VARIOUS EXTENSIONS OF ABEL'S LEMMA

By T. J. I'A. BROMWICH.

[Received June 5th, 1907.—Read June 13th, 1907.]

THE following paper contains a collection of various inequalities which are all, in a certain sense, extensions of Abel's lemma, that if the sequence of factors (v_n) is real, positive and decreasing, then

$$hv_1 < \sum_1^p a_n v_n < Hv_1.$$

where H, h are the upper and lower limits of

$$a_1 + a_2 + \ldots + a_n,$$

as n varies from 1 to p.

These results do not seem to have been published in a general form hitherto, although no doubt special cases have been used by many authors. A systematic use of them has enabled me to shorten the proofs of a number of known theorems on limits, and to obtain various extensions of such theorems. Some of these applications are given in connexion with each of the inequalities obtained below; of these the only actual novelties appear to be the theorems on divergent series given in §§ 1 and 4–7, and some of the results on double series in § 5.

1. *Real, Decreasing Positive Factors.*

Suppose that the sequence (v_n) consists of positive terms only, and *never increases*, then by the familiar transformation (due to Abel) we have

$$(1) \qquad \sum_1^p a_n v_n = s_1(v_1 - v_2) + s_2(v_2 - v_3) + \ldots + s_{p-1}(v_{p-1} - v_p) + s_p v_p,$$

where

$$s_n = a_1 + a_2 + \ldots + a_n.$$

Let m be any index less than p and take H, h to denote the upper and lower limits of $s_1, s_2, \ldots, s_{m-1}$, while H_m, h_m denote those of $s_m, s_{m+1}, \ldots, s_p$.

Then the sum on the right of (1) is increased if we put H in place of

$s_1, s_2, \ldots, s_{m-1}$, and H_m in place of $s_m, s_{m+1}, \ldots, s_p$, because all the factors $v_1-v_2, v_2-v_3, \ldots, v_{p-1}-v_p, v_p$ are positive. Thus

$$\sum_1^p a_n v_n \leqslant H[(v_1-v_2)+(v_2-v_3)+\ldots+(v_{m-1}-v_m)]$$
$$+H_m[(v_m-v_{m+1})+(v_{m+1}-v_{m+2})+\ldots+(v_{p-1}-v_p)+v_p]$$

or $$\sum_1^p a_n v_n \leqslant H(v_1-v_m)+H_m v_m.$$

By a similar argument with regard to h, h_m, we establish the complete inequality

$$(2) \qquad h(v_1-v_m)+h_m v_m \leqslant \sum_1^p a_n v_n \leqslant H(v_1-v_m)+H_m v_m,$$

which is *the extended form of Abel's inequality*.* We get back to Abel's result by taking $m=1$.

Applications.—The inequality (2) leads at once to the cases of chief practical interest of the generalized form of Abel's theorem given by Mr. Hardy.† Suppose, in fact, that the factor v_n is a function of a variable x, and that $v_n(x)$ tends to the limit 1 as x tends to 1, while $v_0 \geqslant v_1 \geqslant v_2 \geqslant \ldots$, for values of x less than 1.

Then, *if* Σa_n *converges to a sum* s, we can choose m so that h_m, H_m lie between $s-\epsilon$, $s+\epsilon$, however small ϵ may be, and however great p may be. Thus (2) leads to

$$h(v_0-v_m)+(s-\epsilon)v_m \leqslant \sum_0^x a_n v_n \leqslant H(v_0-v_m)+(s+\epsilon)v_m.$$

Now, as x tends to 1, the right and left sides of the last inequality tend respectively to $(s-\epsilon)$ and $(s+\epsilon)$, since v_0 and v_m both tend to 1. We have therefore

$$s-\epsilon \leqslant \varliminf_{x\to 1} \Sigma a_n v_n \leqslant \varlimsup_{x\to 1} \Sigma a_n v_n \leqslant s+\epsilon.$$

Since ϵ is arbitrarily small, these inequalities cannot be true unless

$$\lim_{x\to 1} \sum_0^\infty a_n v_n = s.$$

But *when* Σa_n *is divergent*, m can be found so that $h_m > N$, however great N is; and so

$$\sum_0^\infty a_n v_n \geqslant h(v_0-v_m)+N v_m.$$

Repeating the foregoing argument we see that

$$\varliminf_{x\to 1} \Sigma a_n v_n \geqslant N.$$

Hence $$\lim_{x\to 1} \sum_0^\infty a_n v_n = \infty,$$

a result which appears to be novel, although an immediate extension of one due to Abel. As a simple example we note that

$$\Sigma \frac{x^n}{1+x^n}, \qquad \Sigma \frac{1}{n}\frac{x^n}{1+x^n},$$

---

* If $\Sigma a_n v_n$ is separated into two parts, from 1 to $m-1$, and from m to p, Abel's inequality can be applied to each part; but the limits obtained are not so close as in (2).

† *Proc. London Math. Soc.*, Ser. 2, Vol. 4, 1906, p. 249 (especially § 3).

tend to infinity as x tends to 1. Of course this conclusion is verified at once by the obvious inequalities

$$\Sigma \frac{x^n}{1+x^n} > \tfrac{1}{2}\Sigma x^n = \tfrac{1}{2}\cdot\frac{1}{1-x};$$

$$\Sigma \frac{1}{n}\cdot\frac{x^n}{1+x^n} > \tfrac{1}{2}\Sigma \frac{x^n}{n} = \tfrac{1}{2}\log\left(\frac{1}{1-x}\right).$$

The inequality (2) can also be used to establish the comparison theorems for divergent series to which we shall be led later (see § 6).

There is an inequality corresponding to (2) in the case of *increasing* factors, but this seems to be of less practical importance; we record the result without proof beyond the remark that the factors $v_1 - v_2$, $v_2 - v_3$, ..., $v_{p-1} - v_p$ are *negative* in (1). We then find

$$Hv_1 - (H - H_m)\, v_m - (H_m - h_m)\, v_p < \sum_1^p a_n v_n$$
$$< hv_1 + (h_m - h)\, v_m + (H_m - h_m)\, v_p.$$

In particular, with $\quad h_m = h \quad$ and $\quad H_m = H,$

we find $\quad Hv_1 - (H-h)\, v_p < \sum_1^p a_n v_n < hv_1 + (H-h)\, v_p.$

2. *Inequalities for Integrals corresponding to* § 1.

