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XVI. On vector differentials

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sufficiently thin to allow the α rays to escape, must *decrease in weight.* Such a decrease has been recently observed by Heydweiler* for radium, but apparently under such conditions that the α rays would be largely absorbed in the glass tube containing the active matter.

In this connexion it is very important to decide whether the loss of weight observed by Heydweiler is due to a decrease of weight of the radium itself or to a decrease of weight of the glass envelope; for it is well known that radium rays produce rapid colourations throughout a glass tube, and it is possible that there may be a chemical change reaching to the surface of the glass which may account for the effects observed.

McGill University,
Montreal, Nov. 10, 1902.

XVI. *On Vector Differentials.* *By* FRANK LAUREN HITCHCOCK.—Second Paper †.

1. THE calculus of Quaternions enables us to represent a vector, or directed quantity, by a single symbol, and to work with it easily and compactly. We are not obliged to resolve into components, nor do we arbitrarily introduce any lines or planes of reference.

One of the simplest vectors is that of a point in space, represented by the symbol ρ. If we have a vector function of ρ continuously distributed throughout a portion of space, we may differentiate it: the result is a linear and vector function of $d\rho$, closely analogous, in a mathematical sense, to a homogeneous strain. Any such strain is fully determined if we know the roots of the strain-cubic, and the three directions which correspond to them.

In an introductory paper on this subject (Phil. Mag. June 1902, p. 576) it was shown that if ν be a vector of unit-length normal to any family of surfaces, and if its differential be $\chi d\rho$, then one of the roots of the cubic in χ is always zero.

The other two roots give directions tangent to the lines of curvature. For a line of curvature may be defined as one such that normals at contiguous points intersect, that is, such that the three vectors ν, $\nu+d\nu$, and $d\rho$ are coplanar; but because ν is a unit-vector $d\nu$ is at right angles to ν, and therefore parallel to $d\rho$. Accordingly $(\chi-g)d\rho=0$, g being a root of the strain-cubic.

* *Phys. Zeit.* 1902.
† Communicated by the Author.

If we take ϵ a unit-vector along this direction, and η another unit-vector such that $\epsilon\eta = \nu$, it is legitimate to write

$$\nabla\nu = \nu\chi\nu + \epsilon\chi\epsilon + \eta\chi\eta\ ;$$

the vector part of $\nabla\nu$ is equal to the term $\nu\chi\nu$, a result of the paper referred to above; $V\epsilon\chi\epsilon = 0$ by the last paragraph; whence $V\eta\chi\eta$ also vanishes and η gives the other root of the strain-cubic.

2. To illustrate further these fundamental facts, take Dupin's theorem that "each member of one of three families of orthogonal surfaces cuts each member of each of the other families along its lines of curvature."

Let the unit-normals be ν ν_1, and ν_2. Then

$$\nabla\nu_1 = \nabla(\nu_2\nu) = \nabla\nu_2 \,.\, \nu - \nu_2\nabla\nu - 2\chi\nu_2.$$

Operate by $S\nu_1$, remembering that

$$S\nu\nabla\nu = S\nu_1\nabla\nu_1 = S\nu_2\nabla\nu_2 = 0;$$

we thus have at once

$$S\nu_1\chi\nu_2 = 0,$$

that is, $\chi\nu_2$ is at right angles to ν_1. But $\chi\nu_2$ is already known to be at right angles to ν, and is therefore parallel to ν_2. This proves the proposition.

3. In order to study certain quantities related to the second differential of the vector ν we may adopt the notation

$$dV\nabla\nu = \psi d\rho,$$

and remembering that $\chi\nu$ and $V\nabla\nu$ have the same tensor we may put

$$\chi\nu = c\lambda\ ;\ \ V\nabla\nu = c\mu.$$

Thus λ, μ, and ν form a rectangular unit system. Differentiation with regard to these three directions may be represented by $\frac{d}{dl}$, $\frac{d}{dm}$, and $\frac{d}{dn}$ respectively. Here λ and μ are not the same as the ϵ and η of Art. 1, except in certain cases, of which families of cylinders are among the simplest.

