfy a certain relation.

Now it is evident that it may be also looked upon as a *locus*; viz., the locus of a pair of points $\frac{S_1}{\alpha} = \frac{S_2}{\beta} = \frac{S_3}{\gamma}$;* where α, β, γ are subject to the relation $(a, b, c, 2f, 2g, 2h)(\alpha, \beta, \gamma)^2 = 0$.

Hence we have the following theorem:—The two circles represented by the quartic $\Omega = (a, b, c, 2f, 2g, 2h)(S_1, S_2, S_3)^2 = 0$ are the locus of a pair of points inverse to each other with respect to the Jacobian J; viz., the points $\frac{S_1}{\alpha} = \frac{S_2}{\beta} = \frac{S_3}{\gamma}$; where α, β, γ satisfy the equation $(a, b, c, 2f, 2g, 2h)(\alpha, \beta, \gamma)^2 = 0$; or, in other words, they are circles inverse to each other with respect to J.

We may briefly notice two special cases of the quartic Ω; viz., $\alpha°$ when S_1, S_2, S_3 meet in a point; $\beta°$ when S_1, S_2, S_3 are coaxal.

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* *Note by* Prof. CAYLEY.—"This is an interesting kind of geometrical correspondence, deserving to be studied; viz., to a point (α, β, γ) there correspond the *pair* of points $\frac{S_1}{\alpha} = \frac{S_2}{\beta} = \frac{S_3}{\gamma}$ (where $S_1=0$, $S_2=0$, $S_3=0$ are arbitrary circles); and, therefore, in general, to the curve $(\alpha, \beta, \gamma)^n = 0$ a curve $(S_1, S_2, S_3)^n = 0$ of the order $2n$."

α°. In this case $\rho = 0$; and the Jacobian reduces to a point, through which the envelope-circle of Ω always passes. Hence it is clear that Ω breaks up into two factors of the forms x^2+y^2 and $x^2+y^2+2gx+2fy+c$. Under these circumstances, therefore, the number of solutions reduces to *four*.

β°. If S_1, S_2, S_3 meet in two common points, a quartic of the form $\Omega=0$ represents a pair of circles, both passing through each of these points,—whatever the values of the coefficients A, B, C, &c.—but in this case there is a certain relation between the cosines of the angles θ, ϕ, ψ at which any circle cuts S_1, S_2, S_3 (see Dr. Salmon's "Conic Sections," p. 103, Ex. 9, 5th edit.); and unless this is satisfied by the given angles θ_1, θ_2, θ_3, the problem is not a possible one. If we were to make use of this relation, and express the quartic Ω in terms of the coordinates x and y, we should find that Ω vanishes identically; the result is, therefore, indeterminate.

Particular cases.—1. If $\theta_1 = 0$ or π, $\theta_2 = 0$ or π, $\theta_3 = 0$ or π, we have Casey's Equations for the four pairs of circles *touching* three given circles. (See the paper by Prof. Cayley, referred to above.)

2. If $\theta_1 = \theta_2 = \theta_3 = \frac{1}{2}\pi$, we shall get the unique equation

$$KJ^2 = \Sigma\,\{(\delta_1^2-r_2^2-r_3^2)^2-4r_2^2r_3^2\}\,S_1^2-2\,.\,\Sigma\,\{(\delta_2^2-r_3^2-r_1^2)(\delta_3^2-r_1^2-r_2^2) + 2r_1^2(\delta_1^2-r_2^2-r_3^2)\}\,S_2S_3 = 0.$$

