

ON INNER LIMITING SETS OF POINTS IN A LINEAR INTERVAL

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THE theory of inner limiting sets of points in a linear interval was suggested by a remark made by Borel,* that, if each rational point p/q , of the interval $(0, 1)$, be enclosed in an interval $\left(\frac{p}{q} - \lambda \frac{1}{q^3}, \frac{p}{q} + \lambda \frac{1}{q^3}\right)$, where λ is a positive number, the same for all the rational points, then, as λ is indefinitely diminished, there are, besides the rational points themselves, certain transcendental points (those of Liouville) which remain in the interior of the set of intervals, however small λ may become.

It appears, as in this example, that, if each point x of an open set is enclosed in a series of intervals $\delta_1(x), \delta_2(x), \dots, \delta_n(x), \dots$ chosen according to some prescribed law, such that, for each point x , $\delta_n(x)$ has a zero limit when n is increased indefinitely, the magnitude and position of $\delta_n(x)$ being assigned for each n and each x , then, in general, some or all of those limiting points of the given set which do not belong to that set remain in the interior of the intervals $\{\delta_n(x)\}$, however great n may be. This consideration led Dr. W. H. Young† to examine the properties of those sets of points for which it is possible, by a proper choice of the system of intervals, to exclude all the limiting points of the set which do not belong to it from the intervals, each such point ceasing to be interior to the set of intervals $\{\delta_n(x)\}$ for some definite value of n , so that the given set of points itself contains all points which are interior to the sets of intervals $\{\delta_n(x)\}$ for every value of n ; such a set Dr. Young has named an "inner limiting set" (*innere Grenzmenge*). He has proved that every such set has either the cardinal number a of the rational numbers, or else the cardinal number c of the continuum; this result will be assumed in the present paper. Dr. Young has also proved that the cardinal number of an inner limiting set is a when the set contains no component that is dense in itself, and is c when it does contain such a component.

In the present communication it is shown that the *necessary* and

* *Leçons sur la théorie des fonctions*, p. 44.

† "Zur Lehre der nicht abgeschlossenen Punktmengen," *Leipziger Berichte*, August, 1903.

sufficient condition that an enumerable set may be an inner limiting set is that it contains no component dense in itself. It is further shown that the most general inner limiting set which is unenumerable consists of a set dense in itself and each point of which is of degree c in the set, together with an enumerable set each point of which is of degree a or zero in the set, and which contains no component that is dense in itself.

The theory of inner limiting sets is closely connected with the theory of sets of the first and of the second category, which is of importance in the theory of functions of a real variable. The problem of determining the necessary and sufficient conditions that a given set of points is such that a pointwise discontinuous function can exist such that its points of discontinuity may be the points of the given set, is identical with the problem of determining the conditions under which a given set of points can be exhibited as the limit of a sequence of non-dense closed sets.

The set of points complementary to a set which has this property is an inner limiting set which is everywhere dense and is of cardinal number c in every interval contained in the domain of the set. It would be desirable that the necessary and sufficient conditions that such a set may be an inner limiting set should be obtained in a form which would be applicable as a criterion to the case of an everywhere dense set defined in any manner. This is equivalent to the determination of the necessary and sufficient conditions, in a direct form, that any given set may be of the second category; I have, however, not succeeded in doing this.

1. Let P denote a set of points contained in the linear interval (a, b) , and let a point x_1 of the set P be enclosed in a sequence of intervals $\delta_1(x_1), \delta_2(x_1), \dots, \delta_n(x_1), \dots$, such that each interval contains the next one, and each contains x_1 in its interior; and let the sequence be such that $\delta_n(x_1)$ converges to the limit zero as n is indefinitely increased. Suppose such a sequence of intervals to be prescribed for each point x of the set P , the length and position of $\delta_n(x)$ being assigned for each value of x and each value of n , subject to the conditions already stated: such a sequence of sets of intervals $\{\delta_1(x)\}, \{\delta_2(x)\}, \dots, \{\delta_n(x)\}, \dots$, we shall speak of as a "convergent sequence of sets" of intervals containing the set P . It may be possible to prescribe the law of formation of the intervals so that, if p be any fixed point whatever not belonging to the set P , there exists a number n_1 such that p is not interior to any interval of the set $\{\delta_{n_1}(x)\}$, and is thus exterior to or at an end-point of all the intervals of the set: if the set P is such that it is possible to determine a convergent sequence

of sets of intervals enclosing P , which has the property specified, then P is said to be an "inner limiting set."