The analogy between Abel's inequality and the so-called *second theorem of the mean* at once suggests the following theorem:—

If the function $v(x)$ never increases with x, but is always positive in an interval (a, b), then

$$(3) \quad h\,[v(a) - v(c)] + h_c v(c) \leqslant \int_a^b v(x) f(x)\, dx \leqslant H\,[v(a) - v(c)] + H_c v(c),$$

where H, h are the upper and lower limits of the integral

$$\int_a^{\xi} f(x)\, dx$$

as ξ ranges from a to c, while H_c, h_c are those found as ξ ranges from c to b. Here $v(a)$ and $v(c)$ are used to denote the limits $v(a+0)$ and $v(c-0)$ respectively.

If the function $v(x)$ is supposed differentiable the inequality (3) is most easily proved by integration by parts (compare p. 65 below); but, in the general case, the inequality can be obtained by a simple modification of Pringsheim's proof* for the case $c = b$.

Let the interval (a, b) be divided into n sub-intervals by inserting

* *Münchener Sitzungsberichte*, Bd. xxx., 1900, p. 209.

points $x_1, x_2, \ldots, x_{n-1}$, and let $x_0 = a$, $x_n = b$; write further $v_r = v(x')$, or if $v(x)$ is discontinuous at x_r, we take v_r as the limit* of $v(x)$ as x approaches x_r from *smaller* values of x.

Then, if $$J = \int_a^b v(x) f(x)\, dx, \quad J_r = \int_{x_r}^{x_{r+1}} v(x) f(x)\, dx,$$

and $$K_r = \int_{x_r}^{x_{r+1}} f(x)\, dx,$$

we find $$J = \sum_{r=0}^{n-1} J_r,$$

and $$J_r - v_{r+1} K_r = \int_{x_r}^{x_{r+1}} [v(x) - v_{r+1}] f(x)\, dx.$$

In the last integral the bracket is positive and less than $v_r - v_{r+1}$, in virtue of the decreasing property of $v(x)$; thus

$$| J_r - v_{r+1} K_r | < (v_r - v_{r+1}) \int_{x_r}^{x_{r+1}} | f(x) |\, dx.$$

Consequently if μ is the maximum value of

$$\int_{x_r}^{x_{r+1}} | f(x) |\, dx$$

for all the sub-intervals, we find

$$| J_r - v_{r+1} K_r | < \mu (v_r - v_{r+1}).$$

Hence $$\left| J - \sum_{r=0}^{n-1} v_{r+1} K_r \right| < \mu v_0.$$

Now, if we take x_m to coincide with c, we see from the inequality (2) of § 1 that

$$h [v(x_1) - v(c)] + h_c v(c) < \sum_{r=0}^{n-1} v_{r+1} K_r < H [v(x_1) - v(c)] + H_c v(c),$$

because $$K_0 + K_1 + \ldots + K_{r-1} = \int_a^{x_r} f(x)\, dx.$$

Consequently we have

$$h [v(x_1) - v(c)] + h_c v(c) - \mu v(a) < J < H [v(x_1) - v(c)] + H_c v(c) + \mu v(a).$$

* That this limit exists follows from the monotonic property of $v(x)$.

Now, let all the sub-intervals tend uniformly to zero, then μ also tends to zero, provided that the integral

$$\int_a^b |f(x)|\, dx$$

is convergent; and $v(x_1)$ tends to the limit $v(a+0)$, or $v(a)$ in our present notation. Then, since J is independent of the mode of choosing the sub-intervals, we find

$$h[v(a)-v(c)]+h_c v(c) \leqslant J \leqslant H[v(a)-v(c)]+H_c v(c).$$

Pringsheim has shewn, however, that the absolute convergence of $\int_a^b f(x)\,dx$ is superfluous; and that the convergence of this integral together with that of $\int_a^b v(x) f(x)\,dx$ will suffice to establish the result.

In fact, under these conditions, we can find a *finite* number (p) of intervals enclosing all the discontinuities of $f(x)$, and such that $|L_s| < \epsilon$ and $|L_s'| < \epsilon$, where L_s, L_s' denote the integrals of $f(x)$ and of $f(x)\,v(x)$ respectively taken over the s-th of these intervals.

For any part of the range (a, b) outside these p intervals we can argue as above, and deduce that the corresponding contribution to $(J-\Sigma v_{r+1} K_r)$ is less than μv_0, where μ can be made as small as we please. But for these special intervals, the difference is numerically less than

$$|\Sigma v_{s+1} L| + |\Sigma L_s'| < p\epsilon + |\Sigma v_0 L_s| < p\epsilon\,(1+v_0),$$

and so we arrive finally at the same inequality as before.

Applications.—The arguments of § 1 need no further alteration in order to establish such theorems as the following:—

If $v(x, t)$ is a decreasing function of x ($t > 0$) which tends to the limit 1, *as t tends to* 0, *then*

$$\lim_{t\to 0} \int_a^\infty v(x, t) f(x)\,dx = \int_a^\infty f(x)\,dx,$$

if the latter is convergent. Also $\lim_{t\to 0} \int_a^\infty v(x, t) f(x)\,dx = \infty$,

if $\int^\infty f(x)\,dx$ *diverges to infinity.*

As another application, we consider Jordan's theorem:*—

Let $v(x)$ be a function decreasing† *as x increases from a to b; and let $f(x, t)$ be a function of x, t, such that*

(1) *The integral* $\left|\int_a^\xi f(x, t)\,dx\right| < K$, *where ξ lies between a, b and K is independent of ξ and t.*

(2) *The limit* $\lim_{t\to\infty} \int_a^\xi f(x, t)\,dx$ *is independent of ξ and equal say to L, provided that ξ belongs to any sub-interval (a', b'), from which a is excluded; and the convergence to the limit is uniform in the sub-interval.*

* *Cours d'Analyse*, t. II., 2me éd., 1894, p. 228.

† By taking the difference of two such functions we pass at once to Jordan's *fonction à variation bornée;* and since the operation of subtraction will not affect the final result, there is no real loss of generality in restricting the function at the start.

