

ON NON-UNIFORM CONVERGENCE AND TERM-BY-TERM INTEGRATION OF SERIES

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[Received January 6th, 1903.—Read January 8th, 1903].

1. *Introductory.*—In the present paper the problem of the classification and distribution of the points of non-uniform convergence of a convergent series of functions

$$F(x) = f_1(x) + f_2(x) + \dots + f_n(x) + \dots \text{ ad inf.}$$

and of the integration of this series term by term is discussed in its most general form, *i.e.*, when the only restriction laid on the functions F, f_1, f_2, \dots is that they are at most pointwise discontinuous.* Without this restriction, one at least of the functions being totally discontinuous, the integration would be impossible.

The results obtained, (§§ 12, 13), are generalizations of those enunciated and proved for continuous functions by Prof. Osgood in the fundamental paper on this subject in the *American Journal of Mathematics*, Vol. xix., and recently extended by Dr. Hobson,† by the application of Baire's methods, to the case when only one of the functions is discontinuous; that is, when the sum of a series of continuous functions, not being itself continuous, is, by a theorem of Baire's,‡ pointwise discontinuous.

It should be noted that the results previously obtained are not assumed in this paper. With certain simplifications, alluded to in their proper place, what follows constitutes a second proof of Dr. Hobson's results, and in the case of continuous functions reduces in all essentials to Prof. Osgood's original proof.

2. Let $f_1(x), f_2(x), \dots, f_n(x), \dots$ be at most pointwise discontinuous functions, and let their sum, denoted by $F(x)$, be also at most pointwise discontinuous. We denote the sum of the first n terms of the

* In Part II., of course, they must also belong the class of *integrable* pointwise discontinuous functions.

† *Proc. London Math. Soc.*, Vol. xxxiv., pp. 254 *et seq.* Dr. Hobson communicated some of his results to me before his paper was ready for presentation, and I succeeded at that time in extending Prof. Osgood's results by the use of the latter's original methods. At Prof. Osgood's suggestion I undertook, before publishing my work, to attempt the further extension of these theorems: the present paper is the result.

‡ "Dissertation," *Ann. di Mat.*, Vol. iii., 1899.

series by $s_n(x)$, and the residue by $R_n(x)$, so that


$$s_n(x) = f_1(x) + f_2(x) + \dots + f_n(x), \quad R_n(x) = f_{n+1}(x) + f_{n+2}(x) + \dots \text{ ad inf.,} \\ F(x) = s_n(x) + R_n(x).$$

3. We begin by defining afresh some of the terms we shall have reason to employ.

Uniform Convergence at a Point.—A series of functions of x is said to be “uniformly convergent at a point ξ ” if, given any positive quantity A , however small, an interval δ can be described, having ξ as internal point, so that, for all points x within the interval δ , $|R_n(x)| < A$ for all values of $n \geq m$ where m is an integer, independent of x , which can always be determined.

Similarly we may define the expression *right-handed*, (or *left-handed*), *uniform convergence* at ξ . In this case, ξ must be, left-hand, (or right-hand), end-point of δ .

Uniform Convergence throughout a Closed Interval or Closed Set.—A series of functions of x is said to be “uniformly convergent throughout a closed interval or closed set” if it is uniformly convergent at every point of the interval or set; the uniform convergence at the extreme left-hand point being in general right-handed only, and at the extreme right-hand point left-handed.

The Heine-Borel theorem and its extension* enable us to put this definition in a form sometimes more convenient for application. Let us take the case of a  FIG. 1.
closed set, as including the other. Let C and D be the extreme points, (C, C') and (D, D') the corresponding intervals. Then the points of the given set in (C', D') form a closed set, every point of which is interior to an interval. Hence, by the theorems referred to, there are a finite number k of the intervals having the same property, and to each of these corresponds a definite integer m . If we take m_A to be the largest of these integers, and of those corresponding to (C, C') and (D, D') , then it is plain that m_A possesses the property that, for all values of $n \geq m_A$, $|R_n(x)| < A$ for all points x of the $k+2$ intervals containing the closed set.

Thus we have a *second (the usual) definition of uniform convergence throughout a closed interval*. A series of functions of x is said to be “uniformly convergent throughout a closed interval” when, given any positive quantity A , however small, an integer m can be found, independent of x , such that, for all values of $n \geq m$, $|R_n(x)| < A$ for all values of x in the interval.