If the set P is not an inner limiting set, then, however the intervals of the sequence are constructed, there exist points belonging to the derivative of P , and not to P itself, each of which is interior to some interval of the set $\{\delta_n(x)\}$, however great n may be. If p be such a point of P' , then, for any fixed x belonging to P , a value of n exists such that p is not interior to $\delta_n(x)$, but such values of n , when taken for every value of x in P , have no upper limit.

When a point p is not in the interior of the set $\{\delta_n(x)\}$ for the value n_1 of n , but is interior to $\{\delta_{n_1-1}(x)\}$, we shall say that p is *shed* from the sequence of sets of intervals at the index n_1 . An inner limiting set is then such that each limiting point of the set which does not belong to it is shed at a definite index from the sequence of sets of intervals, provided the intervals of this sequence are properly chosen.

2. If $P_1, P_2, P_3, \dots, P_n, \dots$ denote a sequence of non-dense closed sets of points, the set $M(P_1, P_2, \dots, P_n, \dots) = Q$, which contains every point that belongs to any of the given closed sets, and no points which do not belong to some one or more of the given sets, has been termed by Baire* a "set of the first category." If we denote the sets $P_1, M(P_1, P_2), M(P_1, P_2, P_3), \dots, M(P_1, P_2, \dots, P_n), \dots$ by $Q_1, Q_2, Q_3, \dots, Q_n, \dots$, then Q is the limit of the sequence $Q_1, Q_2, \dots, Q_n, \dots$ of closed sets each one of which contains the preceding one; it is in this form that a set Q of the first category appears as the set of points of discontinuity of a pointwise discontinuous function. The set which is complementary to a set of the first category is said to be "of the second category"; it is known that every set of points of the second category is such that in every interval contained in the interval (a, b) for which the set is defined, there exists a part of the set which has the cardinal number of the continuum.

If P is a set of the second category, the points of P can be enclosed in a set of intervals $\{\delta_n(x)\}$ which is such that those points which are not internal to any of the intervals of $\{\delta_n(x)\}$ form a non-dense closed set Q_n ; all the points which do not belong to any of the closed sets $Q_1, Q_2, \dots, Q_n, \dots$ are points of P . It thus appears that a set of the second category is an inner limiting set, and thus that an investigation of the nature and properties of inner limiting sets will throw light upon the nature of sets of

* *Annali di Mat.* [3], Vol. III., p. 67 (1899).

the second category, and will therefore have an influence upon the theory of pointwise discontinuous functions.

Every enumerable set $(p_1, p_2, \dots, p_n, \dots)$ of points forms a set of the first category; for it may be exhibited as the limit of a sequence $(p_1), (p_1, p_2), (p_1, p_2, p_3), \dots, (p_1, p_2, \dots, p_n), \dots$ of finite and therefore closed sets; the complementary set is therefore of the second category, and consequently an inner limiting set.

If P is an enumerable set, those of its limiting points which do not belong to P form a set $P' - D(P, P')$ which is an inner limiting set; this set is not identical with the set complementary to P , unless P is everywhere dense. To prove this we observe that the set $P' - D(P, P')$ has no limiting points which belong neither to itself nor to P ; hence, if a sequence of sets of intervals can be found enclosing the points of $P' - D(P, P')$ which shed each of the points of P at some definite index, the set $P' - D(P, P')$ is an inner limiting set; the set of intervals $\{\delta_1(x)\}$ enclosing the points of $P' - D(P, P')$ can be so chosen that the point p_1 is exterior to all of them; then $\{\delta_2(x)\}$ can be so chosen that p_2 is exterior to them, and so on; thus each point p_n of P is shed at a definite index, and the theorem is thus established.

The irrational points of $(0, 1)$ form an inner limiting set, but the rational points of $(0, 1)$ do not form an inner limiting set, since, as is well known, the irrational points do not form a set which can be exhibited as the limit of a sequence of non-dense closed sets.

It is easily seen that an inner limiting set remains one, if a finite number of points is added to or subtracted from the set. Also the sum of a finite number of inner limiting sets is itself an inner limiting set; this is not, in general, true of the sum of an indefinitely great number of limiting sets, as may be seen, for example, by considering the case in which each set consists of a single point.

