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Sums of Greatest Integers. By G. B. MATHEWS. Received
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1. This note is submitted not so much for the sake of the results themselves—though they are curious enough—as on account of their connexion with a well known *crux* in the theory of numbers.

Let p be any odd prime, and let r_1, r_2, \dots, r_{p-1} be the least positive residues (mod p) of $1^2, 2^2, 3^2, \dots, (p-1)^2$. Each distinct residue occurs twice, because $s^2 \equiv (p-s)^2$, so that $r_s = r_{p-s}$. The residues may be divided into two classes (α) and (β), according as they are less or greater than $\frac{1}{2}p$. For instance, if $p = 7$, the classes are

(α) 1, 2,

(β) 4.

I shall write $n(\alpha), n(\beta)$ for the number of residues in the classes (α), (β) respectively. In every case $n(\alpha) + n(\beta) = \frac{1}{2}(p-1)$; if $p \equiv 1 \pmod{4}$, $n(\alpha) = n(\beta)$; but, if $p \equiv 3 \pmod{4}$, $n(\alpha) > n(\beta)$, and the difference $n(\alpha) - n(\beta)$ is precisely the number of properly primitive forms of determinant $-p$. The difficulty is to prove this second result without the use of transcendental analysis; in what follows the difficulty is not removed, but an expression for $n(\alpha) - n(\beta)$ is given in a purely arithmetical shape which is quite

independent of the theory of forms, and may possibly suggest a new proof that $n(\alpha) > n(\beta)$, when $p \equiv 3 \pmod{4}$.

2. The residue of x^2 will belong to the class (α) so long as

$$0 < x^2 < \frac{1}{2}p,$$

that is, so long as

$$0 < x < \sqrt{\frac{1}{2}p}.$$

If we write $E\left(\frac{p}{2}\right)^{\frac{1}{2}}$ for the greatest integer in $\sqrt{\frac{1}{2}p}$, the series of residues begins with $E\left(\frac{p}{2}\right)^{\frac{1}{2}}$ numbers belonging to the class (α) . These are followed by $E(p)^{\frac{1}{2}} - E\left(\frac{p}{2}\right)^{\frac{1}{2}}$ residues of class (β) ; these again by $E\left(\frac{3p}{2}\right)^{\frac{1}{2}} - E(p)^{\frac{1}{2}}$ residue of class (α) , and so on alternately. If we observe that the last square taken into account is

$$(p-1)^2 = (p-2)p+1,$$

and that its residue belongs to the class (α) , it will follow that we may write

$$\begin{aligned} 2n(\alpha) = & E\left(\frac{p}{2}\right)^{\frac{1}{2}} + E\left(\frac{3p}{2}\right)^{\frac{1}{2}} - E(p)^{\frac{1}{2}} + E\left(\frac{5p}{2}\right)^{\frac{1}{2}} - E(2p)^{\frac{1}{2}} + \dots \\ & \dots + E\left(\frac{2p-5}{2}p\right)^{\frac{1}{2}} - E(\overline{p-3}.p)^{\frac{1}{2}} \\ & + (p-1) - E(\overline{p-2}.p)^{\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} 2n(\beta) = & E(p)^{\frac{1}{2}} - E\left(\frac{p}{2}\right)^{\frac{1}{2}} + E(2p)^{\frac{1}{2}} - E\left(\frac{3p}{2}\right)^{\frac{1}{2}} + \dots \\ & \dots + E(\overline{p-2}.p)^{\frac{1}{2}} - E\left(\frac{2p-5}{2}p\right)^{\frac{1}{2}}. \end{aligned}$$

Hence

$$n(\alpha) - n(\beta) = X - Y,$$

where $X = E\left(\frac{p}{2}\right)^{\frac{1}{2}} + E\left(\frac{3p}{2}\right)^{\frac{1}{2}} + \dots + E\left(\frac{2p-5}{2}p\right)^{\frac{1}{2}} + \frac{p-1}{2},$

$$Y = E(p)^{\frac{1}{2}} + E(2p)^{\frac{1}{2}} + E(3p)^{\frac{1}{2}} + \dots + E(\overline{p-2}.p)^{\frac{1}{2}}.$$

