On some General Classes of Multiple Definite Integrals.

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1. The methods of my former paper may be employed as follows to obtain some more general theorems including those before found as particular cases.

Let $S(p, q)$ be any symmetric function of p and q which does not become infinite for any positive values of p and q, or when either or both of them is zero or infinite. Then, using the fact that

$$\int_0^\infty \frac{\phi(ax)-\phi(a'x)}{x}\,dx = \log\frac{a}{a'}\{\phi(\infty)-\phi(0)\},$$

the double integral, in which a and b, a' and b' are positive constants,

$$\int_0^\infty\int_0^\infty \frac{S(ax, by)-S(a'x, b'y)}{xy}\,dx\,dy$$

$$= \int_0^\infty\int_0^\infty \frac{S(ax, by)-S(ax, b'y)}{xy}\,dx\,dy + \int_0^\infty\int_0^\infty \frac{S(ax, b'y)-S(a'x, b'y)}{xy}\,dx\,dy$$

$$= \log\frac{b}{b'}\int_0^\infty \frac{S(ax, \infty)-S(ax, 0)}{x}\,dx + \log\frac{a}{a'}\int_0^\infty \frac{S(\infty, b'y)-S(0, b'y)}{y}\,dy,$$

which if $\log\frac{a}{a'}+\log\frac{b}{b'}=0$, *i.e.*, if $ab=a'b'$, may be written

$$= \log\frac{a}{a'}\left\{\int_0^\infty \frac{S(ax, 0)-S(b'x, 0)}{x}\,dx - \int_0^\infty \frac{S(ax, \infty)-S(b'x, \infty)}{x}\,dx\right\}$$

$$= \log\frac{a}{a'}\log\frac{a}{b'}\{S(\infty, 0)-S(0, 0)-S(\infty, \infty)+S(0, \infty)\}$$

$$= -\log\frac{a}{a'}\log\frac{a}{b'}\{S(\infty, \infty)-2S(\infty, 0)+S(0, 0)\}\text{......(1).}$$

In precisely the same way, or by transformation, if $S'(p, q)$ be a symmetric function not made infinite by any vanishing or negative, finite or infinite, values of p and q,

$$\int_{-\infty}^{0}\int_{-\infty}^{0}\frac{S'(ax,by)-S'(a'x,b'y)}{xy}\,dx\,dy$$
$$=-\log\frac{a}{a'}\log\frac{a}{b}\{S'(0,0)-2S'(0,-\infty)+S'(-\infty,-\infty)\}\;\ldots\;(2),$$

a, b, a', b' being positive constants connected by the same relation $ab=a'b'$.

In the same way each of the theorems proved below has a precisely corresponding one with regard to integrals between negative limits. These will be omitted for brevity, as they are easily written down.

Other general theorems may be obtained as in my former paper by the substitution of ϵ^x, ϵ^y, &c., for x, y, &c., and again by the substitution of $\log x$, $\log y$, &c., for x, y, &c.

2. Again $S(p,q,r)$ being any such symmetric function of p, q, r as not to be made infinite by any vanishing or positive values of its arguments, and a, b, c, a', b', c' being positive constants; then, introducing a seventh positive constant β, the triple integral

$$\int_0^\infty\int_0^\infty\int_0^\infty\frac{S(ax,by,cz)-S(a'x,b'y,c'z)}{xyz}\,dx\,dy\,dz$$
$$=\int_0^\infty\int_0^\infty\int_0^\infty\frac{S(ax,by,cz)-S(ax,\beta y,c'z)}{xyz}\,dx\,dy\,dz$$
$$+\int_0^\infty\int_0^\infty\int_0^\infty\frac{S(ax,\beta y,c'z)-S(a'x,b'y,c'z)}{xyz}\,dx\,dy\,dz$$
$$=-\log\frac{b}{\beta}\log\frac{b}{c'}\int_0^\infty\frac{S(ax,\infty,\infty)-2S(ax,\infty,0)+S(ax,0,0)}{x}\,dx$$
$$-\log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty\frac{S(\infty,\infty,c'z)-2S(\infty,0,c'z)+S(0,0,c'z)}{z}\,dz,$$

by (1), provided that $bc=\beta c'$ and $a\beta=a'b'$. If the additional condition holds, $\log\frac{a}{a'}\log\frac{a}{b'}+\log\frac{b}{\beta}\log\frac{b}{c'}=0$, this may be written

$$=\log\frac{a}{a'}\log\frac{a}{b'}\left\{\int_0^\infty\frac{S(ax,\infty,\infty)-S(c'x,\infty,\infty)}{x}\,dx\right.$$
$$\left.-2\int_0^\infty\frac{S(ax,\infty,0)-S(c'x,\infty,0)}{x}\,dx+\int_0^\infty\frac{S(ax,0,0)-S(c'x,0,0)}{x}\,dx\right\}$$
$$=\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\{S(\infty,\infty,\infty)-S(\infty,\infty,0)$$
$$-2[S(\infty,\infty,0)-S(0,\infty,0)]+S(\infty,0,0)-S(0,0,0)\}$$
$$=\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\{S(\infty,\infty,\infty)-3S(\infty,\infty,0)$$
$$+3S(\infty,0,0)-S(0,0,0)\}\;\ldots\ldots\ldots\ldots\;(3).$$