3. If α_1, α_2, α_3 be the angles of intersection (*i.e.*, the definite angles between the radii at a common point) of the pairs of circles S_2, S_3; S_3, S_1; S_1, S_2; and $\alpha_1+\alpha_2+\alpha_3 = 2p$; the equations of the four pairs of circles which pass through the intersections of S_1, S_2, S_3 are

$$\Sigma\,.\,r_1 \sin(p-\alpha_1)\,.\,S_2S_3 = 0,$$

$$-r_1 \sin p\,.\,S_2S_3 + r_2 \sin(p-\alpha_3)\,.\,S_3S_1 + r_3 \sin(p-\alpha_2)\,.\,S_1S_2 = 0,$$

$$r_1 \sin(p-\alpha_3)\,.\,S_3S_1 - r_2 \sin p\,.\,S_1S_2 + r_3 \sin(p-\alpha_1)\,.\,S_1S_2 = 0,$$

$$r_1 \sin(p-\alpha_2)\,.\,S_3S_1 + r_2 \sin(p-\alpha_1)\,.\,S_1S_2 - r_3 \sin p\,.\,S_1S_2 = 0;$$

where, as before, r_1, r_2, r_3 are the radii of S_1, S_2, S_3.

These results may be obtained by putting the coefficients of S_1^2, S_2^2, S_3^2 in the quartic Ω, each equal to zero; and so eliminating θ_1, θ_2, θ_3.

I may conclude with the remark that the above method may be employed with advantage to find the equations of the spheres which can be drawn to cut four given spheres at given angles θ_1, θ_2, θ_3, θ_4. For we may show that a pair of spheres cutting four spheres given by the equations

$$S_1 = x^2+y^2+z^2+2g_1x+\&c. = 0,$$
$$S_2 =$$
$$\&c.,$$

at the angles $\theta_1, \theta_2, \theta_3, \theta_4$, and their supplements respectively (measured in the definite way indicated before) will be always touched by a variable sphere
$$\alpha S_1+\beta S_2+\gamma S_3+\delta S_4,$$
where $\alpha, \beta, \gamma, \delta$ satisfy the relation
$$\Sigma A\alpha^2+2L\beta\gamma+2M\gamma\alpha+2N\alpha\beta+2P\alpha\delta+2Q\beta\delta+2R\gamma\delta = 0;$$
and, therefore, conversely, we have to find the envelope of the variable sphere in question when $\alpha, \beta, \gamma, \delta$ are subject to the relation just given.

The result is, that there are in all eight pairs of spheres satisfying the given conditions, and the radii of any pair may be found by writing down the relation which connects the mutual distances of five points in space.

APPENDIX.

1. Since writing the above, I have succeeded in arriving at the following equations for the groups of circles cutting three given small circles S_1, S_2, S_3 on a sphere at given angles $\theta_1, \theta_2, \theta_3$; viz.,

If the equations of the given circles be written in the forms
$$S_1 = \sqrt{S}-\alpha_1 \sec r_1,$$
$$S_2 = \sqrt{S}-\alpha_2 \sec r_2,$$
$$S_3 = \sqrt{S}-\alpha_3 \sec r_3,$$
(see Dr. Salmon's "Geometry of Three Dimensions," p. 195); and
$$\tan^2 r_1 \sin^2\theta_1 = A, \quad \tan^2 r_2 \sin^2\theta_2 = B, \quad \tan^2 r_3 \sin^2\theta_3 = C,$$
$$-2\,(-\cos\delta_1+\cos r_2\cos r_3+\sin r_2\sin r_3\cos\theta_2\cos\theta_3)\sec r_2\sec r_3 = 2F,$$
$$\ldots\ldots\ldots = 2G,$$
$$\ldots\ldots\ldots = 2H;$$
where r_1, r_2, r_3 are the radii of the given circles, and $\delta_1, \delta_2, \delta_3$ the distances between their centres; then the equation
$$\begin{vmatrix} A, & H, & G, & S_1 \\ H, & B, & F, & S_2 \\ G, & F, & C, & S_3 \\ S_1, & S_2, & S_3, & 0 \end{vmatrix} = 0$$
belongs to a group of circles cutting S_1, S_2, S_3 at the angles $\theta_1, \theta_2, \theta_3$ respectively.