3. As a numerical example, when $p = 11$ (omitting the sign E for brevity),

$$X = \left(\frac{11}{2}\right)^{\dagger} + \left(\frac{33}{2}\right)^{\dagger} + \left(\frac{55}{2}\right)^{\dagger} + \dots + \left(\frac{187}{2}\right)^{\dagger} + 5$$

$$= 2 + 4 + 5 + 6 + 7 + 7 + 8 + 9 + 9 + 5 = 62,$$

$$Y = (11)^{\dagger} + (22)^{\dagger} + (33)^{\dagger} + \dots + (99)^{\dagger}$$

$$= 3 + 4 + 5 + 6 + 7 + 8 + 8 + 9 + 9 = 59,$$

whence

$$X - Y = 3,$$

which is right, the distinct residues being 1, 3, 4, 5, 9.

The values of X , Y for the first few values of p are given by the table:—

$$p = 3 \quad 5 \quad 7 \quad 11 \quad 13 \quad 17 \quad 19,$$

$$X = 2 \quad 8 \quad 20 \quad 62 \quad 88 \quad 160 \quad 206,$$

$$Y = 1 \quad 8 \quad 19 \quad 59 \quad 88 \quad 160 \quad 203.$$

4. Expressions for Y may be deduced from formulæ given by J. Hacks (*Acta Math.*, Vol. x., p. 39); thus, when

$$p \equiv 1 \pmod{4},$$

$$Y = \frac{2}{3}(p-1)(p-2),$$

and X has then the same value. If

$$p \equiv 3 \pmod{4},$$

$$Y = \frac{(p-1)(4p-11)}{6} + \frac{2R}{p},$$

where

$$2R = r_1 + r_2 + \dots + r_{p-1};$$

and hence, if $h(p)$ is the number of properly primitive classes of determinant $-p$,

$$X = \frac{(p-1)(4p-11)}{6} + \frac{2R}{p} + h(p).$$

As a numerical example, when $p = 19$, the residues are 1, 4, 5, 6, 7, 9, 11, 16, 17; whence

$$h(19) = 3, \quad 2R/19 = 8;$$

$$\text{and therefore} \quad X = \frac{18.65}{6} + 8 + 3 = 206,$$

$$Y = 203,$$

agreeing with the values calculated independently.

The Genesis of the Double Gamma Functions. By E. W. BARNES, B.A., Fellow of Trinity College, Cambridge. Received December 5th, 1899. Communicated December 14th, 1899.

1. The following paper is the natural sequence of results obtained in two previous papers.

The "Theory of the Gamma Function" * contained a discussion of the function defined by the formula

$$e^{\frac{1}{2}\pi z} \Gamma(z+1) = \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n} \right) e^{-\frac{z}{n}} \right\},$$

and it is evident that the expression on the right-hand side of this equality may be regarded as the positive half of the product expression for $\sin \pi z$; we may, in fact, term it the "halb-sinus" with Betti.†

Again, in the "Theory of the G Function," ‡ it was shown that

$$G(z) = e^{r(z)} z \prod_0^{\infty} \prod_n' \left\{ \left(1 + \frac{z}{m+n} \right) e^{-\frac{z}{m+n} + 2 \frac{z^2}{(m+n)^2}} \right\},$$

where $r(z)$ is a quadratic function of z .

If now we can associate with the letter m , each time that it occurs in this product, a complex constant τ which is not real and negative, we shall obtain a product which may be regarded as the positive quarter of the product expression for Weierstrass's function $\sigma(z)$, and which will be therefore a natural extension of the Γ function.

* *Messenger of Mathematics*, Vol. xxix., pp. 64 *et seq.*

† Klein (quoting Botti), *Ueber die hypergeometrische Function* (1894), p. 126.

‡ *Quarterly Journal of Mathematics*, Vol. xxxi., pp. 264 *et seq.*