The conditions in the constants under which this is true are, by elimination of β, easily written as the two independent ones $abc = a'b'c'$ and $\log\frac{a}{a'}\log\frac{a}{b'} = \log\frac{b}{c'}\log\frac{c}{c'}$. By a is of course denoted, in these results and conditions, any one of a, b, c, and by c' any one of a', b', c'. Notice that the second condition is the one by which the coefficient $\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}$ is made a symmetric function of a, b, c, as it of course must be.

3. Again, the corresponding quadruple integral

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{S(ax, by, cz, dw) - S(a'x, b'y, c'z, d'w)}{xyzw}\,dx\,dy\,dz\,dw$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{S(ax, by, cz, dw) - S(ax, by, c'z, d'w)}{xyzw}\,dx\,dy\,dz\,dw$$

$$+\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{S(ax, by, c'z, d'w) - S(a'x, b'y, c'z, d'w)}{xyzw}\,dx\,dy\,dz\,dw$$

$$= -\log\frac{c}{c'}\log\frac{c}{d'}\int_0^\infty\int_0^\infty \frac{S(ax, by, \infty, \infty) - 2S(ax, by, \infty, 0) + S(ax, by, 0, 0)}{xy}\,dx\,dy$$

$$-\log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty\int_0^\infty \frac{S(\infty, \infty, c'z, d'w) - 2S(\infty, 0, c'z, d'w) + S(0, 0, c'z, d'w)}{zw}\,dz\,dw,$$

by (1), provided that $cd = c'd'$ and $ab = a'b'$. If in addition we have $\log\frac{a}{a'}\log\frac{a}{b'} + \log\frac{c}{c'}\log\frac{c}{d'} = 0$, this may be written

$$= \log\frac{a}{a'}\log\frac{a}{b'}\left\{\int_0^\infty\int_0^\infty \frac{S(ax, by, \infty, \infty) - S(c'x, d'y, \infty, \infty)}{xy}\,dx\,dy - 2\int_0^\infty\int_0^\infty \frac{S(ax, by, \infty, 0) - S(c'x, d'y, \infty, 0)}{xy}\,dx\,dy\right.$$

$$\left.+\int_0^\infty\int_0^\infty \frac{S(ax, by, 0, 0) - S(c'x, d'y, 0, 0)}{xy}\,dx\,dy\right\}$$

$$=-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\{S(\infty,\infty,\infty,\infty)-2S(\infty,0,\infty,\infty)+S(0,0,\infty,\infty)-2S(\infty,\infty,\infty,0)$$
$$+4S(\infty,0,\infty,0)-2S(0,0,\infty,0)+S(\infty,\infty,0,0)-2S(\infty,0,0,0)+S(0,0,0,0)\},$$

provided that also $ab=c'd'$,

$$=-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\{S(\infty,\infty,\infty,\infty)-4S(\infty,\infty,\infty,0)+6S(\infty,\infty,0,0)-4S(\infty,0,0,0)+S(0,0,0,0)\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Collected, the conditions for the truth of this are the four

$$ab=a'b'=cd=c'd',\qquad \log\frac{a}{a'}\log\frac{a}{b'}+\log\frac{c}{c'}\log\frac{c}{d'}=0,$$

the first three of which are those by virtue of which the value is symmetric in a, b, c, d, as well as in a', b', c', d'.

4. The corresponding integral of the fifth order may now be considered.

By (3),
$$\int_0^\infty\int_0^\infty\ldots\int_0^\infty\frac{S(ax,by,cz,du,ev)-S(ax,by,\gamma z,d'u,e'v)}{xyzuv}\,dx\,dy\,dz\,du\,dv$$
$$=\log\frac{c}{\gamma}\log\frac{c}{d'}\log\frac{c}{e'}\int_0^\infty\int_0^\infty\frac{S(ax,by,\infty,\infty,\infty)-3S(ax,by,\infty,\infty,0)+3S(ax,by,\infty,0,0)-S(ax,by,0,0,0)}{xy}\,dx\,dy,$$

provided that $cde=\gamma d'e'$ and $\log\frac{c}{\gamma}\log\frac{c}{d'}=\log\frac{d}{e'}\log\frac{e}{e'}$.