Then
$$\lim_{t\to\infty}\int_a^\xi v(x)f(x,t)\,dx = Lv(a),$$
where $v(a)$ denotes the limit of $v(x)$ as x approaches a through larger values.

For, suppose c to be any number greater than a, then we have, from (3),
$$\int_a^\xi v(x)f(x,t)\,dx \leqslant H[v(a)-v(c)]+H_c v(c) = (H-H_c)[v(a)-v(c)]+H_c v(a),$$
where, for brevity, we suppress the left-hand sides of the inequalities.

Now, in virtue of condition (1), $H-H_c < 2K$, and choose c so as to make $2K[v(a)-v(c)]<\epsilon$, then, since $\lim_{t\to\infty} H_c = L$, we have
$$\overline{\lim_{t\to\infty}}\int_a^\xi v(x)f(x,t)\,dx \leqslant Lv(a)+\epsilon.$$
Similarly the other sides of the inequalities give
$$\underline{\lim_{t\to\infty}}\int_a^\xi v(x)f(x,t)\,dx \geqslant Lv(a)-\epsilon.$$
Thus
$$\lim_{t\to\infty}\int_a^\xi v(x)f(x,t)\,dx = Lv(a).$$

Clearly in the foregoing $f(x,t)$ may be complex, since the argument can be applied to the real and imaginary parts separately. Thus we have, for example,
$$\lim_{t\to\infty} t\int_0^\xi e^{-tx}dx = \lim_{t\to\infty}(1-e^{-t\xi}) = 1, \quad\text{and so}\quad \lim_{t\to\infty} t\int_0^\xi e^{-tx}v(x)\,dx = v(0), \quad (\xi>0),$$
where t is complex and tends to infinity along any path which makes its real part tend to positive infinity (compare Picard, *Traité d'Analyse*, t. II., 1er éd., p. 171).

3. *Complex Factors.*

If the factors v_n are complex, we assume (following Dirichlet) that the series
$$\sum_1^\infty |v_n-v_{n+1}|$$
is convergent. It follows that the series $\sum_1^\infty (v_n-v_{n+1})$ converges, and therefore v_n tends to a definite limit as n tends to infinity. Write then
$$V_n = \{|v_n-v_{n+1}|+|v_{n+1}-v_{n+2}|+\dots\text{ to }\infty\}+\lim_{\nu\to\infty}|v_\nu|,$$
and it follows that
$$V_n-V_{n+1} = |v_n-v_{n+1}|.$$
Hence
$$V_n-V_p \geqslant |v_n-v_p|, \quad\text{if } p>n,$$
and so
$$V_n \geqslant |v_n|,$$
by making p tend to infinity.

It follows from (1) of § 1 that, if σ is any number (real or complex)
$$(4)\quad \left|\sum_1^p a_n v_n-\sigma v_1\right| < \sum_1^{m-1}\eta(V_n-V_{n+1})+\sum_m^{p-1}\eta_m(V_n-V_{n+1})+\eta_m V_p$$
$$= \eta(V_1-V_m)+\eta_m V_m,$$
where η, η_m are the upper limits to $|s_n-\sigma|$ as n ranges from 1 to $m-1$, and from m to p respectively.

Applications.—We can extend the argument of the small type on p. 59 to this case, *provided that* Σa_n *is convergent.*

For suppose that $\sum\limits_0^\infty a_n = \sigma$, and that $\lim\limits_{x\to 1} v_n = 1$, so that

$$\lim_{x\to 1} (V_0 - V_m) = \lim_{x\to 1} \{ |v_0-v_1| + |v_1-v_2| + \ldots + |v_{m-1}-v_m| \} = 0.$$

Then, if $\lim\limits_{x\to 1} V_0$ *is finite, we have*

$$\lim_{x\to 1} \sum_0^\infty a_n v_n = \sigma.$$

For we can choose m so as to make $\eta_m V_1$ less than ϵ, and when m is fixed, since η is finite, we have

$$\lim_{x\to 1} \eta (V_0 - V_m) = 0;$$

thus we find

$$\overline{\lim_{x\to 1}} \, | \Sigma a_n v_n - \sigma | < \epsilon,$$

which gives the desired result.

The only fresh condition introduced is that $\lim\limits_{x\to 1} V_0$ must be finite. Thus, for example, with $v_n = x^n$, we find that $\lim\limits_{x\to 1} \dfrac{|1-x|}{1-|x|}$ must be finite, which implies that the path by which x tends to 1 must lie within the inner loop of a certain limaçon.

For, if we write $$x = 1 - \rho e^{i\phi},$$
we find from the condition $$|1-x| \leqslant k\{1-|x|\} \quad (k>1),$$
the equivalent form $$\rho(k^2-1) \leqslant 2k(1-k\cos\phi),$$
which represents the inner loop of a limaçon, with a node at $\rho = 0$ (*i.e.*, $x = 1$). Stolz and Gmeiner have used the limaçon $k\rho = 2(1-k\cos\phi)$, which is similar to the above curve, but of smaller linear dimensions.

In Pringsheim's paper* the area used is bounded by a circle and two lines which intersect at the point $x = 1$: it will be seen that this area falls within the limaçon.

Similarly, if $$v_n = r^n P_n(\cos\theta),$$

it is proved in my paper just quoted (see § 2, p. 206) that

$$V_0 \leqslant \sqrt{(1-2r\cos\theta + r^2)}/(1-r),$$

and so the path of approach to the point $r = 1$, $\theta = 0$ must lie within an area of the unit-circle which is bounded in the same way as for a power-series.

* *Münchener Sitzungsberichte*, Bd. XXXI., 1901, p. 514. Pringsheim's figure is given also in my paper (Fig. 1), on "Series of Zonal Harmonics" (*Proc. London Math. Soc.*, Ser. 2, Vol. 4, 1906, p. 204). The limaçon used here is drawn on p. 211 of my book on *Infinite Series*.

