"On the Mathematical Theory of Combined Streams." By W. J. Macquorn Rankine, C.E., LL.D., F.R.SS. Lond. and Edin.

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1. *Object of this Investigation.*—The principles of the action of combined streams were to a certain extent investigated by Venturi, and stated in his essay 'Sur la Communication latérale du Mouvement dans les Fluides' (Paris, 1798). The principle of the conservation of momentum, so far as I know, was first explicitly applied to combined streams by Mr. William Froude, F.R.S., in a paper on Giffard's Injector, read to the British Association at Oxford, in 1860, and published in the Transactions of the Sections, p. 211. Various other authors have treated the same problem by different methods, based virtually on the same principle. A very complete and precise investigation of the theory of combined streams, in every case in which two streams only are combined, is contained in Professor Zeuner's treatise 'Das Locomotivenblasrohr" (Zürich, 1863), The theoretical conclusions are tested by comparison with experiment, and applied to practical questions, especially those relating to the apparatus from which the treatise takes its name. The object of the present investigation is to apply similar principles to the combination of any number of streams; and the demonstration of the fundamental dynamic equation differs from that given by Zeuner in method, though not in principle, being effected at one operation by the direct application of the principle of the equality of impulse and momentum, instead of by the consideration of the loss of energy that takes place during the combination of the streams.

2. *Terms and Notation used, and Suppositions made.*—The several streams which are combined will be called before their junction, the *component streams*; the stream formed by their combination will be called the *resultant stream*. The passages through which the component and resultant streams flow will be called respectively the *supply-tubes* and the *discharge-tube*. The combination of the streams will be supposed to take place in a short cylindrical chamber, with its axis parallel to the direction of flow, which will be called the *junction-chamber*.

At one end of the junction-chamber are the outlets of the supply-tubes, which will be called the *nozzles*; at the other end, the inlet of the discharge-tube, which will be called the *throat*. It will be supposed, further, that the supply-tubes are so formed as to direct the component streams at the nozzles, so that they shall all flow sensibly parallel to each other and to the resultant stream. The principal symbols used are as follows: for any one of the component streams:—

a, area of nozzle;

v, velocity of flow at nozzle;

s_0, bulkiness, or reciprocal of density at nozzle.

The several component streams may be distinguished from each other, when required, by suffixes; as 1, 2, 3, &c.

For the resultant stream:

A, area of throat;

V, velocity of flow at throat;

S_0, bulkiness, or reciprocal of density at throat.

Intensities of pressure, in *absolute units* on the unit of area:

p_0, at the nozzle end of junction-chamber;

P_0, at the throat.

(These may be converted into *units of weight* on the unit of area, by dividing by g).

The flow of each stream is supposed to be steady. The fluids may be either liquid, vaporous, gaseous, or mixed.

3. *Equation of Continuity.*—The mass of fluid that enters the junction-chamber through a given nozzle in a unit of time is $\frac{av}{s_0}$. The mass discharged in the same time at the throat is $\frac{AV}{S_0}$. The flow being steady, the following equation must at every instant be fulfilled:

$$\frac{AV}{S_0} = \Sigma \cdot \frac{av}{s_0}. \quad \quad (1)$$

If S_0 and the several values of s_0 are given, that equation gives the velocity of the resultant stream in terms of those of the component streams; viz.

$$V = \frac{S_0}{A} \cdot \Sigma \cdot \frac{av}{s_0}. \quad \quad (1\,A)$$

If all the fluids are liquids, each of sensibly invariable bulkiness, we have also $AV = \Sigma \cdot av$; that is, the volume of flow of the resultant stream is equal to the aggregate of the volumes of flow of the component streams; but if any or all of the streams are vaporous or gaseous, the values of s_0 will depend upon that of p_0, and the value of S_0 upon that of P_0, and upon the changes of bulkiness of the fluids which may take place in the junction-chamber, through change of temperature, change of condition, or chemical action.

In any case S_0 may be regarded as a given function of P_0, and of the mutual proportions of the several values of $\frac{av}{s_0}$; in other words, of the ingredients in the resultant stream.