And in like manner
$$\int_0^\infty\int_0^\infty\ldots\int_0^\infty\frac{S(ax,by,\gamma z,d'u,e'v)-S(a'x,b'y,c'z,d'u,e'v)}{xyzuv}\,dx\,dy\,dz\,du\,dv$$
$$=\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\int_0^\infty\int_0^\infty\frac{S(\infty,\infty,\infty,d'u,e'v)-3S(\infty,\infty,0,d'u,e'v+3S(\infty,0,0,d'u,e'v)-S(0,0,0,d'u,e'v)}{uv}\,du\,dv,$$

provided that $ab\gamma=a'b'c'$ and $\log\frac{a}{a'}\log\frac{a}{b'}=\log\frac{b}{c'}\log\frac{\gamma}{c'}$.

If then also $\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}+\log\frac{c}{\gamma}\log\frac{c}{d'}\log\frac{c}{e'}=0$, by addition we have

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\frac{S(ax,\,by,\,cz,\,du,\,ev)-S(a'x,\,b'y,\,c'z,\,d'u,\,e'v)}{xyzuv}\,dx\,dy\,dz\,du\,dv$$

$$=-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\left\{\int_0^\infty\int_0^\infty\frac{S(ax,by,\infty,\infty,\infty)-3S(ax,by,\infty,\infty,0)+3S(ax,by,\infty,0,0)-S(ax,by,0,0,0)}{xy}\,dx\,dy\right.$$
$$\left.-\int_0^\infty\int_0^\infty\frac{S(d'x,e'y,\infty,\infty,\infty)-3S(d'x,e'y,\infty,\infty,0)+3S(d'x,e'y,\infty,0,0)-S(d'x,e'y,0,0,0)}{xy}\,dx\,dy\right\}$$

$$=\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\Big\{S(\infty,\infty,\infty,\infty,\infty)-3S(\infty,\infty,\infty,\infty,0)+3S(\infty,\infty,\infty,0,0)-S(\infty,\infty,0,0,0)$$
$$-2\,[S(\infty,0,\infty,\infty,\infty)-3S(\infty,0,\infty,\infty,0)+3S(\infty,0,\infty,0,0)-S(\infty,0,0,0,0)]$$
$$+S(0,0,\infty,\infty,\infty)-3S(0,0,\infty,\infty,0)+3S(0,0,\infty,0,0)-S(0,0,0,0,0)\Big\}$$

by (1), provided that also $ab=d'e'$,

$$=\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\{S(\infty,\infty,\infty,\infty,\infty)-5S(\infty,\infty,\infty,\infty,0)+10S(\infty,\infty,\infty,0,0)$$
$$-10S(\infty,\infty,0,0,0)+5S(\infty,0,0,0,0)-S(0,0,0,0,0)\}\quad\ldots\ldots\ldots(5).$$

By elimination of γ, the conditions in the ten constants for the truth of this may be written as the five

$$ab=d'e',\qquad cde=a'b'c',$$

$$\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}=-\log\frac{c}{e'}\log\frac{d}{e'}\log\frac{e}{e'}=\log\frac{a}{c'}\log\frac{b}{c'}\log\frac{a'b'}{ab}=-\log\frac{c}{d'}\log\frac{c}{e'}\log\frac{d'e'}{de}.$$

5. Again, the methods used in proving (8) and (9) below may both be applied to evaluate the sextuple integral

$$\int_0^\infty\int_0^\infty\ldots\int_0^\infty \frac{\mathrm{S}\,(ax,\,by,\,cz,\,du,\,ev,\,fw)-\mathrm{S}\,(a'x,\,b'y,\,c'z,\,d'u,\,e'v,\,f'w)}{xyzuvw}\,dx\,dy\,dz\,du\,dv\,dw.$$

It is thus found that, under the same seven conditions in the twelve constants, for which either of these results is true, the value of this integral is

$$-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'}\{\mathrm{S}\,(\infty,\infty,\infty,\infty,\infty,\infty)-6\mathrm{S}\,(\infty,\infty,\infty,\infty,\infty,0)+15\,\mathrm{S}\,(\infty,\infty,\infty,\infty,0,0)$$
$$-20\mathrm{S}\,(\infty,\infty,\infty,0,0,0)+15\mathrm{S}\,(\infty,\infty,0,0,0,0)-6\mathrm{S}\,(\infty,0,0,0,0,0)+\mathrm{S}\,(0,0,0,0,0,0)\}\ldots\ldots(6).$$