On the other hand, *when Σa_n is divergent, we cannot infer that*

$$\lim_{x \to 1} \sum_0^\infty a_n v_n = \infty .$$

In fact the argument of § 1 obviously depends on the fact that v_n is real, and in the simplest case ($v_n = x^n$) Pringsheim has proved that, even when a_n is real and positive and Σa_n diverges, the limit may depend on the path by which x approaches 1. Pringsheim gives as an example the series obtained by rearranging in powers of x the series

$$\exp\left\{\frac{1}{(1-x)^2}\right\} = 1 + \frac{1}{(1-x)^2} + \frac{1}{2!}\frac{1}{(1-x)^4} + \frac{1}{3!}\frac{1}{(1-x)^6} + \dots .$$

If this series is denoted by $\Sigma a_n x^n$, it is clear that a_n is positive; and Σa_n diverges, because, if x tends to 1 along the real axis, $1/(1-x)^2$ tends to infinity, so that

$$\lim_{x \to 1} \Sigma a_n x^n = \infty \quad (0 < x < 1).$$

Now, since a_n is positive, Σa_n must either converge or diverge; and if convergent we should have, by the familiar form of Abel's theorem,

$$\lim_{x \to 1} \Sigma a_n x^n = \Sigma a_n \quad (0 < x < 1),$$

but this limit is infinity, so that Σa_n must diverge.

But yet, if we write $1-x = \rho e^{i\phi}$, as above (p. 64), we find

$$\left| \exp\left\{\frac{1}{(1-x)^2}\right\} \right| = \exp\left(\frac{1}{\rho^2}\cos 2\phi\right),$$

which tends to zero with ρ, if $\cos 2\phi$ is negative, or if $\phi > \frac{1}{4}\pi$.

It is perhaps natural to enquire if the inequality (4) cannot be modified so as to apply to a *complex integral*; in this case the result is obtained most rapidly by the method of integration by parts. This is permissible here because the function $v(x)$ is supposed analytic and $v(x)$ is therefore differentiable. If we write

$$g(z) = \int_a^z f(x)\,dx,$$

it follows that $$\int_a^b f(x)\,v(x)\,dx = g(b)\,v(b) - \int_a^b g(x)\,v'(x)\,dx,$$

and so if H is the upper limit of $|g(x)|$ on the path of integration, we have

$$\left| \int_a^b f(x)\,v(x)\,dx \right| < HV,$$

where $$V = \int_a^b |v'(x)| . |dx| + |v(b)| .$$

This method has been recently used by Mr. Berry* to prove that

$$\lim_{R \to \infty} \int_{-R}^{R} e^{ix} \frac{dx}{x} = 0,$$

* *Messenger of Mathematics*, Vol. XXXVII., 1907, p. 61.

when the path of integration is a semicircle joining the points $-R$, R, and passing through the upper half of the complex plane.

In fact, if $f(x) = e^{ix}$ and $v(x) = 1/x$, we find that

$$V = (\pi+1)/R,$$

and
$$\left| \int_R^z f(x)\,dx \right| = \left| \frac{1}{i}(e^{iz}-e^{iR}) \right| < 2,$$

because
$$|e^{iz}| \leqslant 1,$$

so that
$$\left| \int_{-R}^{R} e^{ix}\frac{dx}{x} \right| < \frac{2(\pi+1)}{R},$$

which gives the desired result.

The same method will give (for the same path)

$$\lim_{R\to\infty} \int_{-R}^{R} e^{ix}\frac{P(x)}{Q(x)}\,dx = 0,$$

if $P(x)$ and $Q(x)$ are polynomials in x of which the first is of degree one less than the second.

4. *Inequalities corresponding to those of § 1 for Double Series.*

Suppose that $v_{m,n}$ is a real positive sequence which decreases with respect to *both* indices, in the sense that

$$v_{m,n}-v_{m+1,n} \geqslant 0, \qquad v_{m,n}-v_{m,n+1} \geqslant 0,$$

$$\Delta_{m,n} = v_{m,n}-v_{m+1,n}-v_{m,n+1}+v_{m+1,n+1} \geqslant 0.$$

Then it is known that*

$$(5) \quad \sum_{m=1}^{p}\sum_{n=1}^{q} a_{m,n}v_{m,n} = \sum_{m=1}^{p-1}\sum_{n=1}^{q-1}\Delta_{m,n}s_{m,n}+\sum_{m=1}^{p-1}\Delta_m s_{m,q}+\sum_{n=1}^{q-1}\Delta_n s_{p,n}+s_{p,q}v_{p,q},$$

where
$$\Delta_m = v_{m,q}-v_{m+1,q}, \qquad \Delta_n = v_{p,n}-v_{p,n+1}.$$

Here, using the ordinary geometrical representation, $s_{m,n}$ denotes the sum of all the terms contained within a rectangle whose sides are m and n. It should, perhaps, be remarked that (5) is an algebraical *identity*, and does not depend on the preceding inequalities.

Now suppose that for all values of m and n between $1, p$ and $1, q$ respectively, the upper and lower limits of $s_{m,n}$ are H, h; then since $\Delta_{m,n}$, Δ_m, Δ_n, $v_{p,q}$ are all positive it follows at once from (5) that

$$(6) \qquad hv_{1,1} < \sum_1^p\sum_1^q a_{m,n}v_{m,n} < Hv_{1,1},$$

which is the immediate extension to double series of the ordinary form of Abel's lemma. To see that (6) is correct, we need only note that to put

* Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 1, 1903, p. 124; from the results given there it is easy to infer the truth of our inequalities for any number of variables of summation.

$s_{m,n} = H$ is equivalent to writing H in place of $a_{1,1}$ and 0 in place of all the other a's.

To obtain the inequality corresponding to (2) of § 1, let us suppose that H_ν, h_ν are the upper and lower limits of $s_{m,n}$ when $m \geqslant \nu$, $n \geqslant \nu$; H, h being the upper and lower limits for $s_{m,n}$ if *either* suffix is less than ν. We then obtain

$$(7) \qquad h(v_{1,1}-v_{\nu,\nu})+h_\nu v_{\nu,\nu} < \sum_1^p \sum_1^q a_{m,n} v_{m,n} < H(v_{1,1}-v_{\nu,\nu})+H_\nu v_{\nu,\nu},$$

since, to obtain the right-hand side, we have to write H_ν for $s_{m,n}$ if $m, n \geqslant \nu$, and otherwise H. But this is equivalent to writing $a_{1,1} = H$, $a_{\nu,\nu} = H_\nu - H$, which gives the right-hand side of (7). Similarly for the left-hand side.

It is possible to extend (7) to complex factors by a method similar to that of § 3.