The constituents of ψ may be arranged according to the following skeleton:—

$$\left.\begin{aligned} \psi\lambda &= P\lambda + r\mu + q'\nu \\ \psi\mu &= r'\lambda + Q\mu + p\nu \\ \psi\nu &= q\lambda + p'\mu + R\nu \end{aligned}\right\},$$

in which if we interchange p and p', q and q', r and r', we shall change ψ into ψ'.

To build up this function notice first that the quantity c is the absolute curvature of the orthogonal trajectories of the given surfaces. If c_1 be the tortuosity of these curves then

$$\frac{d\mu}{dn}=c_1\lambda\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

cf. Tait's 'Quaternions,' §§ 299, 300. Hence

$$\begin{aligned}\psi\nu &= \frac{d(c\mu)}{dn}\\ &= \frac{dc}{dn}\mu + c\frac{d\mu}{dn}\\ &= cc_1\lambda + \frac{dc}{dn}\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)\end{aligned}$$

which gives definite values for q and p', and shows that R vanishes. Again

$$\begin{aligned}dc &= d\mathrm{TV}\nabla\nu\\ &= -\mathrm{S}\mu\psi d\rho, \text{ by Tait, § 140 (1)}\\ &= -\mathrm{S}d\rho\psi'\mu,\end{aligned}$$

so that

$$\begin{aligned}\nabla c &= \psi'\mu\\ &= \frac{dc}{dl}\lambda + \frac{dc}{dm}\mu + \frac{dc}{dn}\nu, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)\end{aligned}$$

giving values for Q and r. Next take

$$\begin{aligned}\psi d\rho &= d\mathrm{V}\nabla\nu\\ &= d(\nu\chi\nu)\\ &= -\mathrm{V}\chi\nu\chi d\rho + \mathrm{V}\nu\phi d\rho, \text{ say ;}\end{aligned}$$

then by taking conjugates,

$$\psi' d\rho = \chi'\mathrm{V}\chi\nu d\rho - \phi'\mathrm{V}\nu d\rho,$$

whence by putting ν for $d\rho$ and remembering that $\chi\mu=\chi'\mu$,

$$\begin{aligned}\psi'\nu &= -c\chi\mu\\ &= c\lambda\mathrm{S}\lambda\chi\mu + c\mu\mathrm{S}\mu\chi\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)\end{aligned}$$

giving values for p and q'.

Furthermore, because $\nabla^2\nu$ is a vector,

$$
\begin{aligned}
S\nabla V\nabla\nu &= 0 \\
&= S(\lambda\psi\lambda + \mu\psi\mu + \nu\psi\nu) \\
&= -(P + Q + R) \\
&= -\left(P + \frac{dc}{dm} + 0\right), \text{ by (2) and (3),}
\end{aligned}
$$

whence we have for the value of P,

$$
P = -\frac{dc}{dm}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)
$$

It remains to get an expression for r'. Identically we have

$$
\nabla V\nabla\nu = (p - p')\lambda + (q - q')\mu + (r - r')\nu \; ; \quad . \quad . \quad (6)
$$

operate by $S\nu$ and put for r its value from (3),

$$
S \, . \, \nu\nabla(c\mu) = r' - \frac{dc}{dl},
$$

but by the ordinary expansion

$$
S \, . \, \nu\nabla(c\mu) = cS\nu\nabla\mu - \frac{dc}{dl} \, ;
$$

whence by equating values

$$
r' = cS\nu\nabla\mu. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)
$$

To sum up results,

$$
\left.
\begin{aligned}
\psi\lambda &= \quad -\frac{dc}{dm}\lambda + \frac{dc}{dl}\mu + c\nu S\lambda\chi\mu \\
\psi\mu &= cS\nu\nabla\mu \, . \, \lambda + \frac{dc}{dm}\mu + c\nu S\mu\chi\mu \\
\psi\nu &= \qquad cc_1\lambda + \frac{dc}{dn}\mu
\end{aligned}
\right\}, \quad . \quad . \quad . \quad (8)
$$

or more compactly

$$
\psi d\rho = \lambda S d\rho\left(\frac{dc}{dm}\lambda - cS\nu\nabla\mu \, . \, \mu - cc_1\nu\right) - \mu S d\rho\nabla c + c\nu S d\rho\chi\mu.
$$

