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Sir W. Thomson

To cite this article: Sir W. Thomson (1887) XXXII. On the equilibrium of a gas under its own gravitation only, Philosophical Magazine Series 5, 23:142, 287-292, DOI: [10.1080/14786448708628007](https://doi.org/10.1080/14786448708628007)

To link to this article: <http://dx.doi.org/10.1080/14786448708628007>



Published online: 29 Apr 2009.



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XXXII. *On the Equilibrium of a Gas under its own Gravitation only.* By Sir W. THOMSON*.

THIS problem, for the case of uniform temperature, was first, I believe, proposed by Tait in the following highly interesting question, set in the Ferguson Scholarship Examination (Glasgow, October 2nd, 1885):—"Assuming Boyle's Law for all pressures, form the equation for the equilibrium-density at any distance from the centre of a spherical attracting mass, placed in an infinite space filled originally with air, Find the special integral which depends on a power of the distance from the centre of the sphere alone."

The answer (in examinational style!) is:—Choose units properly; we have

$$\frac{d\rho}{dr} = - \frac{\rho \int_0^r \rho r^2 dr}{r^2} \quad \dots \dots \dots (1),$$

where ρ is the density at distance r from the centre. Assume

$$\rho = A r^\kappa \quad \dots \dots \dots (2).$$

We find $A=2$, $\kappa=-2$; and therefore

$$\rho = \frac{2}{r^2} \quad \dots \dots \dots (3)$$

satisfies the equation in the required form.

Tait informs me that this question occurred to him while writing for 'Nature' a review of Stokes's Lecture † on Inferences from the Spectrum Analysis of the Lights of Sun, Stars, Nebulæ, and Comets; and in the 'Proceedings of the Edinburgh Mathematical Society' he has given some Transformations of the equation of Equilibrium. The same statistical problem has recently been forced on myself by con-

* Communicated by the Author, having been read before the Royal Society of Edinburgh on the 7th and 21st February, 1887.

Note of February 22, 1887.—Having yesterday sent a finally revised proof of this paper for press, I have today received a letter from Prof. Newcomb, calling my attention to a most important paper by Mr. J. Homer Lane, "On the Theoretical Temperature of the Sun," published in the American Journal of Science for July 1870, p. 57, in which precisely the same problem as that of my article is very powerfully dealt with, mathematically and practically. It is impossible now, before going to press, for me to do more than refer to Mr. Lane's paper; but I hope to profit by it very much in the continuation of my present work which I intended, and still intend, to make.—W. T.

† Lecture III. of Second Course of "Burnet Lectures," Aberdeen, Dec. 1884; published, London, 1885 (Macmillan).

siderations which I could not avoid in connection with a lecture which I recently gave in the Royal Institution of London, on "The Probable Origin, the Total Amount, and the Possible Duration of the Sun's Heat."

Helmholtz's explanation, attributing the Sun's heat to condensation under mutual gravitation of all parts of the Sun's mass, becomes not a hypothesis but a statement of fact, when it is admitted that no considerable part of the heat emitted from the Sun is produced by present in-fall of meteoric matter from without. The present communication is an instalment towards the gaseous dynamics of the Sun, Stars, and Nebulæ.

To facilitate calculation of practical results, let a kilometre be the unit of length; and the terrestrial-surface heaviness of a cubic kilometre of water at unit density, taken as the maximum density under ordinary pressure, be the unit of force (or, approximately, a thousand million tons heaviness at the earth's surface). If p be the pressure, ρ the density, and t the temperature from absolute zero, we have, by Boyle and Charles's laws,

$$p = H\rho t \quad . \quad . \quad . \quad . \quad . \quad . \quad (4);$$

where t denotes absolute (thermodynamic*) temperature, with 0° Cent. taken as unit; and H denotes what is commonly, in technical language, called "the height of the homogeneous atmosphere" at 0° C. For dry common air, according to Regnault's determination of density,

$$H = 7.985 \text{ kilometres} \quad . \quad . \quad . \quad . \quad (4').$$

Let β be the gravitational coefficient proper to the units chosen; so that $\beta mm'/D^2$ is the force between m , m' at distance D . The earth's mean density being 5.6, and radius 6370 kilometres, we have

$$\frac{4\pi}{3} \cdot 6370 \cdot 5.6\beta = 1; \text{ and therefore } 4\pi\beta = 1/11890 \quad . \quad . \quad (5).$$

Let now the p , ρ , t of (4) be the pressure, density, and temperature at distance r from the centre of a spherical shell containing gas in gross-dynamic† equilibrium. We have, by

* The notation of the text is related to temperature Centigrade on the thermodynamic principle (which is approximately temperature Centigrade by the air-thermometer), as follows:—

$$= \frac{1}{273} (\text{temperature Centigrade} + 273);$$

see my Collected Mathematical and Physical Papers, vol. i. Arts. xxxix., and xlviii. part vi. §§ 99, 100; and article "Heat," §§ 35-38 & 47-67, *Encyc. Brit.*, and vol. iii. (soon to be published) of Collected Papers.