Applications.—The inequality (7) enables us to give a new proof and extension of results already communicated to the Society.*

Suppose, in fact, that the series $\sum_0^\infty \sum_0^\infty a_{m,n}$ is convergent in Pringsheim's sense and satisfies the condition of finitude,† then if $v_{m,n}$ is a function of x, y which satisfies the inequalities prescribed at the beginning of this article, and tends to the limit 1 as x, y tend to 1, we have

$$\lim_{x,y\to 1} \sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n} = s,$$

where s is Pringsheim's sum of the double series $\Sigma\Sigma a_{m,n}$.

For, in fact, we can find ν, so that

$$s-\epsilon \leqslant h_\nu < H_\nu \leqslant s+\epsilon,$$

and

$$-C < h, \qquad H < C,$$

by the condition of finitude.

Thus (7) yields‡

$$-C(v_{0,0}-v_{\nu,\nu})+(s-\epsilon)v_{\nu,\nu} < \sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n} < C(v_{0,0}-v_{\nu,\nu})+(s+\epsilon)v_{\nu,\nu}.$$

Since $v_{0,0}$ and $v_{\nu,\nu}$ both tend to 1 as x, y tend to 1, we find

$$s-\epsilon \leqslant \overline{\underline{\lim}}_{x,y\to 1} \sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n} \leqslant s+\epsilon.$$

Since ϵ is arbitrarily small, these inequalities can only be true if

$$\lim_{x,y\to 1} \sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n} = s.$$

* Bromwich and Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, 1904, p. 161 (see § 3, p. 164); the case discussed there is given by writing $v_{m,n} = x^m y^n$ and supposing $\Sigma\Sigma a_{m,n}$ convergent.

† So that $|s_{m,n}| < C$, where C is independent of m, n.

‡ The convergence in Pringsheim's sense of the double series $\Sigma\Sigma a_{m,n} v_{m,n}$ follows from Hardy's paper quoted on p. 66 above, or can be proved by a direct application of the inequality (6).

Pass next to the case of divergence, say to $+\infty$; *it will be assumed that the divergence is not due to the presence of any singly divergent row or column.* Thus, when ν is fixed we can determine a constant C_ν, such that

$$|s_{m,n}| < C_\nu,$$

provided that *either* of m, n is less than ν; thus, for example, we may have m increasing without limit, provided that $n < \nu$.

Let ν be now found so that

$$s_{m,n} > N, \quad \text{if} \quad m, n \geqslant \nu;$$

this is possible in view of the divergence of the double series $\Sigma\Sigma a_{m,n}$; thus $h_\nu \geqslant N$. Also

$$h \geqslant -C_\nu,$$

and so we have
$$\sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n} > N v_{\nu,\nu} - C_\nu (v_{0,0} - v_{\nu,\nu}).$$

Thus repeating the former argument, we find

$$\varliminf_{x, y \to 1} \sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n} \geqslant N,$$

and so we must have
$$\lim_{x, y \to 1} \sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n} = \infty.$$

5. *Inequalities for a Quotient.*

We consider the quotient $X_p = R_p/Q_p$,

where
$$R_p = \sum_1^p b_n v_n, \qquad Q_p = \sum_1^p a_n v_n.$$

For brevity write
$$A_n = a_1 + a_2 + \ldots + a_n,$$
$$B_n = b_1 + b_2 + \ldots + b_n.$$

Then, as in § 1, we find

$$R_p = B_1(v_1 - v_2) + B_2(v_2 - v_3) + \ldots + B_{p-1}(v_{p-1} - v_p) + B_p v_p.$$

Now, suppose that $a_1, a_2, \ldots, a_n, \ldots$ are all positive and consider the sequence of quotients
$$B_1/A_1, \; B_2/A_2, \; \ldots, \; B_p/A_p.$$

Let H, h be the upper and lower limits of the *whole** set of quotients, while H_m, h_m are those for which the suffix is not less than m; so that

$$H \geqslant H_m, \quad \text{and} \quad h \leqslant h_m.$$

Thus, *if the sequence* (v_n) *is positive and decreasing*, we find

$$R_p < H[A_1(v_1 - v_2) + A_2(v_2 - v_3) + \ldots + A_{m-1}(v_{m-1} - v_m)]$$
$$+ H_m[A_m(v_m - v_{m+1}) + A_{m+1}(v_{m+1} - v_{m+2}) + \ldots + A_{p-1}(v_{p-1} - v_p) + A^p v_p].$$

Thus
$$R_p < H_m Q_p + (H - H_m)(Q_m - A_m v_m),$$

* Note the distinction between this case and that of § 1.

and since $H \geqslant H_m$, we find (on including the corresponding expression with h, h_m),

$$h_m Q_p - (h - h_m) Q_m < R_p < H_m Q_p + (H - H_m) Q_m.$$

That is

$$h_m - (h - h_m) \frac{Q_m}{Q_p} < \frac{R_p}{Q_p} < H_m + (H - H_m) \frac{Q_m}{Q_p}. \tag{8}$$

Again, *if the sequence* (v_n) *is positive and increasing*, we find that

$$\begin{aligned} R_p < h\,[A_1(v_1 - v_2) + A_2(v_2 - v_3) + \ldots + A_{m-1}(v_{m-1} - v_m)] \\ + h_m[A_m(v_m - v_{m+1}) + A_{m+1}(v_{m+1} - v_{m+2}) + \ldots + A_{p-1}(v_{p-1} - v_p] \\ + H_m A_p v_p, \end{aligned}$$

because here all the differences are *negative*, but v_p is still positive.