3. It has been shown by Dr. W. H. Young (*loc. cit.*) that an inner limiting set which is enumerable has no component which is dense in itself: it will here be shown that in the case of an enumerable set P this condition is sufficient as well as necessary.

First, let us suppose that the derivative P' of P is also enumerable; in this case P cannot contain a component which is dense in itself; for the derivative of such a component would be perfect and would be a component of the closed set P' , which is impossible when P' is enumerable. Divide P into two parts P_1 and P_2 ; of these let P_1 consist of those points which are not limiting points of the set $P' - D(P, P')$, which

consists of those points of P' which do not belong to P ; the other part P_2 consists of those points which are limiting points of $P' - D(P, P')$. The set $P' - D(P, P')$ is such that those of its limiting points which do not belong to the set itself belong to P_2 ; hence, $P' - D(P, P')$ being enumerable, the set P_2 is, in accordance with what has been proved in § 2, an inner limiting set. Let the points of P_1 be $x_1, x_2, x_3, \dots, x_n, \dots$; then, since these points are not limiting points of $P' - D(P, P')$, each of the points x_n can be enclosed in an interval $\delta_1(x_n)$ which contains no points of $P' - D(P, P')$; it follows that the points of $P' - D(P, P')$, not being contained in properly chosen intervals enclosing P_1 , and being shed each at a definite index from a properly chosen sequence of sets of intervals enclosing P_2 , the set $P_1 + P_2$ or P is an inner limiting set.

A set P whose derivative P' is enumerable is such that a derivative $P^{(a)}$ must vanish, where a is some number of the first or second class; the set is therefore said to be "reducible"; we thus have the following :—

THEOREM I.—*Every reducible set is an inner limiting set.*

4. Next, let us suppose that P' , the derivative of the enumerable set P , has the power of the continuum. It can be shown that, if P' is continuous in any interval, P cannot be an inner limiting set; for suppose P' to be continuous in any interval (α, β) ; then the part of P which is in this interval is everywhere dense in (α, β) and is consequently also dense in itself. We can establish a (1, 1) correspondence between the points of P which are in (α, β) and the rational points of the interval $(0, 1)$, the two sets having necessarily the same order-type; the interval (α, β) can be so chosen that the points α, β are points of P and will correspond to the points 0, 1; the order of the points in $(0, 1)$ may be made the same as in (α, β) , the irrational points of $(0, 1)$ corresponding to those points of (α, β) which do not belong to P . To a convergent sequence of sets of intervals in (α, β) there will correspond a similar sequence in $(0, 1)$; now it has been shown in § 2 that the rational points in $(0, 1)$ do not form an inner limiting set; hence the part of P in (α, β) is not an inner limiting set, and therefore P itself is not an inner limiting set. The set P' , having thus been shown to be continuous in no interval, can be resolved into two parts G_1 and L_1 , of which G_1 is a non-dense perfect set, and L_1 is an enumerable set contained in the interiors of the intervals complementary to G_1 .

The set P may be divided into two parts P_1 and Q_1 , where P_1 consists of those points which are interior to the free intervals of G_1 , and Q_1 of those points which belong to G_1 ; it may happen that Q_1 does not exist.

We can show that P_1 is an inner limiting set whether Q_1 exists or not; for P_1 consists of a series of sets $P_{11}, P_{12}, \dots, P_{1n}, \dots$ interior to the free intervals $(a_1, b_1), \dots, (a_n, b_n), \dots$, which are complementary to G_1 ; the set P_{1n} interior to (a_n, b_n) has all its limiting points in that interval, and these since they belong to L_1 are enumerable, and therefore, by Theorem II., $P_{1,n}$ is an inner limiting set. The series of sets of intervals which enclose the points of $P_{1,n}$ may be chosen so that every interval of every set is interior to (a_n, b_n) ; thus no limiting points of P not belonging to P , except those belonging to $P'_{1,n}$, are ever interior to any interval of the sequence assigned to $P_{1,n}$; as this holds for every n , it follows that P_1 is an inner limiting set, and its points are such that they can be enclosed in a sequence of series of intervals which from the beginning contain no point of G_1 .