* W. H. Young, *Proc. London Math. Soc.*, Vol. xxxv., p. 387.

A similar definition might be given in the case of the closed set, the words after " $|R_n(x)| < A$ " being replaced by the following:—"for all values of x within a finite number of intervals, having the points of the given set as internal points."

Uniform Convergence throughout an Open Interval or Set.—A series of functions of x is said to be "uniformly convergent throughout an open interval or an open set" if it be uniformly convergent at every point of the open interval or open set; or, which is the same thing, if it be uniformly convergent throughout every closed set contained in the open interval or set.

4. It will be noticed that uniform convergence at a point does not imply, in general, uniform convergence in any interval, however small, containing the point.

Given any interval in which the series is convergent, the points of the interval will consist of the set of points at which the series is uniformly convergent, and those at which it is not; these latter are called *the points of non-uniform convergence*.

A point of non-uniform convergence must, then, be such that when A is chosen sufficiently small, but finite, there is a set of points $P_1, P_2, \dots, P_i, \dots$ all lying above the line $y = A$, the coordinates of P_i being $x_i, |R_{n_i}(x_i)|$, where n_1, n_2, \dots is a series of positive integers, constantly increasing without limit, and the limiting value of x_i , for $i = \infty$, is the abscissa of P .

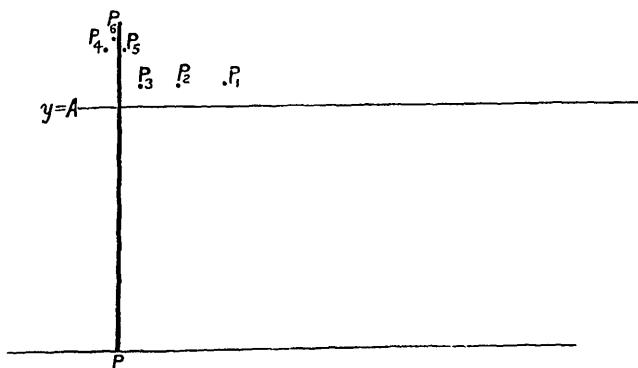


FIG. 2.

Osgood uses the graphic expression:—"A point P of non-uniform convergence is one in the neighbourhood of which the loci $y = s_n(x)$ have peaks." This expression, which, in the case when $y = s_n(x)$ is a graphic curve, is at once evident, can still be used *cum grano salis* in the general case: we shall see later what modifications have to be borne in mind.

5. *Measure of Non-uniform Convergence.*—It is convenient to classify the points of non-uniform convergence in two different ways: (1) quantitatively, and (2) qualitatively. First, quantitatively: Suppose P to be a point of non-uniform convergence, either left-handed or right-handed, or both. Then a smallest quantity B must exist, such that, for all values of $A \leq B$, it is impossible to find an interval δ , however small, having P as right-hand, left-hand, or interior point respectively, corresponding to a given integer m , so that, for all values of x in the interval δ , and for all values of $n \geq m$, $|R_n(x)| < A$. This quantity B may be called *the measure of convergence* at the point, the words “right-handed” or “left-handed” being added, if we are dealing with those cases. Notice that at a point at which the non-uniform convergence is both right- and left-handed the measure of non-uniform convergence obtained by this rule is the larger of the two measures of right- and left-handed non-uniform convergence.

It may happen that the measure of non-uniform convergence, so defined, is infinite at some points. Such a point may be, but is not of necessity, an infinite discontinuity of one or more of the functions.

6. We now proceed to a convenient qualitative classification of the points of non-uniform convergence.

It was first pointed out by Stokes* that, when the sum of a series of continuous functions is discontinuous at a point P , that point is necessarily a point of non-uniform convergence of the series. It was at first supposed that the converse was true, i.e., that there were no other points of non-uniform convergence. Prof. Osgood's paper, however, deals with the sum of continuous functions only when it is itself continuous; so that no point of the type considered by Stokes occurs at all.