Continuing in this way, conditions in the $2n$ involved constants $a_1, a_2, \ldots a_n, a'_1, a'_2, \ldots a'_n$ may be found for which the value of the corresponding n-tuple integral has a value which may be expressed symbolically as

$$(-1)^{n-1}\Pi_{r=1}^{r=n}\log\frac{a_1}{a'_r}\,\mathrm{S}\,(\infty-0)^n.$$

6. Now, again, let $s\,(p,q)$ denote any symmetric function whatever of p and q, and $\mathrm{S}\,(s,s')$ any symmetric function of s and s', which does not become infinite when s and s' are any quantities whose values $s\,(p,q)$ may assume for positive or vanishing values of p and q; and let $a, b, a', b', c, d, c', d'$ be positive constants. Then the quadruple integral

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\mathrm{S}\,[s\,(ax,\,by),\ s\,(cz,\,du)]-\mathrm{S}\,[s\,(a'x,\,b'y),\ s\,(c'z,\,d'u)]}{xyzu}\,dx\,dy\,dz\,du$$

$$=-\log\frac{c}{c'}\log\frac{c}{d'}\int_0^\infty\int_0^\infty \frac{\mathrm{S}\,[s\,(ax,\,by),\ s\,(\infty,\infty)]-2\mathrm{S}\,[s\,(ax,\,by),\ s\,(\infty,0)]+\mathrm{S}\,[s\,(ax,\,by),\ s\,(0,0)]}{xy}\,dx\,dy$$

$$-\log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty\int_0^\infty \frac{\mathrm{S}\,[s\,(\infty,\infty),\ s\,(c'z,\,d'u)]-2\mathrm{S}\,[s\,(\infty,0),\ s\,(c'z,\,d'u)]+\mathrm{S}\,[s\,(0,0),\ s\,(c'z,\,d'u)]}{zu}\,dz\,du,$$

by two applications of (1), provided that $ab = a'b'$ and $cd = c'd'$. If in addition $\log\frac{a}{a'}\log\frac{a}{b'} + \log\frac{c}{c'}\log\frac{c}{d'} = 0$, this may be written

$$\log\frac{a}{a'}\log\frac{a}{b'}\left\{\int_0^\infty\int_0^\infty \frac{S[s(ax, by),\ s(\infty,\infty)] - S[s(c'x, d'y),\ s(\infty,\infty)]}{xy}\,dx\,dy\right.$$
$$-2\int_0^\infty\int_0^\infty \frac{S[s(ax, by),\ s(\infty, 0)] - S[s(c'x, d'y),\ s(\infty, 0)]}{xy}\,dx\,dy$$
$$\left.+\int_0^\infty\int_0^\infty \frac{S[s(ax, by),\ s(0,0)] - S[s(c'x, d'y),\ s(0,0)]}{xy}\,dx\,dy\right\}$$

$$= -\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\left\{\begin{array}{l} S[s(\infty,\infty),\ s(\infty,\infty)] - 2S[s(\infty,0),\ s(\infty,\infty)] + S[s(0,0),\ s(\infty,\infty)] \\ -2S[s(\infty,\infty),\ s(\infty,\ 0)] + 4S[s(\infty,0),\ s(\infty,\ 0)] - 2S[s(0,0),\ s(\infty,\ 0)] \\ +\ S[s(\infty,\infty),\ s(\ 0,\ 0)] - 2S[s(\infty,0),\ s(\ 0,\ 0)] + S[s(0,0),\ s(\ 0,\ 0)] \end{array}\right\},$$

if the fourth condition $ab = c'd'$ be satisfied,

$$= -\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\left\{\begin{array}{l} S[s(\infty,\infty),\ s(\infty,\infty)] - 4S[s(\infty,\infty),\ s(\infty,0)] + 2S[s(\infty,\infty),\ s(0,0)] \\ +4S[s(\infty,\ 0),\ s(\infty,\ 0)] - 4S[s(\infty,\ 0,\ s(0,\ 0)] + S[s(\ 0,\ 0),\ s(0,0)] \end{array}\right\} \quad (7).$$

7. Again, by two applications of (3), the sextuple integral, in which $s(p, q, r)$ is any symmetric function, and $S(s_1, s_2, s_3)$ a symmetric function not made infinite by any values of its arguments that $s(p, q, r)$ can assume for positive or vanishing values of p, q, r,