† Not in molecular equilibrium of course; and not in gross-thermal equilibrium, except in the case of t uniform throughout the gas.

elementary hydrostatics,

$$\frac{dp}{dr} = -\rho \left(M + \int_a^r dr 4\pi r^2 \rho \right) \beta / r^2 \quad . \quad . \quad . \quad (6),$$

whence

$$\frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dp}{dr} \right) = -4\pi \beta r^2 \rho \quad . \quad . \quad . \quad (7),$$

where M denotes the whole quantity of matter within radius a from the centre; which may be a nucleus and gas, or may be all gas.

If the gas is enclosed in a rigid spherical shell, impermeable to heat, and left to itself for a sufficiently long time, it settles into the condition of gross-thermal equilibrium, by "conduction of heat," till the temperature becomes uniform throughout. But if it were stirred artificially all through its volume, currents not considerably disturbing the static distribution of pressure and density will bring it approximately to what I have called convective equilibrium* of temperature—that is to say, the condition in which the temperature in any part P is the same as that which any other part of the gas would acquire if enclosed in an impermeable cylinder with piston, and dilated or expanded to the same density as P . The *natural stirring* produced in a great free fluid mass like the Sun's, by the cooling at the surface, must, I believe, maintain a somewhat close approximation to convective equilibrium throughout the whole mass. The known relations between temperature, pressure, and density for the ideal "perfect gas," when condensed or allowed to expand in a cylinder and piston of material impermeable to heat, are†

$$p = HT \rho^k \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

$$t = T \rho^{k-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9);$$

where k denotes the ratio of the thermal capacity of the gas, pressure constant, to its thermal capacity, volume constant, which is approximately equal to 1.41 or 1.40 (we shall take it 1.4) for all gases, and all temperatures, densities, and pressures; and T denotes the temperature corresponding to unit density in the particular gaseous mass under consideration.

Using (8) to eliminate p from (7) we find

$$\frac{d}{dr} \left[r^2 \frac{d(\rho^{k-1})}{dr} \right] = - \frac{4\pi \beta (k-1)}{HT k} r^2 \rho \quad . \quad . \quad . \quad (10);$$

* See "On the Convective Equilibrium of Temperature in the Atmosphere," Manchester Phil. Soc. vol. ii. 3rd series, 1862; and vol. iii. of Collected Papers.

† See my Collected Mathematical and Physical Papers, vol. i. Art. xlviii. note 3.

which, if we put $\rho^{k-1} = u$ (11),

$$1/(k-1) = \kappa \quad . \quad . \quad . \quad . \quad . \quad (12),$$

and

$$r^{-1} \sqrt{\frac{HTk}{4\pi\beta(k-1)}} = x \quad . \quad . \quad . \quad . \quad . \quad (13)$$

takes the remarkably simple form

$$\frac{d^2u}{dx^2} = -\frac{u^\kappa}{x^4} \quad . \quad . \quad . \quad . \quad . \quad (14).$$

Let $f(x)$ be a particular solution of this equation ; so that

$$\left. \begin{aligned} f''(x) &= -[f(x)]^\kappa x^{-4} \\ \text{and therefore } f''(mx) &= -[f(mx)]^\kappa m^{-4} x^{-4} \end{aligned} \right\} \quad . \quad . \quad . \quad (15).$$

From this we derive a general solution with one disposable constant, by assuming

$$u = Cf(mx) \quad . \quad . \quad . \quad . \quad . \quad (16);$$

which, substituted in (14), yields, in virtue of (15),

$$m^2 = C^{-\kappa+1} \quad . \quad . \quad . \quad . \quad . \quad (17);$$

so that we have, as a general solution,

$$u = Cf[xC^{-\frac{1}{2}(\kappa-1)}] \quad . \quad . \quad . \quad . \quad . \quad (18).$$

Now the class of solutions of (14) which will interest us most is that for which the density and temperature are finite and continuous from the centre outwards, to a certain distance, finite as we shall see presently, at which both vanish. In this class of cases u increases from 0 to some finite value, as x increases from some finite value to ∞ . Hence if $u=f(x)$ belongs to this class, $u=Cf(mx)$ also belongs to it; and (18) is the general solution for the class. We have therefore, immediately, the following conclusions :—

(1) The diameters of different globular* gaseous stars of the same kind of gas are inversely as the $\frac{1}{2}(\kappa-1)$ th powers (or $\frac{3}{2}$ powers) of their central temperatures, at the times when, in the process of gradual cooling, their temperatures at places of the same densities are equal (or “T” the same for the different masses). Thus, for example, one sixteenth central temperature corresponds to eight-fold diameter : one eighty-first central temperature corresponds to twenty-seven fold diameter.