The set Q_1 consists of points which belong to G_1 , and therefore Q_1 has no limiting points in L_1 . If every point of G_1 is a limiting point of Q_1 , let the points of G_1 be made to correspond in order of magnitude to the points of the continuum $(0, 1)$; the irrational points of $(0, 1)$ correspond to those points of G_1 which do not belong to Q_1 ; the whole of a complementary interval of G_1 , including its two end-points, corresponds to a single point in $(0, 1)$. To an interval in $(0, 1)$ there corresponds an interval which contains in its interior an indefinitely great number of the intervals complementary to G_1 ; to a convergent sequence of sets in $(0, 1)$ there corresponds a convergent sequence. Since the set of rational points in $(0, 1)$ is dense in itself and not a limiting set, it follows that the set Q_1 is dense in itself and not a limiting set. It has thus been shown that, if Q_1 is dense in G_1 , it is not a limiting set, and is dense in itself; in that case P is not a limiting set.

If Q_1 is not identical with G_1 , let Q'_1 be resolved into an enumerable set L_2 and a perfect set G_2 ; the latter may be absent. The set Q_1 may then be resolved into a component P_2 contained in L_2 , and a component Q_2 contained in G_2 ; thus $P = P_1 + P_2 + Q_2$. The same argument applied to P_2 as was applied to P_1 shows that P_2 is an inner limiting set, and the intervals of the convergent sequence which encloses its points may be taken to be all interior to the intervals complementary to G_2 . The set Q_2 contained in G_2 may be treated as Q_1 in G_1 was treated, and we thus have $Q_2 = P_3 + Q_3$, where P_3 is an inner limiting set, and Q_3 is contained in a perfect set G_3 ; this will be the case unless Q_2 is dense in G_2 . Proceeding in this manner, it may happen that, for some integer n , Q_n does not exist, and then P is expressed as the sum of a finite number n of inner limiting sets, and is itself therefore an inner limiting set, and contains no component which is dense in itself; it may, however, happen that, for some number n , Q_n is dense in G_n ; in that case, as has been shown above, Q_n

is not an inner limiting set, and is dense in itself; P is in that case not an inner limiting set. If no integer n exists for which either of these things happens, we consider the set $M(P_1, P_2, \dots, P_n, \dots)$, where n has every integral value; it may happen that this set contains every point of P , but, if it does not contain every point of P , we take the set

$$P - M(P_1, P_2, \dots, P_n, \dots),$$

and resolve it as before into an inner limiting set P_ω and a set Q_ω contained in a perfect set G_ω ; we then proceed as before to resolve Q_ω into $P_{\omega+1}$ and a set $Q_{\omega+1}$ contained in a perfect set $G_{\omega+1}$. We proceed further in this manner, and may obtain sets with index any transfinite ordinal number of the second class; in the case of any such number β , we proceed just as in the case of a finite index, provided β is not a limiting number; if β is a limiting number, we obtain the resolution from the preceding numbers as in the case of ω .

We thus have P resolved into $P_1 + P_2 + \dots + P_\omega + P_{\omega+1} + \dots + P_\beta + Q_\beta$, where β is a non-limiting ordinal number of the first or second class, or else P is resolved into $P_1 + P_2 + \dots + P_\omega + \dots + P_\beta + \dots$ with no last term. Since P is enumerable, this process must come to an end at or before some definite number α of the first or second class; the end of the process comes either (1) when Q_α is dense in G_α , in which case, as we have seen, Q_α is dense in itself and is not an inner limiting set, and thus P is also not an inner limiting set, or (2) when there is no component Q_α in G_α , or (3) when there is no G_α . It has thus been shown that, for P to be an inner limiting set, it is necessary that it should contain no component which is dense in itself, and that, when this condition is satisfied, P is the sum of a finite or infinite number of inner limiting sets, of which there may or may not be a last set.

Let us now assume that P contains no component that is dense in itself; it can then be shown that P must be an inner limiting set. Let P_γ be any one of the components into which P has been resolved, γ denoting an ordinal number of the first or second class; we fix a convergent sequence of sets of intervals enclosing the points of P_γ such that all the intervals are interior to the intervals complementary to G_γ ; the set $P_{\gamma+1} + P_{\gamma+2} + \dots$, which is contained in G_γ , has no limiting points in any of the intervals which enclose the points of P_γ , for all its limiting points must be in G_γ . The convergent sequence of sets of intervals having been determined in the manner described for every P_γ , we can now show that each limiting point p of P which does not belong to P is shed from the whole convergent sequence of sets of intervals, at a definite index. The point p is either a limiting point of P_1 or is contained in G_1 ; in the former case it is shed