The following examples, due to Osgood, illustrate this point:—

$$\text{Ex. 1.} \quad s_n(x) = \frac{n^2 x + a}{1 + n^2 x^2},$$

$$F(0) = a, \quad F(x) = 0, \quad (0 < x \leq 1).$$

When a is not zero we have a Stokes point at the origin. When a is zero the sum is continuous. In any case, however, the origin is a point of non-uniform convergence, with what Osgood calls “infinite peaks,” that is to say, the measure of non-uniform convergence at the origin is infinite.

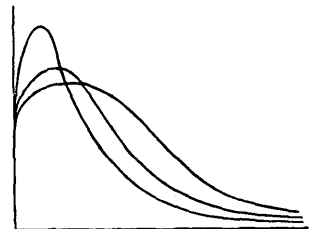


FIG. 3.

* *Camb. Phil. Trans.*, Vol. VII., 1847.

$$\text{Ex. 2.} \quad s_n(x) = \frac{nx+a}{1+n^2x^2},$$

$$F(0) = a, \quad F(x) = 0, \quad (0 < x \leq 1).$$

Here again when a is not zero we have a Stokes point at the origin, while when a is zero the sum is continuous. The origin is, however, for all values of a , a point of non-uniform convergence with

peaks of height $B = \frac{\sqrt{1+a^2}+a}{2}$, B being the measure of convergence (Fig. 4).

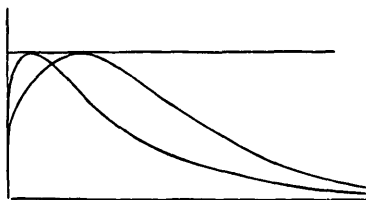


FIG. 4.

7. The set of points discovered by Stokes is, so to speak, visible; it consists of the discontinuities of $F(x)$; $f_1(x), f_2(x), \dots$ being continuous. The remaining points of non-uniform convergence might be said to be invisible.

When we come to consider the case when $f_1(x), f_2(x), \dots$ are pointwise discontinuous as well as $F(x)$, the matter becomes slightly more complicated. We have, however, still two classes of points of non-uniform convergence: (1) visible ones, due to discontinuities of the functions involved; and (2) invisible ones, which have no particular connection with discontinuities.

Referring to the diagram of Fig. 2, P may be a discontinuity of the functions $R_n(x)$ for an infinite number of values of n , that is to say, a discontinuity of one or more of the functions $F(x), f_1(x), f_2(x), \dots$, or else the points P_i may be points of the corresponding loci, due to discontinuities of the successive functions $R_{n_i}(x)$. In this case P would be among the limiting points of the set formed by the discontinuities of all the functions $R_n(x)$. Such a point P would be a visible point of non-uniform convergence.

On the other hand, P may be a point of continuity of all the functions, and the points P_i may belong to what may almost be called connected parts of their corresponding loci, due to the fact that, P_i being a point of continuity, we can assign a definite interval containing P_i , throughout which the oscillation of the function R_{n_i} is as small as we please. We shall then have, as in the former case, an invisible point of non-uniform convergence.

The following example illustrates the theory of the invisible points in the case when the functions $f_1(x), f_2(x), \dots$, are discontinuous.

Ex. 3.—
$$s_n(x) = \frac{n^2 x + a}{1 + n^3 x^3}$$

at all points except the points $1/n, 2/n, 3/n, \dots, n-1/n$, and at all these points has the value zero.

$$F(0) = a, \quad F(x) = 0, \quad (0 < x \leq 1).$$

The origin is an invisible point of non-uniform convergence when a is zero, and a visible one when a is not zero. There are no other visible points of non-uniform convergence at all, the discontinuities of the functions $s_n(x)$ and their limiting points giving rise always to points of uniform convergence of the sum $F(x)$.

Ex. 4.—To illustrate the second type of visible points of non-uniform convergence consider the following series of functions:— f_1 is zero at all the points between 0 and 1 except the point $\frac{1}{2}$, at which it is 1; f_2 is -1 at the point $\frac{1}{2}$, 1 at $\frac{1}{4}$ and $\frac{3}{4}$, and zero at all other points; f_3 is -1 at $\frac{1}{4}$ and $\frac{3}{4}$, 1 at $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$, and zero at all the other points; and so on. Thus s_1 is zero at all points except $\frac{1}{2}$, where it is 1; s_2 is zero at all points except $\frac{1}{4}$ and $\frac{3}{4}$, where it is 1; s_3 is zero at all points except $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$, where it is 1; and so on. F is zero everywhere, and is therefore continuous.