* This adjective excludes stars or nebulae *rotating steadily* with so great angular velocities as to be much flattened, or to be annular ; also nebulae revolving circularly with different angular velocities at different distances from the centre, as may be approximately the case with spiral nebulae. It would approximately enough include the sun, with his small angular velocity of once round in 25 days, were the fluid not too dense through a large part of the interior to approximately obey gaseous law. It no doubt applies very accurately to earlier times of the sun’s history, when he was much less dense than he is now.

(2) Under the same conditions as (1) (that is, H and T the same for the different masses), the central densities are as the κ th powers (or $\frac{5}{2}$ powers) of the central temperatures; and therefore inversely as the $\frac{2\kappa}{\kappa-1}$, or $\frac{2}{2-k}$, or $\frac{10}{3}$, powers of the diameters.

(3) Under still the same conditions as (1) and (2), the quantities of matter in the two masses are inversely as the $\left(\frac{2}{2-k} - 3\right)$ th powers, (inversely as the cube roots) of their diameters.

(4) The diameters of different globular gaseous stars, of the same kind of gas, and of the same central densities, are as the square roots of their central temperatures.

(5) The diameters of different globular gaseous stars of different kinds of gas, but of the same central densities and temperatures, are inversely as the square roots of the specific densities of the gases.

(6) A single curve $[y=f(r^{-1})]$ with scale of ordinate (r) and scale of abscissa (y) properly assigned according to (18), (17), and (11) shows for a globe of any kind of gas in molecular equilibrium, of given mass and given diameter, the absolute temperature at any distance from the centre. Another curve, $\{y=[f(r^{-1})]^\kappa\}$, with scales correspondingly assigned, shows the distribution of density from surface to centre.

It is easy to find, with any desired degree of accuracy, the particular solution of (13), for which

$$u=A, \text{ and } \frac{du}{dx}=A', \text{ where } x=a \quad . \quad . \quad (19),$$

a denoting any chosen value of x , and A and A' any two arbitrary numerics, by successive applications of the formula

$$u_{i+1}=A-\int_a^x dx \left(A' - \int_a^x dx \frac{u_i^\kappa}{x^4} \right) \quad . \quad . \quad (20);$$

the quadratures being performed with labour moderately proportional to the accuracy required, by tracing curves on "section"-paper (paper ruled with small squares) and counting the squares and parts of squares in their areas. To begin, u_0 may be taken arbitrarily; but it may conveniently be taken from a hasty graphic construction by drawing, step by step, successive arcs* of a curve with radii of curvature calculated from (13) with the value of du/dx found from the step-by-step process. If this preliminary construction is done with

* This method of graphically integrating a differential equation of the second order, which first occurred to me many years ago as suitable for finding the shapes of particular cases of the capillary surface of revolution, was successfully carried out for me by Prof. John Perry, when a student in

care, by aid of good drawing-instruments, u_1 calculated from u_0 by quadratures will be found to agree so closely with u_0 , that u_0 itself will be seen to be a good solution. If any difference is found between the two, u_1 is the better: u_2 is a closer approximation than u_1 ; and so on, with no limit to the accuracy attainable.

Mr. Magnus Maclean, my official assistant in the University of Glasgow, has made a successful beginning of working-out this process for the case $u=16$ where $x=\infty$; and has already obtained a somewhat approximate solution, of which the produce useful for our problem is expressed in the following table.

Numerical Solution of $\frac{d^2u}{dx^2} + x^{-4} u^{2.5} = 0$.

Distance from centre = $r=1/x$.	Reciprocal of distance from centre = $x=1/r$.	Temperature = u .	Density = $u^{2.5}$.	Mass within dis- tance r from the centre = du/dx = $\int_x^\infty dx u^{2.5} x^{-4}$.
0	∞	16.00	1024	.00
.100	10	14.46	795.2	.28
.111	9	14.14	751.6	.38
.125	8	13.71	695.8	.52
.143	7	13.10	621.2	.731
.167	6	12.20	520.0	1.056
.200	5	10.92	394.1	1.566
.250	4	9.00	243.0	2.336
.333	3	6.15	93.81	3.436
.500	2	2.25	7.595	4.366
.667	1.5	0	0	4.49

The deduction from these numbers, of results expressing in terms of convenient units the temperature and density at any point of a given mass of a known kind of gas, occupying a sphere of given radius, must be reserved for a subsequent communication.

One interesting result which I can give at present, derived from the first and last numbers of the several columns of the preceding table, is, that the central density of a globular gaseous star is $22\frac{1}{2}$ times its average density.

my laboratory in 1874, in a series of skilfully executed drawings representing a large variety of cases of the capillary surface of revolution, which have been regularly shown in my Lectures to the Natural Philosophy Class of the University of Glasgow. These curves were recently published in the Proc. Roy. Instit. (Lecture of Jan. 29, 1886), and 'Nature,' July 22 and 29, and Aug. 19, 1886; also to appear in a volume of Lectures now in the press, to be published in the 'Nature' series.