The quantity r' may be expanded thus

$$
\begin{aligned}
r' &= cS\nu\nabla\mu \\
&= cS\nu\nabla(-\lambda\nu) \\
&= -cS\nu(\nabla\lambda \, . \, \nu - \lambda\nabla\nu - 2\chi\lambda) \\
&= c(S\nabla\lambda - c). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)
\end{aligned}
$$

4. Quantities such as $S\nabla\lambda$ involve operating on ν by both ∇ and d. These operators are not always commutative. In fact if P be any scalar, and σ and τ any vectors, whose differentials we may call $\phi d\rho$ and $\theta d\rho$, we shall have

$$\nabla S\tau\nabla P = S\tau\nabla . \nabla P - \theta'\nabla P, \text{ by (5) of the first paper,}$$
$$= S\tau\nabla . \nabla P + iS\theta i\nabla P + jS\theta j\nabla P + kS\theta k\nabla P,$$

and this extended to a vector by the usual method gives

$$\nabla\phi\tau = -S\tau\nabla . \nabla\sigma + i\phi\theta i + j\phi\theta j + k\phi\theta k. \quad . \quad (10)$$

This equation may be obtained in a quite different way. Write

$$d\phi\tau = d\phi . \tau + \phi d\tau,$$

where $d\phi . \tau$ indicates the result of differentiating $\phi\tau$ as if τ were a constant vector. With this understanding

$$d\phi\tau = Sd\rho\nabla . S\tau\nabla . \sigma + \phi\theta d\rho$$
$$= S\tau\nabla . Sd\rho\nabla . \sigma + \phi\theta d\rho,$$

provided we do not substitute for $d\rho$ any but constant vectors. If now we call the two terms on the right $\phi_1 d\rho$ and $\phi_2 d\rho$, we shall obtain from each a part of $\nabla\phi\tau$. The first term gives

$$q_1 = iS\tau\nabla . Si\nabla . \sigma + jS\tau\nabla . Sj\nabla . \sigma + kS\tau\nabla . Sk\nabla . \sigma$$
$$= S\tau\nabla . (iSi\nabla . \sigma + jSj\nabla . \sigma + kSk\nabla . \sigma)$$
$$= -S\tau\nabla . \nabla\sigma,$$

and the second term gives

$$q_2 = i\phi\theta i + j\phi\theta j + k\phi\theta k,$$

leading to the same result as before.

5. From (10), by putting ν for τ and χ for ϕ,

$$\nabla\chi\nu = \frac{d}{dn}\nabla\nu + \lambda\chi^2\lambda + \mu\chi^2\mu + \nu\chi^2\nu. \quad . \quad . \quad (11)$$

Here the first term on the right is the same as $\psi\nu - \frac{dm_2}{dn}$; and because for any direction at right angles to ν

$$\chi^2 - m_2\chi + m_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

it follows that

$$\begin{aligned}\lambda\chi^2\lambda+\mu\chi^2\mu &= \lambda(m_2\chi-m_1)\lambda+\mu(m_2\chi-m_1)\mu \\ &= 2m_1+m_2(\lambda\chi\lambda+\mu\chi\mu) \\ &= 2m_1+m_2 S(\lambda\chi\lambda+\mu\chi\mu+\nu\chi\nu) \\ &= 2m_1-m_2^2\,;\end{aligned}$$

the last term of (11) may be written $c\nu\chi\lambda$; therefore

$$\left.\begin{aligned}V\nabla\chi\nu &= \psi\nu+c\nu\chi\lambda \\ S\nabla\chi\nu &= 2m_1-m_2^2-\frac{dm_2}{dn}\end{aligned}\right\}. \quad . \quad . \quad . \quad (11a)$$

$S\nabla\chi\nu$ may also be expanded thus

$$\begin{aligned}S\nabla\chi\nu &= S\nabla(c\lambda) \\ &= cS\nabla\lambda-\frac{dc}{dl},\end{aligned}$$

which by comparison gives

$$S\nabla\lambda=\frac{1}{c}\left(2m_1-m_2^2-\frac{dm_2}{dn}+\frac{dc}{dl}\right), \quad . \quad . \quad (13)$$

and so from (9)

$$r'=2m_1-m_2^2-\frac{dm_2}{dn}+\frac{dc}{dl}-c^2. \quad . \quad . \quad . \quad (14)$$