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\left[s(ax, by, cz), s(du, ev, fw)\right]-\mathrm{S}\left[s(a'x, b'y, c'z), s(d'u, e'v, f'w)\right]}{xyzuvw}\,dx\,dy\,dz\,du\,dv\,dw$$

$$=\log\frac{d}{d'}\log\frac{d}{e'}\log\frac{d}{f'}\int_0^\infty\int_0^\infty\int_0^\infty \frac{\left\{\begin{array}{l}\mathrm{S}\left[s(ax, by, cz), s(\infty, \infty, \infty)\right]-3\mathrm{S}\left[s(ax, by, cz), s(\infty, \infty, 0)\right]\\ +3\mathrm{S}\left[s(ax, by, cz), s(\infty, 0, 0)\right]-\mathrm{S}\left[s(ax, by, cz), s(0, 0, 0)\right]\end{array}\right\}}{xyz}\,dx\,dy\,dz$$

$$+\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\int_0^\infty\int_0^\infty\int_0^\infty \frac{\left\{\begin{array}{l}\mathrm{S}\left[s(\infty, \infty, \infty), s(d'u, e'v, f'w)\right]-3\mathrm{S}\left[s(\infty, \infty, 0), s(d'u, e'v, f'w)\right]\\ +3\mathrm{S}\left[s(\infty, 0, 0), s(d'u, e'v, f'w)\right]-\mathrm{S}\left[s(0, 0, 0), s(d'u, e'v, f'w)\right]\end{array}\right\}}{uvw}\,du\,dv\,dw,$$

subject to the four conditions $def=d'e'f'$, $abc=a'b'c'$, $\log\frac{d}{d'}\log\frac{d}{e'}=\log\frac{e}{f'}\log\frac{f}{f'}$, $\log\frac{a}{a'}\log\frac{a}{b'}=\log\frac{b}{c'}\log\frac{c}{c'}$.

If, in addition, $\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}+\log\frac{d}{d'}\log\frac{d}{e'}\log\frac{d}{f'}=0$, this may be written

$$=-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\left\{\int_0^\infty\int_0^\infty\int_0^\infty \frac{\mathrm{S}\left[s(ax, by, cz), s(\infty, \infty, \infty)\right]-\mathrm{S}\left[s(d'x, e'y, f'z), s(\infty, \infty, \infty)\right]}{xyz}\,dx\,dy\,dz\right.$$

$$-3\int_0^\infty\int_0^\infty\int_0^\infty \frac{\mathrm{S}\left[s(ax, by, cz), s(\infty, \infty, 0)\right]-\mathrm{S}\left[s(d'x, e'y, f'z), s(\infty, \infty, 0)\right]}{xyz}\,dx\,dy\,dz$$

$$+3\int_0^\infty\int_0^\infty\int_0^\infty \frac{\mathrm{S}\left[s(ax, by, cz), s(\infty, 0, 0)\right]-\mathrm{S}\left[s(d'x, e'y, f'z), s(\infty, 0, 0)\right]}{xyz}\,dx\,dy\,dz$$

$$\left.-\int_0^\infty\int_0^\infty\int_0^\infty \frac{\mathrm{S}\left[s(ax, by, cz), s(0, 0, 0)\right]-\mathrm{S}\left[s(d'x, e'y, f'z), s(0, 0, 0)\right]}{xyz}\,dx\,dy\,dz\right\},$$

which, if $abc=d'e'f'$, and $\log\frac{a}{d'}\log\frac{a}{e'}=\log\frac{b}{f'}\log\frac{c}{f'}$, becomes

$$= -\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'}$$

$$\times\Big\{ S[s(\infty,\infty,\infty), s(\infty,\infty,\infty)] - 3S[s(\infty,\infty,0), s(\infty,\infty,\infty)] + 3S[s(\infty,0,0), s(\infty,\infty,\infty)] - S[s(0,0,0), s(\infty,\infty,\infty)]$$

$$-3S[s(\infty,\infty,\infty), s(\infty,\infty,0)] + 9S[s(\infty,\infty,0), s(\infty,\infty,0)] - 9S[s(\infty,0,0), s(\infty,\infty,0)] + 3S[s(0,0,0), s(\infty,\infty,0)]$$

$$+3S[s(\infty,\infty,\infty), s(\infty,0,0)] - 9S[s(\infty,\infty,0), s(\infty,0,0)] + 9S[s(\infty,0,0), s(\infty,0,0)] - 3S[s(0,0,0), s(\infty,0,0)]$$

$$- S[s(\infty,\infty,\infty), s(0,0,0)] + 3S[s(\infty,\infty,0), s(0,0,0)] - 3S[s(\infty,0,0), s(0,0,0)] + S[s(0,0,0), s(0,0,0)]\Big\}$$