Hence, as before, we get

$$R_p < h_m Q_p + (H_m - h_m) A_p v_p + (h_m - h)(A_m v_m - Q_m);$$

and since $h_m - h$ and Q_m are *positive*, we may omit the last term in the last bracket. Thus, summing up, we find

$$\begin{cases} H_m - (H_m - h_m) \dfrac{A_p v_p}{Q_p} - (H - H_m) \dfrac{A_m v_m}{Q_p} < \dfrac{R_p}{Q_p}, \\ \dfrac{R_p}{Q_p} < h_m + (H_m - h_m) \dfrac{A_p v_p}{Q_p} + (h_m - h) \dfrac{A_m v_m}{Q_p}. \end{cases} \tag{9}$$

Finally, *if the sequence* (v_n) *first increases to a maximum* v_μ *and afterwards steadily decreases*, there is no difficulty in modifying the foregoing work to prove that, if $m < \mu$,

$$\begin{cases} h_m - (H - H_m) \dfrac{A_m v_m}{Q_p} - (H_m - h_m) \dfrac{A_\mu v_\mu}{Q_p} < \dfrac{R_p}{Q_p}, \\ \dfrac{R_p}{Q_p} < H_m + (h_m - h) \dfrac{A_m v_m}{Q_p} + (H_m - h_m) \dfrac{A_\mu v_\mu}{Q^p}. \end{cases} \tag{10}$$

We note that the method of § 2 can be at once applied to deduce inequalities for the quotient of two integrals from (8)–(10). Thus, if $f(x)$ is a positive function from a to b and $v(x)$ decreases in the same interval, we can obtain limits for the quotient

$$\int_a^b g(x)\, v(x)\, dx \Big/ \int_a^b f(x)\, v(x)\, dx$$

in terms of those of $\displaystyle \int_a^\xi g(x)\, dx \Big/ \int_a^\xi f(x)\, dx.$

I do not stay to write these out, as the reader should have no difficulty in recognizing the necessary changes in (8)–(10); and up to the present I have not made any practical use of these inequalities.

Applications.—*Comparison Theorem for Divergent Series.*

Suppose that Σa_n is a divergent series of positive terms, and that (v_n) is a decreasing sequence of functions of x, such that

$$\lim_{x\to 1} v_n = 1.$$

Then, if

$$\lim_{n\to\infty} (B_n/A_n) = l,$$

we have also

$$\lim_{x\to 1} \left(\sum_0^\infty b_n v_n \Big/ \sum_0^\infty a_n v_n\right) = l.$$

For then we can choose m so that

$$l-\epsilon \leqslant h_m < H_m < l+\epsilon,$$

and then (8) gives

$$l-\epsilon-(h-l+\epsilon)\frac{Q_m}{Q_p} < \frac{R_p}{Q_p} < l+\epsilon+(H-l-\epsilon)\frac{Q_m}{Q_p}.$$

If $\sum_0^\infty a_n v_n$ is divergent, Q_p will tend to infinity with p, and then the inequality becomes

$$l-\epsilon \leqslant \varlimsup_{p\to\infty} \frac{R_p}{Q_p} \leqslant l+\epsilon,$$

and since these limits are independent of m, we must have

$$\lim_{p\to\infty} \left(\sum_0^p b_n v_n \Big/ \sum_0^p a_n v_n\right) = l,$$

so that $\sum_0^\infty b_n v_n$ is also divergent, and the quotient of $\sum_0^p b_n v_n$ by $\sum_0^p a_n v_n$ tends to the limit l.

On the other hand, if (as happens in the most interesting special cases) $\sum_0^\infty a_n v_n$ converges, it follows from § 1 that

$$\lim_{x\to 1} \left(\sum_0^\infty a_n v_n\right) = \infty,$$

so that

$$\lim_{x\to 1} \left(\sum_0^m a_n v_n \Big/ \sum_0^\infty a_n v_n\right) = 0.$$

If we apply this result to the inequality for R_p/Q_p, first allowing p to tend to infinity, we find that

$$l-\epsilon \leqslant \overline{\lim_{x\to 1}} \left(\sum_0^\infty b_n v_n \Big/ \sum_0^\infty a_n v_n\right) \leqslant l+\epsilon,$$

or

$$\lim_{x\to 1} \left(\sum_0^\infty b_n v_n \Big/ \sum_0^\infty a_n v_n\right) = l.$$

This is an extension of the well known result, due to Cesàro, that

$$\lim_{x\to 1} \left(\sum_0^\infty b_n x^n \Big/ \sum_0^\infty a_n x^n\right) = l,$$

when b_n, a_n are related as already specified.

As another simple example, we take

$$\lim_{x\to 1} \left(\sum_0^\infty b_n \frac{x^n}{1+x^n} \Big/ \sum_0^\infty a_n \frac{x^n}{1+x^n}\right) = l.$$

Another simple application is to establish a result given recently by Mr. Hardy.* In fact, if we write

$$b_n = a_n\sigma_n, \quad v_n = c_n/a_n,$$

and suppose that $\sum_0^\infty a_n$, $\sum_0^\infty c_n$ are both divergent, we find

$$R_p = c_0\sigma_0 + c_1\sigma_1 + \ldots + c_p\sigma_p,$$

$$Q_p = c_0 + c_1 + \ldots + c_p,$$

$$B_p = a_0\sigma_0 + a_1\sigma_1 + \ldots + a_p\sigma_p.$$

Suppose that B_p/A_p has a definite limit l as p tends to infinity, then we can choose m so that

$$l-\epsilon \leqslant h_m < H_m \leqslant l+\epsilon.$$

Thus, *if c_n/a_n is a decreasing sequence*, we have, from (8),

$$l-\epsilon-(h-l+\epsilon)\frac{Q_m}{Q_p} < \frac{R_p}{Q_p} < l+\epsilon+(H-l-\epsilon)\frac{Q_m}{Q_p}.$$

Thus, since Q_p tends to infinity with p, we find as in the last piece of work, that

$$\lim_{p\to\infty} (R_p/Q_p) = l.$$

This result is due to Cesàro;† but Hardy has succeeded in extending it to the case when c_n/a_n *is an increasing sequence subject to the condition*

$$(a_0+a_1+\ldots+a_p)/a_p < K(c_0+c_1+\ldots+c_p)/c_p$$

for all values of p.

For the last condition gives $A_p v_p < KQ_p$,

and so the inequality (9) leads to

$$l-(2K-1)\epsilon-(H-l-\epsilon)\frac{A_m v_m}{Q_p} < \frac{R_p}{Q_p} < l+(2K-1)\epsilon+(l-h-\epsilon)\frac{A_m v_m}{Q_p},$$

from which we get as before $\lim_{p\to\infty} (R_p/Q_p) = l.$

---

* *Quarterly Journal*, Vol. XXXVIII., 1907, p. 269.

† *Bulletin des Sciences mathématiques*, (2), t. XIII., 1889, p. 51.