6. Because $d(c\mu)=dc\,.\,\mu+cd\mu$ and μ is a unit-vector, it is clear we may write the value of $d\mu$ by inspection of (8), dropping the component along μ and dividing the rest by c. This gives

$$dUV\nabla\nu=\lambda Sd\rho\left(\frac{1}{c}\frac{dc}{dm}\lambda-S\nu\nabla\mu\,.\,\mu-c_1\nu\right)+\nu Sd\rho\chi\mu. \quad (15)$$

The differentials of $\chi\nu$ and $U\chi\nu$, that is of $c\lambda$ and λ, are easily expressed in terms of ψ and χ. For

$$d\chi\nu=d(V\nabla\nu\,.\,\nu)=cV\mu\chi d\rho-V\nu\psi d\rho\,;$$

the first term on the right is the same as $c\nu S\lambda\chi d\rho$ and the last term is the same as $-\lambda Sd\rho\psi'\mu+\mu Sd\rho\psi'\lambda$: therefore

$$d\chi\nu=-\lambda Sd\rho\nabla c+\mu Sd\rho\psi'\lambda+c\nu S\lambda\chi d\rho. \quad . \quad (16)$$

For $dU\chi\nu$ we have only to drop the component of $d\chi\nu$ along λ, and divide the rest by c. This gives

$$dU\chi\nu=\frac{1}{c}\,\mu Sd\rho\psi'\lambda+\nu Sd\rho\chi'\lambda. \quad . \quad . \quad . \quad (17)$$

7. As an application of some of these expressions, let us examine the criterion that the state of affairs contemplated in Dupin's theorem may exist: in other words, find the differential equation which must be satisfied by the unit-normal to a family of surfaces in order that there may be two other orthogonal families.

One form of the condition is that $S\epsilon\nabla\epsilon$ and $S\eta\nabla\eta$ shall both vanish, ϵ and η having the same meaning as in Art. 1. Furthermore,

$$\nabla\epsilon = \nabla(\eta\nu)$$
$$= \nabla\eta \,.\, \nu - \eta\nabla\nu - 2\chi\eta,$$

and by operating with $S\epsilon$ we obtain for *all* families of surfaces

$$S\epsilon\nabla\epsilon = S\eta\nabla\eta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Hence, if the condition just mentioned is fulfilled,

$$S\epsilon\nabla\epsilon + S\eta\nabla\eta = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19a)$$

It is here not essential that ϵ and η shall be of constant length. We may, therefore, put for them any other vectors to which they are respectively parallel. If g and g' be the roots of the quadratic equation

$$\chi^2 - m_2\chi + m_1 = 0,$$

so that $(\chi - g)\epsilon = 0$ and $(\chi - g')\eta = 0$, and if we operate on any vector at right angles to ν with $\chi - g'$ and with $\chi - g$, the two results will be parallel, in order, to ϵ and to η. Choosing as a convenient operand the unit-vector μ, that is $UV\nabla\nu$, we shall have

$$S(\chi - g')\mu\nabla(\chi - g')\mu + S(\chi - g)\mu\nabla(\chi - g)\mu = 0, \,. \quad (19b)$$

and by expanding and rearranging

$$S(2\chi - m_2)\mu\nabla\chi\mu + S(m_2^2 - 2m_1 - m_2\chi)\mu\nabla\mu - S\mu\chi\mu\nabla m_2 = 0. \quad (19c)$$

From (10), by writing χ for ϕ and μ for τ and $d\mu = \theta_\mu d\rho$,

$$\nabla\chi\mu = \frac{d}{dm}\nabla\nu + \lambda\chi\theta_\mu\lambda + \mu\chi\theta_\mu\mu + \nu\chi\theta_\mu\nu.$$

