Small Oscillations to any Degree of Approximation.

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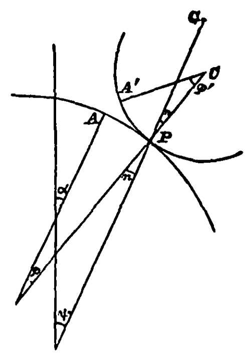
In discussing the small oscillations of a dynamical system, we are usually content to reject the squares of all small quantities. It is clear, however, that in many cases the terms of the higher orders may rise in importance, and may even alter the period of the principal oscillation. It is proposed to investigate an easy method of determining the oscillation of a heavy body moving in any manner with *one independent motion* to any degree of approximation. This we shall do, first, for any small oscillation in two dimensions; and, secondly, for an oscillation in three dimensions about a fixed point.

Let P be the instantaneous centre of rotation at the time t, AP its path in space, A′P in the body. Then the motion may be constructed by making the curve A′P, fixed in the body, roll without sliding on the curve AP fixed in space. Let AP = A′P = s. Let G be the centre of gravity of the body GP = r. Let ψ be the angle GP makes with the vertical, n the angle it makes with the normal, to the curves AP, A′P at P. Let θ be the angle any straight line fixed in the body makes with a straight line fixed in space, κ the radius of gyration of the body about the centre of gravity.



Taking moments in the usual way about the instantaneous centre of rotation considered as a moving point, the equation of motion is

$$(r^2+k^2)\frac{d^2\theta}{dt^2}+rz\sin n\left(\frac{d\theta}{dt}\right)^2=gr\sin\psi.$$

The method of proceeding is as follows:—To solve the equation, we must expand each coefficient by Taylor's theorem in powers of θ, which is to be so chosen as to vanish in the position of equilibrium. To do this we require the successive differentials of the coefficients to any order expressed in terms of the *initial* values only of ψ, n, and r. We find by inspection of the figure

$$\frac{d\psi}{d\theta}=1-z\frac{\cos n}{r},$$

$$\frac{dn}{d\theta}=z\left(\frac{\cos n}{r}-\frac{1}{\rho'}\right),$$

$$\frac{dr}{d\theta}=z\sin n,$$

$$\frac{1}{z}=\frac{1}{\rho}+\frac{1}{\rho'},\quad z=\frac{ds}{d\theta},$$

where ρ' is the radius of curvature of A′P, and ρ of AP at P, and are both supposed to be given functions of s. All that is necessary is to differentiate each coefficient in the differential equation, and to substitute in the result from these equations the values of $\frac{d\psi}{d\theta}$, $\frac{dn}{d\theta}$, $\frac{dr}{d\theta}$, as they enter. In equilibrium, GP is vertical; hence the initial value of ψ is zero. Let the initial or equilibrium values of n and r be a and h. These equations we shall refer to as "the subsidiary equations."

To solve the equation to the first order.—We have

$$(h^2+k^2)\frac{d^2\theta}{dt^2}=gr\sin\psi.$$

We have only to calculate $r\sin\psi$ to the first power of θ,

$$\begin{aligned} r\sin\psi &= \frac{d}{d\theta}(r\sin\psi)\,\theta \\ &= \left\{\sin n\sin\psi + r\cos\psi\left(\frac{1}{z}-\frac{\cos n}{r}\right)\right\}z\theta \\ &= \{r\cos\psi - z\cos(\psi-n)\}\,\theta. \end{aligned}$$

Hence the equation of motion is

$$\frac{d^2\theta}{dt^2}+\frac{z\cos a-h}{h^2+k^2}\theta=0.$$

To solve the equation to the second order.—We have just found

$$\frac{d}{d\theta}(r\sin\psi)=r\cos\psi-z\cos(\psi-n).$$

We must differentiate this, and retain only the terms which do not vanish when $\psi=0$. We get

$$\frac{d^2}{d\theta^2}(r\sin\psi)=z^2\left\{z\cos a\frac{d\frac{1}{z}}{ds}+\frac{\sin 2a}{h}-\frac{\sin a}{\rho'}\right\}.$$

Hence the equation of motion to the second order is

$$\{k^2+h^2+2hz\sin a\theta\}\frac{d^2\theta}{dt}+hz\sin a\left(\frac{d\theta}{dt}\right)^2$$

$$=-(z\cos a-h)\,g\theta+gz^2\left\{z\cos a\frac{d\frac{1}{z}}{ds}+\frac{\sin 2a}{h}-\frac{\sin a}{\rho'}\right\}\frac{\theta^2}{2}.$$