Given any point, we can always assign an integer m such that, for all values of $n \geq m$, the value of $R_n(x)$ there is zero. Hence the series is always convergent. At all their points of continuity all the functions f_r vanish; hence there are no invisible points of non-uniform convergence. The series is, however, *non-uniformly convergent at every point of the continuum* (0, 1), since in the neighbourhood of every assigned point there are discontinuities, equal to 1, of an infinite number of the functions $s_n(x)$, and therefore, since F is continuous, of $R_n(x)$. The points of discontinuity are therefore all visible and belong to the limiting points of the discontinuities of all the functions $R_n(x)$, and are, in fact, identical with them.

8. Two things are to be noted with respect to the visible points of non-uniform convergence. The first is that such a point is not necessarily itself a discontinuity of any of the functions concerned; but, if not, it is necessarily a *limiting* point of discontinuities of the functions f_1, f_2, \dots . The other is that all the discontinuities of F are not necessarily points of non-uniform convergence; some might be due to discontinuities of one or more of the functions $f_1(x), f_2(x), \dots$. To take a simple example, we might have $F = u + f_1 + f_2 + \dots = u + s_n + R_n$ where u is zero everywhere

except at the point $\frac{1}{2}$, where it is 1, and s_n has the value $(n^2x+a)/(1+n^2x^2)$. Here the point $\frac{1}{2}$ is a discontinuity of F , but is not a point of non-uniform convergence. Similarly we see from Ex. 3 that *all* the limiting points of discontinuities of an infinite number of the functions $f_1(x), f_2(x), \dots$ are not of necessity points of non-uniform convergence, but only (*of necessity*) those where the value of each $R_n(x)$ at its particular discontinuity remains always greater than some positive quantity.

9. It is evident that there is nothing to prevent our writing the equation in the form

$$-f_r(x) = f_1(x) + \dots + f_{r-1}(x) - F(x) + f_{r+1}(x) + \dots,$$

and we see that points of discontinuity of $f_r(x)$ may be points of non-uniform convergence of the equation so written.

All the series so obtained have the same $R_n(x)$, for sufficiently large n , and this remains the case if we replace any finite number of the functions by any other at most pointwise discontinuous functions, in such a way that the equation holds. The points of non-uniform convergence are in all these cases the same. Thus we see that the discontinuities of the function $f_r(x)$, for any value of r , have just the same importance from this point of view as those of F itself. It is evident from the above that, in general, a discontinuity of a single function $f_r(x)$ will only cause a discontinuity of $F(x)$, and will not coincide with a point of non-uniform convergence. The two important classes of points to be investigated are those in the neighbourhood of which there are peaks at points of continuity of the successive residues $R_n(x)$, [it may happen casually that one of these points coincides with a discontinuity of one or more of the functions $f_r(x)$ or $F(x)$]; and those points which are limiting points of the discontinuities of an infinite number of the functions $f_r(x)$, such that the value of each $R_n(x)$ at its particular discontinuity is always greater than some positive quantity.

In the cases disposed of by Prof. Osgood and Dr. Hobson, this second class of points cannot come in, since at most one of the functions is discontinuous. Hence in these, and a variety of other cases, only the former class of points of non-uniform convergence can occur.

10. *A-points*.—The former class of points we subdivide into classes of points which we call *A-points*, the formal definition being as follows:—

P is said to be an A-point if, assigned any integer m and any interval δ , however small, containing P as internal point, we can find*

* In Osgood's case this is the same as what he calls a γ_A -point. In the general case all the invisible points of non-uniform convergence are *A-points*, but some of the *A-points* may be discontinuities of one or more of the functions F, f_1, f_2, \dots

inside δ a point x which is, at least on one side, a point of continuity of $R_n(x)$, at which $|R_n(x)| > A$, n being some integer $\geq m$.

That is to say, P is a point in the neighbourhood of which the locus of the points of continuity of $y = |R_n(x)|$ has a peak greater than or equal to A . These points being, as we know, dense everywhere, the corresponding figure is of the type given in Fig. 5.