This result is obtained in the same manner as in the case of circles on a plane. In the first place, if the angle at which two circles cut is measured by the angle which the great circle joining their centres subtends at the point of meeting, and U denote the circle cutting S_1, S_2, S_3 at the angles $\theta_1, \theta_2, \theta_3$ (measured in this way), we may show that U is always touched by a variable circle $\lambda S_1+\mu S_2+\nu S_3$, where λ, μ, ν satisfy a quadric relation of the form
$$A\lambda^2+B\mu^2+C\nu^2+2F\mu\nu+2G\nu\lambda+2H\lambda\mu = 0;$$

where A, B, C, 2F, 2G, 2H are invariant functions of the radii and distances between the centres of the given circles.

To simplify our investigation as much as possible, let us take for the equations of U, S_1, S_2, S_3 the following; viz.,

$$U = \sqrt{S} - z\sec R = 0,$$
$$S_1 = \sqrt{S} - (l'x + m'y + n'z)\sec r_1 = 0,$$
$$S_2 = \sqrt{S} - (l''x + m''y + n''z)\sec r_2 = 0,$$
$$S_3 = \sqrt{S} - (l'''x + m'''y + n'''z)\sec r_3 = 0;$$

where
$$l''l''' + m''m''' + n''n''' = \cos\delta_1,$$
$$\quad . \quad . \quad . \quad = \cos\delta_2,$$
$$\quad . \quad . \quad . \quad = \cos\delta_3;$$

and, according to the conditions of the problem,

$$n' = \cos R\cos r_1 + \sin R\sin r_1\cos\theta_1,$$
$$n'' = \cos R\cos r_2 + \sin R\sin r_2\cos\theta_2,$$
$$n''' = \cos R\cos r_3 + \sin R\sin r_3\cos\theta_3.$$

If, therefore, we write for $\lambda S_1 + \mu S_2 + \nu S_3 = 0$ the equation

$$(\lambda+\mu+\nu)\sqrt{S} - \Sigma\lambda l'\sec r_1 . x - \Sigma\lambda m'\sec r_1 . y - \Sigma\lambda n'\sec r_1 . z = 0;$$

where $\quad \Sigma\lambda l'\sec r_1 = \lambda l'\sec r_1 + \mu l''\sec r_2 + \nu l'''\sec r_3$, &c.;

we see that

$$\Sigma\lambda l'\sec r_1 . x + \Sigma\lambda m'\sec r_1 . y$$
$$+ \{\lambda(n'\sec r_1 - \sec R) + \mu(n''\sec r_2 - \sec R)$$
$$+ \nu(n'''\sec r_3 - \sec R)\} . z = 0;$$

or, say, $Lx + My + Nz = 0$ represents the plane through the origin and the two points common to U and $\lambda S_1 + \mu S_2 + \nu S_3$. Hence the condition that these circles shall *touch* becomes

$$\frac{N}{\sqrt{L^2+M^2+N^2}} = \sin R, \quad \text{or} \quad N^2\cos^2 R = (L^2+M^2)\sin^2 R;$$

which, ultimately, since

$$n'\sec r_1 = \cos R + \sin R\tan r_1\cos\theta_1,$$
$$n''\sec r_2 = \quad . \quad . \quad . \quad . \quad . \quad ,$$
$$n'''\sec r_3 = \quad . \quad . \quad . \quad . \quad . \quad ,$$

reduces to $\quad A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu = 0,$

where $A = \tan^2 r_1\sin^2\theta_1, \quad B = \tan^2 r_2\sin^2\theta_2, \quad C = \tan^2 r_3\sin^2\theta_3$, and

$$2F = -2(-\cos\delta_1 + \cos r_2\cos r_3 + \sin r_2\sin r_3\cos\theta_2\cos\theta_3)\sec r_2\sec r_3,$$
$$2G = \quad . \quad . \quad . \quad . \quad . \quad ,$$
$$2H = \quad . \quad . \quad . \quad . \quad . \quad .$$

Again, if we take a circle V, which cuts S_1, S_2, S_3 at the supplementary angles $\pi-\theta_1$, $\pi-\theta_2$, $\pi-\theta_3$ (measured in the way above explained), it may be proved in the same manner that V is always touched by the circle $\lambda S_1+\mu S_2+\nu S_3$, where λ, μ, ν satisfy the quadric relation just obtained.