This is the same as $\frac{d^2\theta}{dt^2}+a^2\theta=-b^2\left(\frac{d\theta}{dt}\right)^2+c\theta^2,$

where
$$a^2=\frac{z\cos a-h}{k^2+h^2}g,$$

$$b^2=\frac{hz\sin a}{k^2+h^2},$$

$$c = 2a^2b^2 + \tfrac{1}{2}g\frac{z^2}{k^2+h^2}\left\{z\cos\alpha\,\frac{d\frac{1}{z}}{ds} + \frac{\sin 2\alpha}{h} - \frac{\sin\alpha}{\rho'}\right\}.$$

Supposing a not to vanish, we find

$$\theta = \mathrm{A}\sin(at+\mathrm{B}) + \frac{c-a^2b^2}{2a^2}\mathrm{A}^2 + \frac{c+a^2b^2}{6a^2}\mathrm{A}^2\cos 2(at+\mathrm{B}),$$

so that the first approximation is substantially correct unless a be small, *i.e.* unless the equilibrium be nearly neutral. The effect of the small terms is to make the extent of the oscillation on the lower side of the position of equilibrium greater than that on the upper side.

To solve the equation to the third order.—We have now to differentiate again the expressions obtained in the last approximation. The process is very easy, but the result is long. If we suppose that $n=0$ in the position of equilibrium, the equation takes the form

$$\frac{d^2\theta}{dt^2} + a^2\theta = -e^2\theta\left(\frac{d\theta}{dt}\right)^2 + f\theta^2 + f'\theta^3,$$

where e^2, f, f' are functions of h, ρ, ρ' obtained by the above process. It is obvious that one effect of these additional terms is to alter the period of the oscillation. The principal term is now

$$\theta = \mathrm{A}\sin(a't+\mathrm{B}),$$

where
$$a'^2 = g\frac{z-h}{h^2+k^2} + ghz^2\left(\frac{1}{h} - \frac{1}{\rho'}\right)\frac{z-h}{(h^2+k^2)^2}\tfrac{5}{4}\mathrm{A}^2$$
$$-g\left\{2z - h - \frac{z^2}{\rho'} + z\left(1 - \frac{2z}{h} + \frac{z}{\rho'}\right)^2 - \frac{z}{2}\frac{d^2z^2}{ds^2}\right\}\frac{1}{h^2+k^2}\cdot\frac{\mathrm{A}^2}{8}.$$

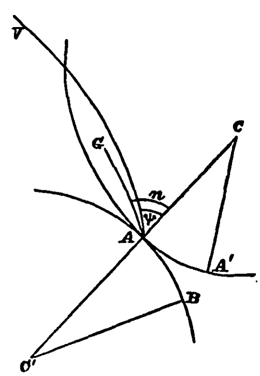
When other forces besides gravity act on the body, the problem may be treated in the same way. Thus, suppose the body acted on by a central force which passes through the centre of gravity and some point fixed in space. A very slight modification of the subsidiary equations already written down will enable us to determine the oscillations to any order.

When the oscillation takes place in three dimensions about a fixed point with only one independent motion, we may represent the geometrical constraints by supposing the body to be attached to a cone without inertia which is constrained to roll on another cone fixed in space. These cones are the surfaces generated in the body and in space by the instantaneous axis. The problem is now reduced to the following simpler form:

To determine to any degree of approximation the oscillations of a heavy conical body on a fixed rough conical surface, the vertices being coincident.

Let I be the moment of inertia of the cone about the instantaneous axis, Ω the angular velocity of the body, N the moment of the forces about the instantaneous axis. Then, by differentiating the equation of Vis Viva, we have $$\mathrm{I}\frac{d\Omega}{dt}+\tfrac{1}{2}\frac{d\mathrm{I}}{dt}\Omega=\mathrm{N}.$$

Let the vertex O of the cones be the centre of a sphere of unit radius. Let the instantaneous axis cut this sphere in A, the line joining O to the centre of gravity cut it in G, and the vertical OV cut it in V. Let OC, OC′ be axes of right circular cones osculating the given cones along the instantaneous axis. Let OA′ be the generating line of the moving cone which will coincide with OB and be the instantaneous axis at time $t+dt$. Let the arcs CA, C′A be ρ, ρ'. Let the angle AOA′ be ds; then we easily find $\Omega=\dfrac{\sin(\rho+\rho')}{\sin\rho\sin\rho'}\dfrac{ds}{dt}$.