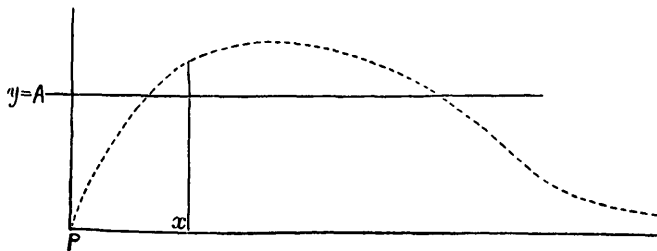


FIG. 5.

In the case disposed of by Prof. Osgood $R_n(x)$ is continuous, and therefore we do not need to postulate that the point x is a point of continuity. In Dr. Hobson's case $R_n(x)$ is not continuous, but the position of the discontinuities as well as their magnitude is the same for all values of n , depending only on $F(x)$. They will certainly be points of non-uniform convergence, but their presence does not seriously complicate affairs.

11. *The set G_A is closed and dense nowhere.*

The set composed of all the A -points, for assigned A , we denote by G_A .

These A -points necessarily form a closed set. For let T be any limiting point, and let us assign an integer m and an interval δ containing T as internal point. Then, as near as we please to T , we can find an A -point P ,

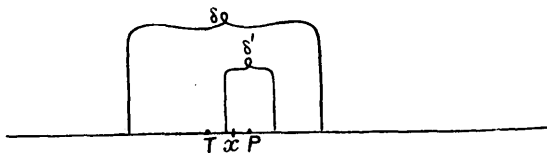


FIG. 6.

and we can assign an interval δ' , lying inside δ , and containing P as internal point. Inside δ' , and therefore inside δ , there is, since P is an A -point, a point x , and a corresponding integer $n \geq m$, such that x is a point of continuity (at least on one side), for $R_n(x)$, for which $|R_n(x)| > A$. Hence T also satisfies the definition of an A -point; so that the set G_A is closed.

To prove that G_A is dense nowhere, let us assume, if possible, the contrary, and let (C, D) be an interval in which, the closed set G_A being

dense everywhere, every point is an A -point. Then in the neighbourhood of any point we please of (C, D) we can assign a point x_1 and an integer n_1 , greater than any assigned m , such that x_1 is a point of continuity of R_{n_1} , at which $|R_{n_1}(x_1)| > A$. Since x_1 is a point of continuity of R_{n_1} , we can assign an interval (C_1, D_1) , having x_1 as internal point or as end-point, such that the oscillation of $R_{n_1}(x)$ in (C_1, D_1) is less than $\frac{A}{2}$.

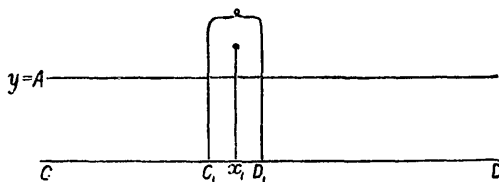


FIG. 7.

This being so, and the value of $|R_{n_1}(x_1)|$ being greater than A , $|R_{n_1}(x)| > \frac{A}{2}$ at every point of (C_1, D_1) .

We can now proceed with (C_1, D_1) as we did with (C, D) , since every point of (C_1, D_1) is an A -point, and determine an integer $n_2 > n_1$ and a point x_2 , such that $|R_{n_2}(x_2)| > A$, x_2 being a point of continuity of $|R_{n_2}(x)|$, and then we can determine an interval (C_2, D_2) interior to (C_1, D_1) , and containing x_2 , such that, for every value of x inside (C_2, D_2) , $|R_{n_2}(x)| > \frac{A}{2}$.

In this way we can determine a sequence of intervals (C, D) , (C_1, D_1) , ..., (C_r, D_r) , ..., and a sequence of integers $n_1, n_2, \dots, n_r, \dots$, such that, at every point of (C_r, D_r) ,

$$|R_{n_r}(x)| > \frac{A}{2}. \quad (1)$$

These intervals have, of course, at least one point ξ interior to all of them, and at that point the limit of $R_n(\xi)$ for n infinite is zero. Hence we can assign a definite integer m , such that

$$|R_n(\xi)| < \frac{A}{2}, \quad (2)$$

for all values of n after the assigned m , and therefore for all values $n = n_r$ greater than m . But this is in contradiction to (1). Hence, by a *reductio ad absurdum*, we are constrained to deny the possibility of finding an interval (C, D) consisting entirely of A -points, which proves the assertion that the set G_A is closed and dense nowhere.