The form of (19c) shows that we are concerned only with that part of $V\nabla\chi\mu$ lying in the tangent plane. The vector part of $\frac{d}{dm}\nabla\nu$ is $\psi\mu$; the terms $\lambda\chi\theta_\mu\lambda$ and $\mu\chi\theta_\mu\mu$ have no tangential component; the term $\nu\chi\theta_\mu\nu$ equals $c_1\nu\chi\lambda$, by (1) or by (15). Thus the first term of (19*c*) becomes

$$S(2\chi\mu - m_2\mu)(\psi\mu + c_1\nu\chi\lambda);$$

multiplying, and noticing that $\mu\nu = \lambda$ while $S\nu\chi\lambda\chi\mu = -m_1$,

$$S(2\chi - m_2)\mu\psi\mu - 2c_1m_1 - c_1m_2S\lambda\chi\lambda$$

is the product. We next obtain from (15)

$$\nabla\mu = \lambda S\mu\chi\mu + \mu(c_1 - S\lambda\chi\mu) - \nu\frac{r'}{c}, \quad . \quad . \quad . \quad (20)$$

and here again we are not concerned with the normal component. Thus the second term of (19*c*) equals

$$S[(m_2^2 - 2m_1)\mu - m_2\chi\mu][\lambda S\mu\chi\mu + \mu(c_1 - S\lambda\chi\mu)]$$
$$= -c_1m_2^2 + m_2^2S\lambda\chi\mu + 2c_1m_1 - 2m_1S\lambda\chi\mu - c_1m_2S\mu\chi\mu.$$

The third term of (19*c*) is the same as $\frac{dm_2}{dn}S\lambda\chi\mu$. Collecting results and noticing that $S(\lambda\chi\lambda + \mu\chi\mu) = -m_2$, we find that all the terms containing c_1 cancel out, and the result is

$$S(2\chi - m_2)\mu\psi\mu + S\lambda\chi\mu\left(m_2^2 - 2m_1 + \frac{dm_2}{dn}\right) = 0, \quad (19d)$$

which by (11*a*) may be written

$$S(2\chi - m_2)\mu\psi\mu - S\lambda\chi\mu S\nabla\chi\nu = 0. \quad . \quad . \quad (19e)$$

Again, because of the identity

$$(\psi - \psi')\mu = V(\nabla V\nabla\nu)\mu$$

we shall have the following expansions:—

$$\begin{aligned} S(2\chi - m_2)\mu\psi\mu &= S(2\chi\mu - m_2\mu)(\psi'\mu - \mu\nabla V\nabla\nu) \\ &= S(2\chi - m_2)\mu\psi'\mu + 2S\mu\chi\mu\nabla V\nabla\nu \\ &= S(2\chi - m_2)\mu\psi'\mu + 2S\lambda\chi\mu S\nu\nabla V\nabla\nu \\ &= S(2\chi - m_2)\mu\psi'\mu + 2S\lambda\chi\mu(S\nabla\chi\nu + V^2\nabla\nu), \end{aligned}$$

and by using this result in (19*e*),

$$S(2\chi - m_2)\mu\psi'\mu + S\lambda\chi\mu(S\nabla\chi\nu + 2V^2\nabla\nu) = 0. \quad . \quad (19f)$$

Finally, by adding (19*e*) and (19*f*)

$$S\mu(\psi + \psi')(2\chi - m_2)\mu + 2S\lambda\chi\mu V^2\nabla\nu = 0, \quad . \quad (19g)$$

where the only operation involving the second differential of the unit-normal is the pure strain $\psi + \psi'$. Thus the equation is of the first order with regard to $V\nabla\nu$.

8. In the paper referred to in Art. 1 it was proved that if P be a scalar such that $\nabla^2 P = 0$ the unit-normal to the equipotential surfaces satisfies the equation

$$V\nabla(\nabla\nu \, . \, \nu) = 0, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

of which various expansions were given. If ν be given satisfying this condition P is determined by the equation

$$\log T\nabla P = \nabla^{-1}(\nabla\nu \, . \, \nu), \quad . \quad . \quad . \quad (22)$$

of which the solution is very direct and obvious. We may thus write, as a set of equations defining orthogonal isothermal surfaces

$$\left.\begin{array}{l} S\nu\nabla\nu = 0 \\ S\mu(\psi + \psi')(2\chi - m_2)\mu + 2S\lambda\chi\mu V^2\nabla\nu = 0 \\ V\nabla(\nabla\nu \, . \, \nu) = 0 \\ \log T\nabla P = \nabla^{-1}(\nabla\nu \, . \, \nu) \end{array}\right\}, \quad . \quad (23)$$

where the first two equations are to be satisfied by one unit-vector in order that there may be three orthogonal families of surfaces, the third equation must be satisfied by each of the three unit-normals in order that these surfaces may all be isotherms, and the last equation serves to determine the three potentials. *Cf.* § 336 of Tait's 'Quaternions.'