6. *Extension of § 5 to the Case of Complex Factors.*

If the factors v_n are complex we suppose, as in § 3, that the series

$$\sum_{1}^{\infty} | v_n - v_{n+1} |$$

is convergent, and we write again

$$V_n = \{ | v_n - v_{n+1} | + | v_{n+1} - v_{n+2} | + \dots \text{ to } \infty \} + \lim_{\nu \to \infty} | v_\nu | .$$

We suppose that the terms a_n which appear in the denominator Q_p are all real and positive, though the terms b_n may be complex; then write η for the upper limit to the differences

$$| B_1/A_1 - \sigma |, \quad | B_2/A_2 - \sigma |, \quad \dots, \quad | B_p/A_p - \sigma |,$$

and η_m for the upper limit when the suffixes are not less than m.

We get at once, since $V_n - V_{n+1} = | v_n - v_{n+1} |$, $V_n \geqslant | v_n |$ (see p. 63),

$$| R_p - \sigma Q_p | < \eta \left[A_1 (V_1 - V_2) + A_2 (V_2 - V_3) + \dots + A_{m-1} (V_{m-1} - V_m) \right]$$
$$+ \eta_m \left[A_m (V_m - V_{m+1}) + \dots + A_{p-1} (V_{p-1} - V_p) + A_p V_p \right].$$

Now, let us write

$$M_n = a_1 V_1 + a_2 V_2 + \dots + a_n V_n$$
$$= A_1 (V_1 - V_2) + A_2 (V_2 - V_3) + \dots + A_{n-1} (V_{n-1} - V_n) + A_n V_n,$$

and then $$| R_p - \sigma Q_p | < \eta M_m + \eta_m (M_p - M_m).$$

Thus

$$\left| \frac{R_p}{Q_p} - \sigma \right| < \eta_m \frac{M_p}{| Q_p |} + (\eta - \eta_m) \frac{M_m}{| Q_p |}. \tag{11}$$

Application.—*The Theorem of Comparison for Complex Divergent Series.*

The direct application of (11) is not so easy as that of (8), owing to the fact (already mentioned on p. 65) that we cannot infer the divergence of $\lim \sum_{0}^{\infty} a_n v_n$ from that of $\sum_{0}^{\infty} a_n$. To avoid this difficulty we introduce the idea of *uniform divergence*, as suggested by Pringsheim; this implies that for all points x under consideration

$$\lim_{x \to 1} \left\{ \left(\sum_{0}^{\infty} a_n V_n \right) \Big/ \left| \sum_{0}^{\infty} a_n v_n \right| \right\} < K,$$

where K is fixed.

Making this hypothesis, it follows that

$$\lim_{x\to 1}\left|\sum_0^\infty a_n v_n\right| = \infty,$$

because

$$\lim_{x\to 1}\sum_0^\infty a_n V_n = \infty,$$

in virtue of § 1.

Then (11) yields at once

$$\left|\left\{\left(\sum_0^\infty b_n v_n\right)\Big/\left(\sum_0^\infty a_n v_n\right) - \sigma\right\}\right| < K\left\{\eta_m + (\eta - \eta_m)\left(\sum_0^m a_n V_n\right)\Big/\left(\sum_0^\infty a_n V_n\right)\right\},$$

and by the usual argument this can be proved to tend to zero as x approaches 1, provided that η_m tends to zero as m tends to infinity. Thus

$$\lim_{x\to 1}\left(\sum_0^\infty b_n v_n\right)\Big/\left(\sum_0^\infty a_n v_n\right) = \lim_{n\to\infty}(B_n/A_n).$$

This result includes Pringsheim's for the case of power series, and also the result proved in § 6 of my paper on "Zonal Harmonics," quoted above.

Thus for power series $v_n = x^n$, and

$$V_n = |x|^n\, |1-x| / \{1-|x|\},$$

so that the above test for uniform divergence gives

$$\lim_{x\to 1}\frac{|1-x|}{1-|x|}\,\frac{\Sigma a_n |x|^n}{|\Sigma a_n x^n|} < K,$$

which in Pringsheim's treatment is divided into two separate conditions

$$\frac{|1-x|}{1-|x|} < K, \qquad \frac{\Sigma a_n |x|^n}{|\Sigma a_n x^n|} < K.$$

Similarly for zonal harmonics, we get

$$v_n = r^n P_n(\cos\theta) \quad \text{and} \quad V_n = \rho r^n/(1-r),$$

where

$$\rho^2 = 1 - 2r\cos\theta + r^2.$$

Then the condition becomes
$$\lim \frac{\rho}{1-r}\,\frac{\Sigma a_n r^n}{|\Sigma a_n r^n P_n(\cos\theta)|} < K,$$

which was also split up into two separate conditions in my paper (see pp. 205, 213).

7. *Extension of § 5 to Quotients of Double Series.*

Let us consider the quotient

$$R_{p,q}/Q_{p,q},$$

where

$$Q_{p,q} = \sum_{m=1}^{p}\sum_{n=1}^{q} a_{m,n} v_{m,n},$$

$$R_{p,q} = \sum_{m=1}^{p}\sum_{n=1}^{q} b_{m,n} v_{m,n},$$

and $a_{m,n}$ is positive, while $v_{m,n}$ is positive and decreasing with respect to both indices (in the sense defined at the beginning of § 4).

We shall now use the notation $A_{m,n}$ and $B_{m,n}$ to denote the sums to m, n terms $\Sigma\Sigma a_{m,n}$ and $\Sigma\Sigma b_{m,n}$; so that $A_{m,n}$ is what was denoted by $s_{m,n}$ in § 4. Then Hardy's equation [see (5), § 4] gives

$$R_{p,q} = \sum_{m=1}^{p-1} \sum_{n=1}^{q-1} \Delta_{m,n} B_{m,n} + \sum_{m=1}^{p-1} \Delta_m B_{m,q} + \sum_{n=1}^{q-1} \Delta_n B_{p,n} + B_{p,q} v_{p,q}.$$