12. *Distribution of Points of non-Uniform Convergence.*—It is evident, from what was pointed out at the end of § 9, that the points of non-

uniform convergence fall into two classes: (1) the A -points for all values of A , and (2) points which are limiting points of discontinuities of an *infinite* number of the functions f_r , including points, if any, which are themselves discontinuities of an *infinite* number of these functions. By what has been proved *the most general distribution of the points of the first of the above classes is when they form a set of the first category,* those for which A has any definite value† forming, like the discontinuities of a pointwise discontinuous function, a closed set, dense nowhere.*

The points of the second of the above classes may, however, as has been shown by an example (§ 7, Ex. 4), fill up the whole continuum.

It will be noticed that in Osgood's and in Hobson's cases the second class of points cannot occur; so that the distribution of the points of non-uniform convergence is strictly limited to form a set of the first category.

II. On Term-by-term Integration.

13. Let us now suppose that $F(x)$, $f_1(x)$, ... are all integrable functions. Then, since the sum of any finite number of integrable functions is integrable, s_n will be integrable for all values of n , and therefore $R_n(x) = F(x) - s_n(x)$ will be integrable also.

We shall now be able to prove that, *in any interval (C, D) in which, for all values of n greater than some assignable integer, $R_n(x)$ has a definite upper limit, the series may be integrated term by term.‡*

Thus, if in the interval considered there are (1) no points of infinite discontinuity of any of the functions, (2) no points in the neighbourhood of which the discontinuities of the functions become larger than any assignable quantity, and (3) no points of non-uniform convergence at which the measure of non-uniform convergence is infinite, we may certainly integrate term by term.

14. The following lemma is an immediate consequence of the definition of integration, and corresponds to the theorem that, when $y = f(x)$ is a graphic curve, $\int_C^D f(x) dx$ is the area bounded by the curve, the axis of x , and the ordinates through C and D .

* Baire, *loc. cit.*

† This may be expressed otherwise by saying: all for which the measure of non-uniform convergence is greater than or equal to a given quantity A , forming, &c.

‡ This result may be deduced as a special case of an important theorem due to Arzelà ("Sulle serie di funzioni," *Rend. di Bologna*, 1900) involving the generalization of the idea of uniform convergence.

LEMMA.—Of Integrals.

If $f(x)$ be ≥ 0 and finite throughout the closed interval (C, D) , and we draw any graphic continuous line whatever not cutting itself, so that no point $[x, f(x)]$ lies above it, the area bounded by that graphic line, the axis of x , and the ordinates through C and D is greater than or equal to $\int_C^D f(x) dx$.

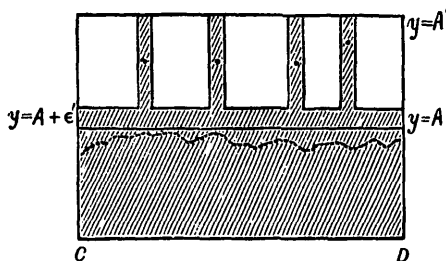


FIG. 8.

15. THEOREM 1.—If (C, D) be a closed interval of length L , entirely free of A -points (for assigned A), in which $|R_n(x)|$ has a definite upper limit A' , for all values of x from C to D , both inclusive, and for all values of n from and after an assignable integer, then we can find an integer m , so that, for all values of $n \geq m$,

$$\int_C^D |R_n(x)| dx < AL + \epsilon,$$

ϵ being as small as we please.

For, since there are no A -points in the closed interval (C, D) , we can assign an integer m , such that, for all values of $n \geq m$, the locus $y = |R_n(x)|$ only arises above the line $y = A$ at points of discontinuity of $R_n(x)$. If $A' \leq A$, the theorem now follows, by the Lemma; if not, we divide these discontinuities into two classes: (1) those at which the discontinuity is less than some small quantity ϵ' , and (2) those for which the discontinuity $\geq \epsilon'$.