9. In studying special cases we have evidently at our disposal a great variety of methods. Equations like (19) appear to be chiefly useful in general investigations. In testing whether any given family of surfaces satisfies the condition discussed in Art. 7 it will usually be easier to find a vector corresponding to one of the non-vanishing roots of the strain-cubic, say parallel to η, and operate on it with $S.\eta\nabla$,—though indeed the nature of the surfaces may be such that (19*g*) takes a very simple form. As a brief example, let a family of rings be denoted by the scalar function

$$P = T^4 q \, S^{-1}\rho\phi\rho,$$

where $q = ix + jy + kz + a$ and $\phi\rho = -(ix + jy)$. Then

$$dTq = -T^{-1}q \, S\rho d\rho,$$

so that by differentiating the given function,

$$dP = -4T^2 q \, S^{-1}\rho\phi\rho \, S\rho d\rho - 2T^4 q \, S^{-2}\rho\phi\rho \, Sd\rho\phi\rho,$$

and because $dP = -Sd\rho\nabla P$,

$$\nabla P = 4\rho T^2 q \, S^{-1}\rho\phi\rho + 2\phi\rho T^4 q \, S^{-2}\rho\phi\rho.$$

Then by taking a unit along $\nabla \mathrm{P}$,

$$\nu = \mathrm{U}\nabla \mathrm{P} = (2\rho \mathrm{S}\rho\phi\rho + \phi\rho \mathrm{T}^2 q)(\mathrm{T}^4 q\ \mathrm{S}\rho\phi\rho - 4a^2 \mathrm{S}^2 \rho\phi\rho)^{-\frac{1}{2}},$$

$\mathrm{T}^2\phi\rho$ being here the same as $\mathrm{S}\rho\phi\rho$. In differentiating again it will be well to put, for brevity, $\mathrm{T}^2 q = t^2$ and $\mathrm{S}\rho\phi\rho = s^2$, so that we have $\mathrm{T}^2\rho = t^2 - a^2$. The result is

$$d\nu = \chi d\rho = \{d\rho(2t^4 s^4 - 8a^2 s^6) + 4s^4 t^2 \rho \mathrm{S}\rho d\rho + \phi\rho \mathrm{S} d\rho\phi\rho(8a^2 s^2 t^2 - t^6)$$
$$+ \phi d\rho(s^2 t^6 - 4a^2 s^4 t^2) + 2s^2 t^4 \rho \mathrm{S} d\rho\phi\rho + 8a^2 s^4 \phi\rho \mathrm{S}\rho d\rho\}\{s^2 t^4 - 4a^2 s^4\}^{-\frac{3}{2}}.$$

This linear and vector function contains six vector terms, of which all but the last two are self-conjugate, and therefore contribute nothing toward $\mathrm{V}\nabla\nu$. The last two terms give

$$\mathrm{V}\nabla\nu = (2s^2 t^4 - 8a^2 s^4)\mathrm{V}\rho\phi\rho(s^2 t^4 - 4a^2 s^4)^{-\frac{3}{2}}$$
$$= 2\mathrm{V}\rho\phi\rho(s^2 t^4 - 4a^2 s^4)^{-\frac{1}{2}}.$$

If this last expression be substituted for $d\rho$ in $\chi d\rho$ above, all the terms vanish except the first and the fourth, giving

$$\chi \mathrm{V}\nabla\nu = (4s^2 \mathrm{V}\rho\phi\rho + 2t^2 \phi \mathrm{V}\rho\phi\rho)(s^2 t^4 - 4a^2 s^4)^{-1}.$$