Conversely, if

$$S_1=\sqrt{S}-a_1 \sec r_1, \quad S_2=\sqrt{S}-a_2 \sec r_2, \quad S_3=\sqrt{S}-a_3 \sec r_3,$$

the envelope $\Sigma\,(BC-F^2)\,S_1^2+2\Sigma\,(GH-AF)\,S_2 S_3=0$

gives a pair of circles cutting S_1, S_2, S_3 at the angles $\theta_1, \theta_2, \theta_3$ and their supplements $\pi-\theta_1$, $\pi-\theta_2$, $\pi-\theta_3$ respectively.

Particular case.—If $\theta_1=0$, $\theta_2=0$, and $\theta_3=0$, for instance, we have Casey's Equation (see Dr. Salmon's "Conic Sections," p. 345) for a pair of circles *touching* S_1, S_2, S_3, either all externally or all internally; viz.,

$$\Sigma\,.\sqrt{\{-\cos\delta_1+\cos(r_2-r_3)\}\sec r_2.\sec r_3\,.\,\{\sqrt{S}-a_1\sec r_1\}}=0.$$

2. If

$$S_1=\sqrt{S}-(l_1x+m_1y+n_1z)=0,$$
$$S_2=\sqrt{S}-(l_2x+m_2y+n_2z)=0,$$
$$S_3=\sqrt{S}-(l_3x+m_3y+n_3z)=0,$$

represent three given conics, each having double contact with a given conic

$$S=x^2+y^2+z^2=0,$$

the above results point to the more general equation

$$\begin{vmatrix} A, & H, & G, & S_1 \\ H, & B, & F, & S_2 \\ G, & F, & C, & S_3 \\ S_1, & S_2, & S_3, & 0 \end{vmatrix}=0,$$

where $A=(l_1^2+m_1^2+n_1^2-1)\sin^2\theta_1$, $B=\ldots\ldots$, $C=\ldots\ldots$,

$$2F=-2\,\{1-(l_2l_3+m_2m_3+n_2n_3)$$
$$+\sqrt{(l_2^2+m_2^2+n_2^2-1)(l_3^2+m_3^2+n_3^2-1)}\,.\cos\theta_2\cos\theta_3\},$$

$2G=\ldots\ldots$, $2H=\ldots\ldots$

This belongs to a group of conics $S-(lx+my+nz)^2$, whose invariants are connected with those of the given conics S_1, S_2, S_3 by the equations

$$\frac{ll_1+mm_1+nn_1-1}{\sqrt{(l^2+m^2+n^2-1)(l_1^2+m_1^2+n_1^2-1)}}=\cos\theta_1, \quad \ldots=\cos\theta_2, \quad \ldots=\cos\theta_3.$$

I am not sure, however, that I can give the geometric meaning of these relations.

The President next requested assistance in the solution of a "Question in the Mathematical Theory of Vibrating Strings," which he had been unable to solve.

A string is said to execute a forced vibration when it is compelled to perform vibrations synchronous with those of a vibrating body to which one end is attached. The amplitude of the forced vibrations is greatest when the length or tension of the string is so adjusted that its natural period of vibration, for its fundamental note, or one of its harmonics, is the same as that of the forcing body. And, within limits, the amplitude diminishes as the period of the string diverges from that of the body.