Then all the points $[x, |R_n(x)|]$, except those at which the discontinuity is greater than or equal to ϵ' , lie below the line $y = A + \epsilon'$. Since, however, $R_n(x)$ is an integrable function, all its discontinuities $\geq \epsilon'$ form a closed set of content zero. Hence we can enclose all the remaining points in small rectangles, bounded by the lines $y = A + \epsilon'$, $y = A'$, the sum of whose breadths is $< \epsilon'$.

We have, in this way, enclosed all the points $[x, |R_n(x)|]$ below a broken line, and the area bounded by this line, the axis of x , and the ordinates through C and D , is less than $(A + \epsilon')L + (A' - A)\epsilon'$.

Now assigning an ϵ , however small, we can choose ϵ' so that

$$\epsilon' = \frac{\epsilon}{L + A' - A}.$$

We shall then have, by the lemma of integrals,

$$\int_C^D |R_n(x)| dx < AL + \epsilon.$$

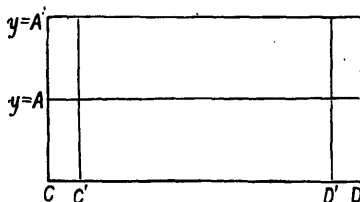


FIG. 9.

COROLLARY.—If (C, D) be as in Theorem 1, only C and D are themselves A -points, the theorem still holds. For we can cut off from each end a length ϵ' , and for the remaining interval (C', D') , we can, as in the proof of Theorem 1, determine m so that, for all values of $n \geq m$, all the points $[x, |R_n(x)|]$ are enclosed within an area $(A + \epsilon')L' + (A' - A)\epsilon'$, L' being the length of (C', D') . Adding on to this the area of the two small rectangles, of height A' on (C, C') and (D', D) as bases, we have included all the points $[x, |R_n(x)|]$ within an area less than $(A + \epsilon')L + 3A'\epsilon' - 5A\epsilon'$.

Choosing therefore $\epsilon' = \frac{\epsilon}{L + 3A' - 5A}$, the corollary is proved.

16. It will now be convenient to express the closed set G_A as the limit of a sequence of closed sets, as Prof. Osgood does in his paper. The sets of the sequence we denote by $G_{A,m}$, and define them in the following way:—

$G_{A,m}$ consists of all the A -points, such that, for all values of $n \geq m$, $|R_n(x)| \leq A$, and their limiting points.

The limiting points, if any, of $G_{A,m}$ which do not satisfy the above conditions are still A -points, since G_A is a closed set. They must therefore, by the above definition and that of an A -point, be discontinuities of $R_n(x)$ for an infinite series of values of n , and, if σ be the discontinuity of $R_n(x)$ at such a point Q , the value of $|R_n(x)|$ at Q will be less than or equal to $A + \sigma$, since there are certainly points in the neighbourhood below the line $y = A$. Hence:—

Those of the limiting points of the closed set $G_{A,m}$ for which $|R_n(x)| \geq A + \sigma$, where σ is any assigned positive quantity, and n any assigned integer $\geq m$, belong to a closed set of content zero.

Since at any particular point x we can always assign an integer m ,

such that, for all values of $n \geq m$, $|R_n(x)| < A$, it is evident that $G_{A,m}$ contains $G_{A,m-1}$, and the limiting set is G_A .

17. To these closed sets $G_{A,m}$ and their limiting set G_A , we propose to apply the Lemma of Content, proved in a paper published in the Society's *Proceedings* (Vol. xxxv., p. 269). The enunciation of the theorem is as follows :—

If G_1, G_2, \dots be a sequence of closed sets of points nowhere dense, and the limiting set G be closed also, and if we assign any small quantity σ , we can determine an integer r and a small quantity ϵ , so that, for all values of $n \geq r$, all the black intervals $\geq \epsilon$ of G_n are identical with all the black intervals $\geq \epsilon$ of G , and the sum of the remaining black intervals of $G_n < \sigma$.

18. THEOREM 2.—*If (C, D) be any closed interval of length L , in which $|R_n(x)|$ has a finite upper limit A' , for all values of x from C to D , both inclusive, and for all values of n from and after an assignable integer, we can determine an integer m , such that, for all values of $n \geq m$,*

$$\int_C^D |R_n(x)| dx < AL + \epsilon,$$

where A and ϵ are any assigned small positive quantities.