But by an elementary transformation (Kelland and Tait's 'Introduction to Quaternions,' p. 190, r), since ϕ is self-conjugate, we have

$$\phi \mathrm{V}\rho\phi\rho = -2\mathrm{V}\rho\phi\rho - \mathrm{V}\rho\phi^2\rho,$$

and also $\phi^2\rho = -\phi\rho$, whence

$$\chi \mathrm{V}\nabla\nu = 2\mathrm{V}\rho\phi\rho(2s^2 - t^2)(s^2 t^4 - 4a^2 s^2)^{-1},$$

which is a scalar multiple of $\mathrm{V}\nabla\nu$. Thus $\mathrm{V}\rho\phi\rho$ is a vector parallel to η and

$$\mathrm{S}\eta\nabla \mathrm{V}\rho\phi\rho = \mathrm{S}(iy - jx)\nabla(iy - jx) = 2\mathrm{S}k(iy - jx) = 0.$$

It is clear also that ($19g$) reduces to

$$\frac{dc}{dm} = 0,$$

a general property of surfaces of revolution, provided the axis is the same for all members of the family.

The following may be taken as further illustrations:—

1. If $\mathrm{S}\sigma\nu = 0$, $\chi\sigma$ differs from $\chi'\sigma$ by a normal vector.

2. When applied to a vector in the tangent plane the operator $\chi[\nu\chi\{\nu(\ )\}]$ or $(\chi \mathrm{V}\nu)^2$ is equivalent to a scalar.

3. If two vectors at right angles to each other and to the normal be operated on with $2\chi - m_2$ they will still be at right angles.

4. If $\chi\epsilon = g\epsilon$ and $\chi\eta = g'\eta$, then $g = -S\eta\nabla\epsilon$ and $g' = +S\epsilon\nabla\eta$.

5. With the notation of Art. 4, it may be proved that

$$-S\tau_2\nabla \,.\, \phi\tau_1 = -S\tau_1\nabla \,.\, \phi\tau_2 + \phi\theta_1\tau_2 - \phi\theta_2\tau_1.$$

6. Putting a for $S\mu\chi\mu$ and b for $S\lambda\chi\mu$, while $d\chi\lambda = \phi_\lambda d\rho$, we may establish these six results:

$$(a)\quad \phi'_\lambda\nu = -\chi'\chi\lambda = \lambda(m_1 - m_2^2) - m_2\nu\chi\mu - c\nu(a + m_2);$$

$$(b)\quad \phi_\lambda\nu = -\chi'\chi\lambda + \lambda\left(\frac{dc}{dl} - c^2\right) + \mu\frac{dc}{dm} - c_1\chi\mu;$$

$$(c)\quad \phi'_\lambda(2\chi - m_2)\lambda = \frac{bm_2}{c}\psi\lambda + \nabla m_2(a + m_2) - \nabla m_1;$$

$$(d)\quad S\nabla\chi\lambda = \left(c - \frac{d}{dl}\right)m_2 - \frac{a}{c}(r' - c^2) + \frac{b}{c}\frac{dc}{dm};$$

$$(e)\quad V\nabla\chi\lambda = \psi\lambda - \frac{\nu}{c}\left(a\frac{dc}{dm} + b\overline{r' + c^2}\right) - c_1\nu\chi\mu - c^2\mu;$$

$$(f)\quad \frac{dm_1}{dn} = \left(ac + 2b\frac{d}{dm} - a\frac{d}{dl}\right)c - m_2m_1 - (a + m_2)r':$$

from which may be deduced the eighteen constituents of $d\chi\lambda$ and $d\chi\mu$.

7. If ν, ν_1, and ν_2 are unit-normals to three orthogonal families of surfaces, so that $\chi\nu_1 = g\nu_1$ and $\chi\nu_2 = g'\nu_2$, with similar expressions for g_1, g_1' and g_2, g_2', $d\nu$ may be expressed in terms of the three normals and the six g's (see Ex. 4).

8. If $S\rho(\phi + P)^{-1}\rho = -1$, where ϕ is self-conjugate with constant constituents, $U\nabla P$ satisfies (21). Thence may be found the distribution of electricity on an ellipsoid by means of (22). (In differentiating we may treat $\phi + P$ like a scalar, that is

$$d[(\phi + P)^{-1}\rho] = (\phi + P)^{-1}d\rho - (\phi + P)^{-2}\rho dP).$$

9. Of $\phi d\rho$ and $\phi' d\rho$ only one can be integrable.