from the intervals enclosing P , at a definite index, and, not being a limiting point of $P_2 + P_3 + \dots$, is shed from the intervals enclosing the points of that set at a definite index; consequently it is shed from the intervals enclosing P at a definite index, the greater of the two former indices; in the latter case, unless p is in G_2 or in P'_2 , it is not a limiting point of $P_2 + P_3 + \dots$, and never comes into any of the intervals enclosing the points of P_1 ; it is therefore shed at a definite index. If p belongs to G_1, G_2, \dots and to every G before G_α , but is not in G_α , it may be a point of P'_α ; in that case it is not a limiting point of the set $P_{\alpha+1} + P_{\alpha+2} + \dots$, and does not come into the interior of any of the intervals which enclose the points of P_1, P_2, \dots , or any P with index less than α ; it is therefore shed at a definite index, from the sequence of sets of intervals enclosing the points of P . It has thus been shown that every limiting point p of P which does not belong to P is shed at a definite index from the convergent sequence of sets of intervals determined in the manner described above. We have thus—

THEOREM II.—*The necessary and sufficient condition that an enumerable set P may be an inner limiting set is that it contains no component which is dense in itself.*

A corollary from the above proof is that every enumerable set is expressible as the sum of an inner limiting set, and of a set which is dense in itself.

A proof of Theorem II. could no doubt be constructed* which would be independent of the use of transfinite numbers; in many cases, theorems which were first obtained by the employment of transfinite numbers have afterwards been proved by methods which do not involve the use of such numbers. This fact illustrates the high value of the theory of these numbers as an instrument of research; they form the natural language to be employed in dealing with non-finite systems.

5. We proceed to consider the case of unenumerable sets. It has been shown† by G. Cantor that every set of points P can be analysed into four components U, V_α, V_x, V_c which are of the following character:—

* Since this paper was written, a paper has been published by Dr. W. H. Young, "On Sequences of Sets of Intervals containing a given Set of Points," *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 262, which contains the following theorem:—"If E contains no component dense in itself, while E' is more than countable, the inner limiting set may be either countably infinite or have the potency c : and we can arrange the intervals so that the inner limiting set consists of E alone." This theorem contains that part of Theorem II. of the present paper which arises when P' is unenumerable, i.e., the second case in the above proof; Dr. Young's proof depends upon the resolution of a set into adherences and coherences.

† See *Acta Mathematica*, Vol. VII.; also Schönflies, *Berichte über die Mengenlehre*, pp. 71-73.

The set U is enumerable and contains no component that is dense in itself; each point of U is of degree zero or of degree a in P , *i.e.*, in a sufficiently small neighbourhood of such a point, there are either no other points of P , or else only an enumerable set of such points.

The set V_a is enumerable and dense in itself, and each point is of degree a in P . The set V_c has the cardinal number c of the continuum, is dense in itself, and each point is of degree c in P , *i.e.*, any neighbourhood, however small, of such a point contains a set of points of P of which the cardinal number is c . The set V_x , if it exists, is dense in itself, and the points of it are all of degree in P , intermediate between a and c ; it is not yet definitely known whether any such set as V_x exists, but it is regarded by Cantor as probable that it cannot.

The expression "degree of a point in a set" is here used as equivalent to what has been denominated by Cantor* the *Mächtigungsgrad* of the point; an isolated point, *i.e.*, one which is not a limiting point, is of degree zero in the set; a point of any degree λ is one such that in a sufficiently small neighbourhood and in all smaller neighbourhoods of the point the cardinal number of the part of the set is λ .

Taking $P = U + V_a + V_x + V_c$, where one or more of the four components of P may be absent, we observe that, if V_c be absent, the necessary and sufficient conditions that P may be an inner limiting set are (1) that $V_x = 0$; this follows from Dr. Young's theorem that every limiting set has either a or c for its cardinal number, and (2) $V_a = 0$, as has been shown in Theorem II., above.

If V_c be present, we observe that no point of $U + V_a + V_x$ can be a limiting point of V_c , for any limiting point of V_c must be of degree c in the set P . If V_c is everywhere dense in (a, b) , it follows that $U + V_a + V_x$ is absent, or $P = V_c$. The set V_c may be non-dense in (a, b) , or it may be dense in some parts of (a, b) and non-dense in other parts.