$$= -\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'}$$

$$\times\Big\{ S[s(\infty,\infty,\infty), s(\infty,\infty,\infty)] - 6S[s(\infty,\infty,\infty), s(\infty,\infty,0)] + 6S[s(\infty,\infty,\infty), s(\infty,0,0)]$$

$$+9S[s(\infty,\infty,0), s(\infty,\infty,0)] - 2S[s(\infty,\infty,\infty), s(0,0,0)] - 18S[s(\infty,\infty,0), s(\infty,0,0)]$$

$$+6S[s(\infty,\infty,0), s(0,0,0)] + 9S[s(\infty,0,0), s(\infty,0,0)] - 6S[s(\infty,0,0), s(0,0,0)]$$

$$+ S[s(0,0,0), s(0,0,0)]\Big\} \quad \ldots\ldots\ldots\ldots (8).$$

Collected, the seven conditions in the twelve constants under which this is true may be written

$$abc = a'b'c' = def = d'e'f',$$

$$\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'} = \log\frac{a}{c'}\log\frac{b}{c'}\log\frac{c}{c'} = -\log\frac{d}{d'}\log\frac{d}{e'}\log\frac{d}{f'} = -\log\frac{d}{f'}\log\frac{e}{f'}\log\frac{f}{f'},$$

$$\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'} = \log\frac{a}{f'}\log\frac{b}{f'}\log\frac{c}{f'}.$$

8. Lastly, by three applications of (1), the sextuple integral

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\left[s(ax,by),\,s(cz,du),\,s(ev,fw)\right]-\mathrm{S}\left[s(a'x,b'y),\,s(c'z,d'u),\,s(e'v,f'w)\right]}{xyzuvw}\,dx\,dy\,dz\,du\,dv\,dw$$

$$=-\log\frac{e}{e'}\log\frac{e}{f'}\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\left[s(ax,by),\,s(cz,du),\,s(\infty,\infty)\right]-2\mathrm{S}\left[s(ax,by),\,s(cz,du),\,s(\infty,0)\right]+\mathrm{S}\left[s(ax,by),\,s(cz,du).\,s(0.0)\right]}{xyzu}\times dx\,dy\,dz\,du$$

$$-\log\frac{c}{c'}\log\frac{c}{d'}\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\left[s(ax,by),\,s(\infty,\infty),\,s(e'v,f'w)\right]-2\mathrm{S}\left[s(ax.by),\,s(\infty,0),\,s(e'v,f'w)\right]+\mathrm{S}\left[s(ax.by),\,s(0,0),\,s(e'v,f'w)\right]}{xyvw}\times dx\,dy\,dv\,dw$$

$$-\log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\left[s(\infty,\infty),\,s(c'z,d'u),\,s(e'v,f'w)\right]-2\mathrm{S}\left[s(\infty,0),\,s(c'z,d'u),\,s(e'v,f'w)\right]+\mathrm{S}\left[s(0,0),\,s(c'z,d'u),\,s(e'v,f'w)\right]}{zuvw}\times dz\,du\,dv\,dw,$$

provided that $ef=e'f'$, $cd=c'd'$, $ab=a'b'$. If, in addition, $\log\frac{c}{c'}\log\frac{c}{d'}=-\log\frac{a}{a'}\log\frac{a}{b'}-\log\frac{e}{e'}\log\frac{e}{f'}$, this may be written

$$=-\log\frac{e}{e'}\log\frac{e}{f'}\left\{\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\left[s(ax,by),\,s(cz,du),\,s(\infty,\infty)\right]-\mathrm{S}\left[s(ax,by),\,s(e'z,f'u),\,s(\infty,\infty)\right]}{xyzu}\,dx\,dy\,dz\,du\right.$$

$$-2\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\left[s(ax,by),\,s(cz,du),\,s(\infty,0)\right]-\mathrm{S}\left[s(ax,by),\,s(e'z,f'u),\,s(\infty,0)\right]}{xyzu}\,dx\,dy\,dz\,du$$

$$\left.+\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\left[s(ax,by),\,s(cz,du),\,s(0,0)\right]-\mathrm{S}\left[s(ax,by),\,s(e'z,f'u),\,s(0,0)\right]}{xyzu}\,dx\,dy\,dz\,du\right\}$$

$$+\log\frac{a}{a'}\log\frac{a}{b'}\left\{\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\left[s(ax,by),\,s(\infty,\infty),\,s(e'v,f'w)\right]-\mathrm{S}\left[s(c'x,d'y).\,s(\infty,\infty).\,s(e'v.f'w)\right]}{xyvw}\,dx\,dy\,dv\,dw\right.$$