If $A' \leq A$, the theorem is an immediate consequence of the Lemma of Integrals. Let then $A' > A$.

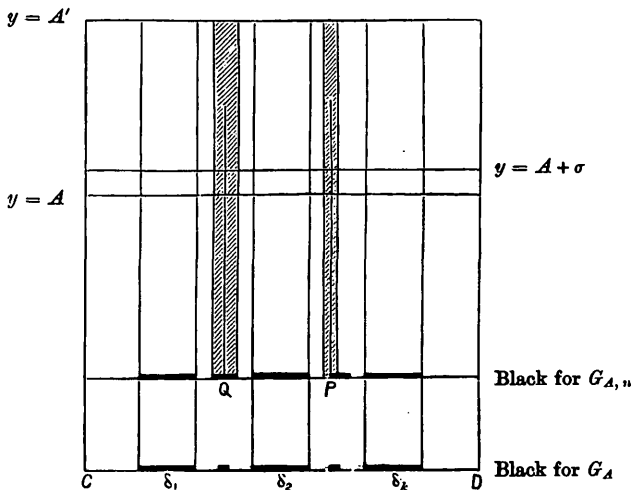


FIG. 10.

Then, by the Lemma of Content, we can determine an integer m' and a small quantity ϵ' , so that, for all values of $n \geq m'$, all the black intervals $\geq \epsilon'$ of $G_{A,n}$ are identical with all the black intervals $\geq \epsilon'$ of G_A , and the sum of the remaining black intervals of $G_{A,n} < \text{any assigned small } \sigma$.

Denoting the black intervals $> \epsilon'$ of G_A by $\delta_1, \delta_2, \dots, \delta_k$, it is an immediate deduction from the corollary to Theorem 1, that we can determine $m \geq m'$, so that, for all values of $n \geq m$, the part of the integral due to these k intervals will be less than $A(\delta_1 + \delta_2 + \dots + \delta_k) + \sigma$. In the remaining segments the locus $y = |R_n(x)|$ can only rise above any given line $y = A + \sigma$ at (1) a point P belonging to $G_{A, n}$, (2) a point Q in a black interval $< \epsilon'$ of $G_{A, n}$.

By § 16 we know that P must be a discontinuity $> \sigma$ of $R_n(x)$, and belongs therefore to a set of content zero. Hence these points $[x, |R_n(x)|]$ can all be enclosed in small rectangles of height A' , the sum of whose breadths is less than σ . The points Q can be enclosed in small rectangles of height A' , the sum of whose lengths, being not greater than that of all the black intervals $< \epsilon'$ of $G_{A, n}$, is certainly less than σ .

Thus the part of the integral due to the complementary segments of $\delta_1, \delta_2, \dots, \delta_k$ is less than $(A + \sigma)(L - \delta_1 - \delta_2 - \dots - \delta_k) + 2\sigma(A' - A)$; hence the whole integral is certainly less than $(A + \sigma)L + \sigma + 2\sigma(A' - A)$. Choosing now

$$\sigma = \frac{\epsilon}{L + 1 + 2(A' - A)},$$

the theorem follows.

19. The theorem enunciated in § 13 is an immediate result of Theorem 2. For, since $\int_C^D |R_n(x)| dx$ can be made as small as we please, by properly choosing A and ϵ , n being sufficiently large, therefore

$$\text{Lt}_{n=\infty} \int_C^D |R_n(x)| dx = 0;$$

a fortiori,
$$\text{Lt}_{n=\infty} \int_C^D R_n(x) dx = 0,$$

that is, since
$$\text{Lt}_{n=\infty} R_n(x) = 0,$$

$$\int_C^D \text{Lt}_{n=\infty} [R_n(x)] dx = \text{Lt}_{n=\infty} \int R_n(x) dx,$$

that is,
$$\int_C^D \text{Lt}_{n=\infty} [F(x) - s_n(x)] dx = \text{Lt}_{n=\infty} \int [F(x) - s_n(x)] dx,$$

or, since
$$\text{Lt}_{n=\infty} s_n(x) = F(x),$$

$$\int_C^D F(x) dx = \text{Lt}_{n=\infty} \int_C^D s_n(x) dx,$$

that is, in words: *The integral of the sum is the limit of the sum of the integrals of the separate terms.*