Suppose that H, h are the upper and lower limits of $B_{m,n}/A_{m,n}$ for all values of m, n between $1, p$ and $1, q$ respectively, while H_ν, h_ν are those when both m, n are greater than ν. Then we see that $R_{p,q}$ will be increased by writing $HA_{m,n}$ or $H_\nu A_{m,n}$ in place of $B_{m,n}$; thus we find

$$R_{p,q} < HQ_{p,q} - (H-H_\nu)(Q_{p,q} + Q_{\nu,\nu} - Q_{p,\nu} - Q_{\nu,q}),$$

or

$$R_{p,q} < H_\nu Q_{p,q} + (H-H_\nu)(Q_{p,\nu} + Q_{\nu,q} - Q_{\nu,\nu}).$$

Now $H-H_\nu$ is positive and so is $Q_{\nu,\nu}$; thus $Q_{\nu,\nu}$ may be omitted from the last inequality, and we find (on including the corresponding lower limit)

$$(12) \qquad h_\nu - (h_\nu - h)\frac{Q_{p,\nu} + Q_{\nu,q}}{Q_{p,q}} < \frac{R_{p,q}}{Q_{p,q}} < H_\nu + (H-H_\nu)\frac{Q_{p,\nu} + Q_{\nu,q}}{Q_{p,q}},$$

which is the extension of (8) given above. The inequalities corresponding to (9) and (10) are necessarily more complicated; and at present I do not see that they are likely to prove of much use in practical applications. I do not, therefore, write them out here.

Application.—*The Theorem of Comparison of Two Divergent Double Series.*

It is evident that (with the same interpretation of $v_{m,n}$ as we have used in § 4) we can infer from (12) the theorem

$$\lim_{(x,y)} \left\{ \left(\sum_0^\infty \sum_0^\infty b_{m,n} v_{m,n} \right) \Big/ \left(\sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n} \right) \right\} = \lim_{(m,n)} (B_{m,n}/A_{m,n}),$$

provided that for any given value of ν,

$$\lim_{(x,y)} \left\{ \left(\sum_{m=0}^\infty \sum_{n=0}^\nu a_{m,n} v_{m,n} \right) \Big/ \left(\sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n} \right) \right\} = 0,$$

and

$$\lim_{(x,y)} \left\{ \left(\sum_{m=0}^\nu \sum_{n=0}^\infty a_{m,n} v_{m,n} \right) \Big/ \left(\sum_0^\infty \sum_0^\infty a_{m,n} v_{m,n} \right) \right\} = 0.$$

There does not seem to be any way of avoiding these two conditions, nor any way of dividing them into simpler forms in general.

Consider now the specially interesting case $v_{m,n} = x^m y^n$, and suppose further that the coefficients $a_{m,n}$ are also divisible into factors; so that

$$a_{m,n} = f_m g_n,$$

where Σf_m, Σg_n are two divergent series of positive terms. Then

$$\sum_0^\infty \sum_0^\infty a_{m,n} x^m y^n = \left(\sum_0^\infty f_m x^m\right)\left(\sum_0^\infty g_n y^n\right),$$

and

$$\sum_{m=0}^\infty \sum_{n=0}^\alpha a_{m,n} x^m y^n = \left(\sum_0^\infty f_m x^m\right)\left(\sum_0^\nu g_n y^n\right),$$

so that our first condition reduces to

$$\lim_{y\to 1} \left(\sum_0^\nu g_n y^n\right) \Big/ \left(\sum_0^\infty g_n y^n\right) = 0,$$

which is certainly satisfied since

$$\lim_{y\to 1} \left(\sum_0^\infty g_n y^n\right) = \infty$$

(a result proved in § 1).

Similarly the second condition is satisfied.

Thus, if we write

$$F_m = f_0 + f_1 + \ldots + f_m, \qquad G_n = g_0 + g_1 + \ldots + g_n,$$

we find $\displaystyle \lim_{(x,y)} \left\{ \left(\sum_0^\infty \sum_0^\infty b_{m,n} x^m y^n\right) \Big/ \left(\sum_0^\infty f_m x^m\right)\left(\sum_0^\infty g_n y^n\right) \right\} = \lim_{(m,n)} (B_{m,n}/F_m G^n).$

This enables us to give an immediate proof of the extension of Frobenius's theorem to double series,* by writing

$$f_m = 1, \qquad g_n = 1.$$

Then $\displaystyle \sum_0^\infty f_m x^m = (1-x)^{-1}, \qquad \sum_0^\infty g_n y^n = (1-y)^{-1},$

and so, if $\displaystyle b_{m,n} = s_{m,n} = \sum_{i=0}^m \sum_{j=0}^n c_{i,j},$

we have $\displaystyle \sum_0^\infty \sum_0^\infty b_{m,n} x^m y^n = (1-x)^{-1}(1-y)^{-1} \sum_0^\infty \sum_0^\infty c_{m,n} x^m y^n,$

and then $\displaystyle \lim_{(x,y)} \left(\sum_0^\infty \sum_0^\infty c_{m,n} x^m y^n\right) = \lim_{(m,n)} s^{(1)}_{m,n},$

if $\displaystyle (m+1)(n+1)\, s^{(1)}_{m,n} = \sum_{i=0}^m \sum_{j=0}^n s_{m,n},$

using the notation of the paper quoted.

---

* Bromwich and Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 161 (see § 8, p. 173).

Similarly we can extend the theorem to cases of greater complexity by writing

$$\sum_0^\infty f_m x^m = (1-x)^{-\alpha}, \qquad \sum_0^\infty g_n y^n = (1-y)^{-\beta},$$

where α, β are positive integers; this gives a kind of extension of Hölder's theorem, although the means employed will correspond to those used for the summation of single series by Cesàro, rather than those introduced by Hölder.* Thus, taking $\alpha = 2 = \beta$, we get

$$\lim_{(x, y)} \left(\sum_0^\infty \sum_0^\infty c_{m,n} x^m y^n\right) = \lim_{(m, n)} \frac{(2!)^2 \sum_{i=0}^{m} \sum_{j=0}^{n} (i+1)(j+1) s_{i,j}}{(m+1)(m+2)(n+1)(n+2)}.$$

The analogue to Hölder's theorem would have on the right the limit

$$\lim_{(m, n)} s^{(2)}_{m,n},$$

where

$$(m+1)(n+1)\, s^{(2)}_{m,n} = \sum_{i=0}^{m} \sum_{j=0}^{n} s^{(1)}_{i,j},$$

the sums $s^{(1)}_{m,n}$ being themselves defined by arithmetic means.

* Compare the form of the theorem given in Art. 123 of my book on *Infinite Series*.