4. *Dynamical Equation.*—The aggregate momentum of the mass of fluid that enters the junction-chamber through the nozzles in a unit of time is $\Sigma \cdot \frac{av^2}{s_0}$. The momentum of the equal mass which leaves the junction-chamber through the throat in the same time is $\frac{AV^2}{S_0}$.

The forward impulse exerted in a unit of time upon the mass of fluid in the junction-chamber by the pressure at the nozzle end of the chamber is $p_0 A$. The backward impulse exerted in the same time on the same mass by the pressure at the throat-end of the chamber is $P_0 A$. By the second law of motion, the difference between those impulses is equal to the change of momentum produced; that is to say,

$$\mathrm{A}(\mathrm{P}_0-p_0)=\Sigma\frac{av^2}{s_0}-\frac{\mathrm{AV}^2}{\mathrm{S}_0}=\Sigma\left\{\frac{av}{s_0}(v-\mathrm{V})\right\}; \quad . \quad . \quad . \quad . \quad (2)$$

or dividing both sides by A,

$$\mathrm{P}_0-p_0=\Sigma\frac{av^2}{\mathrm{A}s_0}-\frac{\mathrm{V}^2}{\mathrm{S}_0}=\Sigma\left\{\frac{av}{\mathrm{A}s_0}(v-\mathrm{V})\right\}. \quad . \quad . \quad . \quad (2\mathrm{A})$$

And this is *the general dynamical equation of the combination of any number of streams of any fluids.*

If the preceding equation, as applied to a combination of two streams only, be compared with the equation not numbered, which immediately precedes equation 60 in Zeuner's treatise, it will be seen that they are virtually identical, although different in form, and demonstrated by different methods.

5. *Loss of Energy at Junction.*—If a given mass of any fluid at the bulkiness s and pressure p is contained in a reservoir, from which it is capable of being expelled by the inward motion of a piston loaded with an external force equivalent to the pressure, it is known that the potential energy of the mass of fluid and of the piston relatively to a point at the level of the centre of mass of the fluid is expressed by multiplying the mass by $\int_0^p sdp$, the relation between s and p being that which is called *adiabatic*; that is to say, such that no heat is received or given out by the fluid. Hence the loss of energy in the junction-chamber in each unit of time is given by the following expression:—

$$\Sigma\left\{\frac{av}{s_0}\left(\frac{v^2}{2}+\int_0^{p_0}s\,dp\right)\right\}-\frac{\mathrm{AV}}{\mathrm{S}_0}\left(\frac{\mathrm{V}^2}{2}+\int_0^{\mathrm{P}_0}\mathrm{S}\,d\mathrm{P}\right), \quad . \quad . \quad . \quad (3)$$

of which the first, or positive term, denotes the aggregate energy, actual and potential, of the component streams as they enter the junction-chamber; and the second, or negative term, expresses the total energy, actual and potential, of the resultant stream as it leaves that chamber. That lost energy takes the form partly of visible eddies and partly of invisible molecular motions—that is, of heat.

The integral expressing the aggregate potential energy of the component streams may be put in the following form:—

$$\int_0^{p_0}\left(\Sigma.\frac{avs}{s_0}\right)dp. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\mathrm{A})$$

If no change of total bulkiness arises from the mixture of the component streams, the volume occupied by a given mass of the mixture is simply the sum of the volumes of its ingredients; so that we have

$$\frac{\mathrm{AVS}}{\mathrm{S}_0}=\Sigma\frac{avs}{s_0}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\mathrm{B})$$

and the expression for the loss of energy becomes

$$\Sigma \cdot \frac{av^3}{2s_0} - \frac{AV^3}{2S_0} - \frac{AV}{S_0}\int_{p_0}^{P_0} S\,dP. \quad . \quad . \quad . \quad . \quad . \quad (3\,C)$$

When the fluids are all liquids, whose compressibility may be neglected, we have $\int_{p_0}^{P_0} S\,dP = S_0\,(P_0 - p_0)$; and substituting for the difference of pressures its value, according to equation (2), the following expression is found for the loss of energy at the junction,

$$\Sigma \cdot \left\{ \frac{av}{s_0} \cdot \frac{(v-V)^2}{2} \right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\,D)$$

that is to say, in the case of liquids all the energy due to the several velocities $(v-V)$ of the component streams relatively to the *resultant stream* is lost.