Let h be the distance of the centre of gravity from the vertex. Let the arcs $\mathrm{GA}=r$, $\mathrm{VA}=z$, $\angle\,\mathrm{GAC}=n$, $\angle\,\mathrm{VAC}=\psi$. Then we have $\mathrm{N}=-gh\sin r\sin z\sin(n-\psi)$. In equilibrium this vanishes when $n=\psi$.

To form the equation of motion to any degree of approximation, we must expand this in powers of s by Taylor's theorem. To accomplish this, we must, as before, find $\dfrac{dr}{ds}, \dfrac{dz}{ds}$, &c., in terms of r, n, &c., so that each differential coefficient may be expressed only in terms of the initial values of the quantities.

These subsidiary equations are easily seen to be

$$\frac{dr}{ds}=\sin n,\qquad \frac{dz}{ds}=\sin\psi,$$
$$\frac{dn}{ds}=\cot r\cos n-\cot\rho,$$
$$\frac{d\psi}{ds}=\cot z\cos\psi+\cot\rho',$$

where $\mathrm{CA}=\rho$, $\mathrm{C'A}=\rho'$.

To solve the equation to the first order.—The equation is

$$\mathrm{I}\frac{\sin(\rho+\rho')}{\sin\rho\sin\rho'}\frac{d^2s}{dt^2}=-gh\sin r\sin z\sin(n-\psi).$$

To expand the right side in powers of s, we differentiate it and substitute from the above subsidiary equations for all differential coefficients as they arise. In equilibrium $n=\psi$; hence, as we have to substitute for all the letters their initial values, we need only differentiate the last term. We get therefore

$$\mathrm{N}=-gh\sin r\sin z\cos(n-\psi)\{(\cot r-\cot z)\cos n-(\cot\rho+\cot\rho')\}\,s.$$

Hence $I\frac{d^2s}{dt^2} = -gh\left\{\sin(z-r)\cos n\frac{\sin\rho\sin\rho'}{\sin(\rho+\rho')} - \sin r\sin z\right\}s$;

therefore, if L be the length of the equivalent pendulum,

$$\frac{I}{hL} = \sin(z-r)\cos n\frac{\sin\rho\sin\rho'}{\sin(\rho+\rho')} - \sin r\sin z.$$

For the next approximation, let us suppose the rolling body to be a right cone on the summit of another right cone. We then require to include the *cubes* of small quantities. By an easy process of differentiation we find the equation becomes

$$\frac{I}{gh}\frac{d^2s}{dt^2} = -\frac{\sin^2\rho\sin z\sin(\rho'+z)}{\sin(\rho+\rho')}s + P\cdot\frac{\sin\rho\sin(\rho'+z)}{\sin^2 z\sin^2\rho'\sin(\rho+\rho')}\cdot\frac{s^3}{6},$$

where $P = \sin\rho\sin^2(\rho'+z) + \sin\rho\sin^2\rho' - 3\sin(\rho'+z)\cos\rho\sin\rho'\sin z$.

The effect of this latter term on the period of the principal term may be easily found.

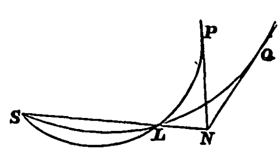
Inversion, with special reference to the Inversion of an Anchor Ring or Torus. By H. M. Taylor, M.A.

[*Read April 9th,* 1874.]

We premise that a straight line inverts into a circle passing through the pole, and *vice versâ;* that a circle inverts into a circle, the two circles being subcontrary sections of a cone of the second degree passing through the pole; and that the angles between lines and surfaces at their points of intersection are the same as the angles between the inverse lines and surfaces at the inverse points.

A normal is a straight line cutting a curve or a surface at right angles; it will therefore invert into a circle through the pole cutting the inverse curve or surface at the inverse point at right angles. Such a circle we will call a normal circle.

We will now prove that, if two normals at any two points of a surface intersect and be equal, the normals at the inverse points of the inverse surface also will intersect and be equal.



Let S be the pole, and let PN, QN, the normals to a surface at P, Q, be equal and intersect in N.

If we draw a circle to touch PN at P and pass through S, this will cut SN in a point L such that $NL\cdot NS = NP^2$; and because NP, NQ are equal, the circle touching QN at Q and passing through S will pass