It will be shown that V_c is in general made up of a part which is non-dense in (a, b) and of a finite or indefinitely great number of parts each of which is everywhere dense in the interval in which it lies. Suppose an interval (a, β) can be found in which V_c is everywhere dense; let x be any point in (a, b) such that $x \geq \beta$; then those values of x for which V_c is everywhere dense in (a, x) , and those values of x for which V_c is not everywhere dense in (a, x) , define a section of all the numbers of the continuum (β, b) ; this section defines a number $\beta_1 \geq \beta$. Similarly we may find a

* Dr. W. H. Young has given an investigation of Cantor's theory of adherences and coherences (see *Quarterly Journal* for 1903), which does not involve the use of transfinite ordinal numbers.

number $\alpha_1 \leq a$, so that (α_1, β_1) is the greatest interval which can be obtained to contain (α, β) and such that V_c is everywhere dense in it. If in the parts of (a, b) outside (α_1, β_1) the set V_c is anywhere dense, we proceed to fix the greatest interval for which it is everywhere dense. If we proceed in this manner, we obtain a finite or enumerably infinite set of detached intervals contained in (a, b) , in each of which V_c is everywhere dense; the remainder of (a, b) may consist of a set of detached intervals, and of a set of points; in this remainder the points of V_c form a non-dense set.

No point of $U + V_a + V_x$ can be in an interval (α_1, β_1) in which V_c is everywhere dense; if \bar{V}_c is the part of V_c which is non-dense, every point of $U + V_a + V_x$ must lie in one of the intervals complementary to the perfect set \bar{V}_c ; it is to be observed that in \bar{V}_c we include the end-points of intervals (α_1, β_1) , in case those end-points belong to V_c . In order that P may be an inner limiting set it is necessary that the part of $U + V_a + V_x$ which is in each interval complementary to \bar{V}_c should be an inner limiting set, and this cannot be the case unless $V_a = 0$ and $V_x = 0$. We have thus proved

THEOREM III.—*In order that a set of points may be an inner limiting set it is necessary that the set should contain no points whose degrees in the set are other than 0, a , or c , and should contain no component which is dense in itself and of which the points are of degree a in the set.*

6. The determination of the necessary and sufficient conditions that any given set of points may be an inner limiting set has now been reduced to the problem of determining the requisite criteria in the case of a set which is dense in itself and all the points of which are of degree c in the set. The case in which such a set is non-dense may be reduced by the method of correspondence to the case in which such a set is everywhere dense; thus the question is reducible to that of the determination of the conditions under which a given everywhere dense set of points of degree c in the set may be a set of the second category.

It is possible to find sets which are everywhere dense and of cardinal number c in every interval, which are of the first category. As an example, let a perfect set G_1 , of which $\theta(b-a)$ is the greatest complementary interval, be constructed in (a, b) ; in each of the complementary intervals of G_1 place a set similar to G_1 ; we then have a new perfect set G_2 , of which the greatest complementary interval is $\theta^2(b-a)$; proceeding in this way, we form a sequence $\{G_n\}$ of perfect sets such that $\theta^n(b-a)$ is the greatest complementary interval of G_n . The limit G of G_n is a

set of the first category; in every interval contained in (a, b) there is, for a sufficiently great value of n , some interval complementary to G_n , lying inside the interval; therefore G is everywhere dense and has points of cardinal number c in every interval.

It thus appears that there exist sets which are everywhere dense and are of cardinal number c in every interval in (a, b) , which are of the first category, and there are others which are of the second category. The question has been raised by Schönflies* whether every such set is necessarily either of the first or else of the second category. This question must certainly be answered in the negative, for, if we divide (a, b) into any finite number of parts, and place in some of them sets of the type indicated which are of the first category, and in the other parts sets which are of the second category, it is clear that the whole set so constituted cannot be either of the first or of the second category.

It is well known that the points which are common to two sets of the second category form a set which is also of the second category. This is true also for non-dense sets contained in a perfect set which are of the second category relatively to the perfect set, as may be seen by the method of correspondence. If we use the results of § 5, we can deduce that the set of points which is the common part of any two inner limiting sets is also an inner limiting set. The theorem is, on using Theorem II., clearly true for inner limiting sets which are enumerable, and the analysis which has been given of limiting sets in general shows that it is true for such limiting sets as are of cardinal number c .

* *Bericht über die Mengenlehre*, p. 110.