The theory of forced vibrations, when the motion of the forcing body is transverse to the length of the string, has been fully discussed by various writers, and especially by Helmholz and Dordini. The latter of these has shown how it is that the amplitude of vibration of the string may vary while that of the body remains constant; and, further, he has given expressions which do not lead to the absurd conclusion (resulting from the ordinary formulæ) of an infinite amplitude when the periods of string and body are absolutely synchronous.

The expressions in question, or others easily deducible from them, when applied to the discussion of the nodes, and length of vibrating segments, explain the phenomena which may be observed in experiment, of nodes of least motion rather than of perfect rest, and of segments of varying length.

But of the forced vibrations, when the forcing body moves in the direction of the string's length, there is, so far as the speaker was aware, as yet no mathematical theory. The principal phenomenon is well known, viz., that when the length and tension of the string are such that it gives out the same note as the forcing body if the forcing motion is transverse, it will give out the octave below when the forcing motion is longitudinal.

In this problem there arise two *primâ facie* difficulties; first, that a forcing motion may be conceived, and, indeed, may be experimentally set up, without producing a vibration in the string; and, secondly, that a mechanical vibration of a given period can give rise to a vibration of a period double of the former. The key to the explanation of these difficulties is probably to be found in the consideration, corroborated by experience, that the motion of the string depends upon small quantities of the second order.

A mathematical solution of the question is the desideratum suggested.

Mr. Strutt made a few remarks on the subject, and mentioned some results he had arrived at. A communication from Prof. Cayley, respecting the extension of the Society's sphere of action, was laid before the meeting by the President; it was determined that the matter should be discussed at the next meeting of the Society. Prof. Clerk-Maxwell asked for information from the members as to the convention established among Mathematicians, with respect to the relation between the positive direction of motion along any axis, and the positive direction of rotation round it. In Sir W. R. Hamilton's Lectures on Quaternions, the coordinate axes are drawn x to South, y to West, and z upwards.

The same system is adopted in Prof. Tait's Quaternions, and in Listing's "Vorstudien zur Topologie." The positive directions of translation and of rotation are thus connected in a left-handed screw, or the tendril of the hop.

On the other hand, in Thomson's and Tait's Natural Philosophy, § 234, the relations are defined with reference to a watch, and lead to the opposite system, symbolized by an ordinary or right-handed screw, or the tendril of the vine. If the actual rotation of the earth from West to East be taken positive, the direction of the earth's axis from South to North is positive in this system. In pure mathematics little inconvenience is felt from this want of uniformity; but in astronomy, electro-magnetics, and all physical sciences, it is of the greatest importance that one or other system should be specified and persevered in. The relation between the one system and the other is the same as that between an object and its reflected image, and the operation of passing from the one to the other has been called by Listing *Perversion.*

Sir W. Thomson and Dr. Hirst stated the arguments in favour of the right-handed system, derived from the motion of the earth and planets and the convention that North is to be reckoned positive, and also from the practice of Mathematicians, in drawing x to the right-hand and y upwards on the plane of the black board, and z towards the spectator. No arguments in favour of the opposite system being given, the right-handed system, symbolized by a corkscrew or the tendril of the vine, was adopted by the Society.

The following presents were received:—

"Crelle's Journal," 73 Band, zweites Heft, April, 1871.

"Proceedings of the Royal Society," vol. xix. No. 127.

"Monatsbericht," Feb., März, 1871.

"Journal of London Institution," No. 5.

"Journal of Institute of Actuaries," No. lxxxii., January, 1871.

"Jahrbuch über die gesammten Fortschritte der Mathematik," erster Band, 1868: from Dr. Karl Ohrtmann and Dr. Felix Müller.

"Theorie und Anwendung des sogenannten Variationscalcus, von Dr. G. W. Strauch," (1849): from Mr. C. R. Hodgson, B.A.

June 8th, 1871.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

Visitors, Messrs. C. Smith, M.A., J. W. L. Glaisher, B.A., and V. Dwelshauvers-Dery, Professeur de Mécanique appliquée à Liège.