When the expression (3 D) is reduced to a single term, it becomes the well-known value of the loss of energy of a single stream of liquid at a sudden enlargement in a tube.

6. *Efficiency of Combined Streams.*—The *efficiency* of a set of combined streams may be defined as the fraction expressing the ratio borne by the total energy of the resultant stream after the combination to the aggregate energy of the component streams before the combination. It is expressed as follows:—

$$\frac{\frac{AV}{S_0}\left\{ \frac{V^2}{2} + \int_0^{P_0} S\,dP \right\}}{\Sigma\left\{ \frac{av}{s_0}\left(\frac{v^2}{2} + \int_0^{p_0} s\,dp \right) \right\}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

7. *General Problem of Combined Streams.*—In most cases the problem of combined streams takes one or other of the two following forms. In each of the two forms the areas of the nozzles a_1, a_2, &c. are given, and also the area of the throat, A.

First Form.—The quantities given, besides the before-mentioned areas, are the pressure at the nozzles, p_0, and the velocities of the component streams, v_1, &c. The functional values given are those of $s_{0,\,1}$, $s_{0,\,2}$, &c., in terms of p_0, and of S_0 in terms of P_0, $\frac{a_1 v_1}{s_{0,\,1}}$, $\frac{a_2 v_2}{s_{0,\,2}}$, &c. Those functional values are to be substituted in the equations (1) and (2); and the solution of these equations will give the numerical values of V and of P_0. In the case of liquids of sensibly constant bulkiness, $s_{0,\,1}$, &c., and S_0 are quantities sensibly independent of p_0 and P_0; and then equations (1) and (2) can be separately solved without elimination, giving respectively V and P_0.

Second Form.—Each of the component streams flows through a passage whose factor of resistance, f, is given, from a separate reservoir in which the pressure p and the elevation z of the surface above the junction-chamber are given. The resultant stream flows through a passage whose factor of resistance, F, is given, into a reservoir in which the pressure P and the elevation Z of the surface above the junction-chamber are given. These, together with the areas A, a_1, a_2, &c., are the quantities given. The functional values given are those of the bulkiness, $s_{0,1}$, $s_{0,2}$, &c., and S_0, as before; also the following values of the velocities, according to well-known principles in hydrodynamics; for any component stream,

$$v=\sqrt{\left\{\frac{2gz+2\int_{p_0}^{p} s\,dp}{1+f}\right\}}; \quad \ldots \ldots \quad (5)$$

and for the resultant stream,

$$V=\sqrt{\left\{\frac{2gZ+2\int_{P_0}^{P} S\,dP}{1+F}\right\}}; \quad \ldots \ldots \quad (6)$$

The functional values given are to be substituted in equations (1) and (2), whose solution will then give the numerical values of p_0 and P_0; and from these and the other data the numerical values of v_1 &c. and of V may be calculated.

November 17, 1870.

General Sir EDWARD SABINE, K.C.B., President, in the Chair.

In pursuance of the Statutes, notice of the ensuing Anniversary Meeting was given from the Chair.

General Boileau, Mr. Busk, Mr. David Forbes, Sir John Lubbock, and Mr. Mivart, having been nominated by the President, were elected by ballot Auditors of the Treasurer's accounts on the part of the Society.

Mr. Andrew Noble, Capt. Sherard Osborn, and Mr. George Frederic Verdon were admitted into the Society.

Anders Jöns Ångström, of Upsala, and Joseph Antoine Ferdinand Plateau, of Ghent, were proposed for election as Foreign Members, and notice was given from the Chair that these gentlemen would be ballotted for at the next Meeting.

The Presents received were laid on the table, and thanks ordered for them.

The following communications were read:—

I. "Researches into the Chemical Constitution of the Opium Bases. —Part IV. On the Action of Chloride of Zinc on Codeia." By AUGUSTUS MATTHIESSEN, F.R.S., Lecturer on Chemistry at St. Bartholomew's Hospital, and W. BURNSIDE, of Christ's Hospital. Received June 23, 1870. (See page 71.)