$$-2\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\left[s(ax,by),\,s(\infty,0),\,s(e'v,f'w)\right]-\mathrm{S}\left[s(c'x,d'y),\,s(\infty,0),\,s(e'v,f'w)\right]}{xyvw}\,dx\,dy\,dv\,dw$$

$$\left.+\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\left[s(ax,by),\,s(0,0),\,s(e'v,f'w)\right]-\mathrm{S}\left[s(c'x,d'y),\,s(0,0),\,s(e'v,f'w)\right]}{xyvw}\,dx\,dy\,dv\,dw\right\};$$

and therefore, by (1), provided that also $cd = e'f'$, and $ab = c'd'$,

$$= \log\frac{e}{e'}\log\frac{e}{f'}\log\frac{c}{e'}\log\frac{c}{f'}$$

$$\times\Big\{\int_0^\infty\int_0^\infty \frac{\mathrm{S}[s(ax,by),s(\infty,\infty),s(\infty,\infty)]-2\mathrm{S}[s(ax,by),s(\infty,0),s(\infty,\infty)]+\mathrm{S}[s(ax,by),s(0,0),s(\infty,\infty)]}{xy}\,dx\,dy$$

$$-2\int_0^\infty\int_0^\infty \frac{\mathrm{S}[s(ax,by),s(\infty,\infty),s(\infty,0)]-2\mathrm{S}[s(ax,by),s(\infty,0),s(\infty,0)]+\mathrm{S}[s(ax,by),s(0,0),s(\infty,0)]}{xy}\,dx\,dy$$

$$+\int_0^\infty\int_0^\infty \frac{\mathrm{S}[s(ax,by),s(\infty,\infty),s(0,0)]-2\mathrm{S}[s(ax,by),s(\infty,0),s(0,0)]+\mathrm{S}[s(ax,by),s(0,0),s(0,0)]}{xy}\,dx\,dy\Big\}$$

$$-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}$$

$$\times\Big\{\int_0^\infty\int_0^\infty \frac{\mathrm{S}[s(\infty,\infty),s(\infty,\infty),s(e'v,f'w)]-2\mathrm{S}[s(\infty,0),s(\infty,\infty),s(e'v,f'w)]+\mathrm{S}[s(0,0),s(\infty,\infty),s(e'v,f'w)]}{vw}\,dv\,dw$$

$$-2\int_0^\infty\int_0^\infty \frac{\mathrm{S}[s(\infty,\infty),s(\infty,0),s(e'v,f'w)]-2\mathrm{S}[s(\infty,0),s(\infty,0),s(e'v,f'w)]+\mathrm{S}[s(0,0),s(\infty,0),s(e'v,f'w)]}{vw}\,dv\,dw$$

$$+\int_0^\infty\int_0^\infty \frac{\mathrm{S}[s(\infty,\infty),s(0,0),s(e'v,f'w)]-2\mathrm{S}[s(\infty,0),s(0,0),s(e'v,f'w)]+\mathrm{S}[s(0,0),s(0,0),s(e'v,f'w)]}{vw}\,dv\,dw\Big\}$$

If then, in addition, we have that $\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'} = \log\frac{e}{e'}\log\frac{e}{f'}\log\frac{c}{e'}\log\frac{c}{f'}$, by taking each term of the first bracket with the corresponding one of the second, and again applying (1), this becomes, since $ab = (c'd' = cd =)\, e'f'$,

$$
\begin{aligned}
= -\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'} \\
\times \Big\{ \; & S[s(\infty,\infty), s(\infty,\infty), s(\infty,\infty)] - 2S[s(\infty,0), s(\infty,\infty), s(\infty,\infty)] + S[s(0,0), s(\infty,\infty), s(\infty,\infty)] \\
& -2S[s(\infty,\infty), s(\infty,0), s(\infty,\infty)] + 4S[s(\infty,0), s(\infty,0), s(\infty,\infty)] - 2S[s(0,0), s(\infty,0), s(\infty,\infty)] \\
& + S[s(\infty,\infty), s(0,0), s(\infty,\infty)] - 2S[s(\infty,0), s(0,0), s(\infty,\infty)] + S[s(0,0), s(0,0), s(\infty,\infty)] \\
& -2S[s(\infty,\infty), s(\infty,\infty), s(\infty,0)] + 4S[s(\infty,0), s(\infty,\infty), s(\infty,0)] - 2S[s(0,0), s(\infty,\infty), s(\infty,0)] \\
& +4S[s(\infty,\infty), s(\infty,0), s(\infty,0)] - 8S[s(\infty,0), s(\infty,0), s(\infty,0)] + 4S[s(0,0), s(\infty,0), s(\infty,0)] \\
& -2S[s(\infty,\infty), s(0,0), s(\infty,0)] + 4S[s(\infty,0), s(0,0), s(\infty,0)] - 2S[s(0,0), s(0,0), s(\infty,0)] \\
& + S[s(\infty,\infty), s(\infty,\infty), s(0,0)] - 2S[s(\infty,0), s(\infty,\infty), s(0,0)] + S[s(0,0), s(\infty,\infty), s(0,0)] \\
& -2S[s(\infty,\infty), s(\infty,0), s(0,0)] + 4S[s(\infty,0), s(\infty,0), s(0,0)] - 2S[s(0,0), s(\infty,0), s(0,0)] \\
& + S[s(\infty,\infty), s(0,0), s(0,0)] - 2S[s(\infty,0), s(0,0), s(0,0)] + S[s(0,0), s(0,0), s(0,0)] \Big\}
\end{aligned}
$$

$$
\begin{aligned}
= -\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'} \\
\times \Big\{ \; & S[s(\infty,\infty), s(\infty,\infty), s(\infty,\infty)] - 6S[s(\infty,\infty), s(\infty,\infty), s(\infty,0)] + 3S[s(\infty,\infty), s(\infty,\infty), s(0,0)] \\
& +12S[s(\infty,\infty), s(\infty,0), s(\infty,0)] - 12S[s(\infty,\infty), s(\infty,0), s(0,0)] - 8S[s(\infty,0), s(\infty,0), s(\infty,0)] \\
& +12S[s(\infty,0), s(\infty,0), s(0,0)] + 3S[s(\infty,\infty), s(0,0), s(0,0)] - 6S[s(\infty,0), s(0,0), s(0,0)] \\
& + S[s(0,0), s(0,0), s(0,0)] \Big\} \quad \ldots\ldots(9).
\end{aligned}
$$

Collected, the conditions in the twelve constants for the truth of this may be written as the seven

$$ab = cd = ef = a'b' = c'd' = e'f',$$

$$\log\frac{a}{a'}\log\frac{a}{b'} + \log\frac{c}{c'}\log\frac{c}{d'} + \log\frac{e}{e'}\log\frac{e}{f'} = 0,$$

$$\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'} + \log\frac{a}{f'}\log\frac{b}{f'}\log\frac{c}{f'}\log\frac{d}{f'}\log\frac{e}{f'}\log\frac{f}{f'} = 0.$$

[ADDITION.—The factors in the above results may be reduced to a very convenient symmetric form, suggested in a note on my former paper, for which I have to thank Professor Cayley.

Taking the factor $\log\frac{a_1}{a_1'}\log\frac{a_1}{a_2'}\log\frac{a_1}{a_3'}\dots\log\frac{a_1}{a_n'}$ of the general formula in Art. 5, write a_1, a_1', &c., for $\log a_1$, $\log a_1'$, &c. The factor becomes $(a_1-a_1')(a_1-a_2')\dots\dots(a_1-a_n')$,

which, by the symmetry of the integral in the two sets of constants, must be capable of reduction to either of the other $2n-1$ forms,

$$\begin{aligned}&(a_2-a_1')(a_2-a_2')\dots\dots(a_2-a_n'),\\ &\dots\quad\dots\quad\dots\quad\dots\dots\quad\dots\quad\dots\\ &(a_n-a_1')(a_n-a_2')\dots\dots(a_n-a_n'),\\ -&(a_1'-a_1)(a_1'-a_2)\dots\dots(a_1'-a_n),\\ &\dots\quad\dots\quad\dots\quad\dots\dots\quad\dots\quad\dots\\ -&(a_n'-a_1)(a_n'-a_2)\dots\dots(a_n'-a_n).\end{aligned}$$

Denote by A the common value of these products.

Now let $f(x)$ denote $(x-a_1')(x-a_2')\dots(x-a_n')$,

and $\phi(x)$ " $(x-a_1)(x-a_2)\dots(x-a_n)$.

Then every root of $\phi(x) = 0$ inserted in $f(x)$ gives it the value A, and every root of $f(x) = 0$ inserted in $\phi(x)$ gives it the value $-A$. From this it follows that

$$f(x) = A + \phi(x) \quad \text{for all values of } x.$$

Therefore, in particular, $A = f(0) - \phi(0)$
$= (-1)^n\{a_1' a_2' \dots a_n' - a_1 a_2 \dots a_n\}$.

Thus the values in Arts. 1—5 are all included in the form which may be symbolized by

$$\{\log a_1 \log a_2 \dots \log a_n - \log a_1' \log a_2' \dots \log a_n'\}\, S(\infty - 0)^n.$$

The same form of factor applies also in the results of Arts. 6—8.]