

(Paper No. 3520.)

“The Velocity of Water Flowing Down a Steep Slope.

By ERNEST PRESCOT HILL, M. Inst. C.E.

THE ordinary equation expressing the relation between the velocity of flow of water and the inclination of the channel in which it is flowing, is applicable only to cases in which the component of the force of gravity in the direction of flow is balanced by the frictional resistance, and in which, therefore, the velocity is constant. It is of no value for determining directly the velocity of flow down a steep slope when the frictional resistance is less than the gravitational force, and when the velocity therefore acquires an acceleration. Such a case occurs when water flows over the waste-weir of a reservoir having a high embankment. The waste-water channel is almost always steep, and its proper design depends upon a knowledge of the velocity of the water at any point in its course. So far as the Author is aware there is no reference to such conditions in the ordinary literature on the flow of water, and no expression is available which would enable an engineer to design such a channel in the most economical manner. It is probable, therefore, that existing channels may have been given a sectional area in excess of that required, in order to provide a margin for uncertainty.

The following investigation involves two assumptions, (1) that the experimental values for the coefficient in the ordinary water-course equation are applicable to high velocities, and (2) that the hydraulic mean depth in any particular case may be taken as constant, either the varying sectional area of the channel making the assumption approximately true, or a mean value being sufficiently accurate. The Author believes that these assumptions are justified, and that a practically useful result is thus obtained.

Let u denote the initial velocity,
 H the height of fall,
 h the loss of head in height H , due to friction,
 l the length of the channel,
 m the hydraulic mean depth,
 v the velocity of flow,
 c the coefficient in the ordinary watercourse formula,
 and θ the angle which the slope makes with the horizontal.

The velocity of a particle sliding down a slope, the effect of friction being neglected, is the same as that of a particle with a free vertical fall.

$$\text{Thus} \quad v = \{u^2 + 2g(H - h)\}^{\frac{1}{2}} \quad . \quad . \quad . \quad (1)$$

The loss of head due to friction is given by the equation,

$$h = \frac{lv^2}{mc^2}$$

but in this case the velocity is a variable quantity,

$$\begin{aligned} \text{and} \quad dh &= \frac{v^2}{mc^2} dl \\ &= \frac{v^2}{mc^2 \sin \theta} dH \quad . \quad . \quad . \quad (2) \end{aligned}$$

Substituting the value of v given in equation (1),

$$\frac{mc^2 \sin \theta}{2g} \frac{dh}{dH} = \frac{u^2}{2g} + H - h.$$

$$\text{Let} \quad \kappa = \frac{2g}{mc^2 \sin \theta};$$

$$\text{then} \quad \frac{dh}{dH} + \kappa h = \kappa \frac{u^2}{2g} + \kappa H$$

Multiplying each side by $e^{\kappa H}$,

$$\frac{dh}{dH} e^{\kappa H} + h \kappa e^{\kappa H} = \frac{u^2}{2g} \kappa e^{\kappa H} + H \kappa e^{\kappa H}.$$

$$\text{Integrating,} \quad h e^{\kappa H} = \frac{u^2}{2g} e^{\kappa H} + \kappa \int H e^{\kappa H} dH + C$$

$$\therefore \quad h = \frac{u^2}{2g} + C e^{-\kappa H} + H - \frac{1}{\kappa}$$

When $h = 0$, $H = 0$,

$$\therefore \quad C = \frac{1}{\kappa} - \frac{u^2}{2g}$$

$$\therefore \quad h = H - (1 - e^{-\kappa H}) \left(\frac{1}{\kappa} - \frac{u^2}{2g} \right) \quad . \quad . \quad (3)$$

Substituting in equation (1)

$$v = \{u^2 + 2g(1 - e^{-\kappa H})\left(\frac{1}{\kappa} - \frac{u^2}{2g}\right)\}^{\frac{1}{2}}. \quad (4)$$

or, more conveniently,

$$v = \left\{\frac{2g}{\kappa} - e^{-\kappa H}\left(\frac{2g}{\kappa} - u^2\right)\right\}^{\frac{1}{2}}. \quad (5)$$

When $h = H$ it is obvious that $v = u$. In this case

$$(1 - e^{-\kappa H})\left(\frac{1}{\kappa} - \frac{u^2}{2g}\right) = 0,$$

and either $H = 0$, or $u = c\sqrt{m \sin \theta}$ which is the ordinary expression where the velocity is constant.

Again, from equation (5) it is clear that v has a limiting value when $H = \infty$; this value is $v = c\sqrt{m \sin \theta}$.

The time of descent is obtained from the equation:—

$$\begin{aligned} \frac{dH}{\sin \theta} &= v dt \\ &= \left\{\frac{2g}{\kappa} - e^{-\kappa H}\left(\frac{2g}{\kappa} - u^2\right)\right\}^{\frac{1}{2}} dt \end{aligned}$$

Let a denote $\frac{2g}{\kappa}$ and let b denote $\frac{2g}{\kappa} - u^2$,

Then

$$\begin{aligned} \frac{dH}{\sin \theta} &= \{a - b e^{-\kappa H}\}^{\frac{1}{2}} dt \\ \sin \theta dt &= \frac{dH}{\{a - b e^{-\kappa H}\}^{\frac{1}{2}}} \end{aligned}$$

Denoting $e^{\frac{\kappa H}{2}}$ by x

$$\begin{aligned} \sin \theta dt &= \frac{2}{\kappa} \frac{dx}{(ax^2 - b)^{\frac{1}{2}}} \\ t \sin \theta &= \frac{2}{\kappa} \int_{x_1}^{x_2} \frac{dx}{(ax^2 - b)^{\frac{1}{2}}} \\ t &= \frac{2}{\kappa \sqrt{a} \sin \theta} \log_e \frac{x_2 \sqrt{a} + \sqrt{ax_2^2 - b}}{x_1 \sqrt{a} + \sqrt{ax_1^2 - b}}. \quad (6) \end{aligned}$$

The following example will serve as an illustration:—

Let the rate of flow be 1,200 cubic feet per second,
the depth of water 2 feet throughout,

u 15 feet per second,

m 1.53 (mean value by preliminary trial),

c 123,

and θ 12°. 40' (a slope of 4.5 to 1).

Then the time of descent $= t = 10.71$ seconds when $H = 100$,

and the following Table gives the velocity and consequent breadth of channel for the values of H stated :—

H	v	Breadth.	H	v	Breadth
0	15.00	40.00	60	53.01	11.32
10	28.34	21.17	70	55.50	10.81
20	36.22	16.56	80	57.60	10.42
30	41.95	14.30	90	59.39	10.10
40	46.42	12.92	100	60.93	9.86
50	50.03	11.99

To take an extreme case, in which m varies considerably, let the channel have a uniform breadth of 40 feet throughout. Let the constant value of m , obtained by a preliminary trial, be 0.7, giving $c = 116$. Then the following Table gives the velocities for the different values of H , the depths of water corresponding to these velocities, and the real values of m :—

H	v For $m = 0.7$	Depth.	m	H	v	Depth.	m
0	15.00	2.00	1.82	60	42.21	0.71	0.69
10	26.82	1.12	1.06	70	43.10	0.70	0.68
20	32.84	0.91	0.87	80	43.74	0.69	0.67
30	36.65	0.82	0.79	90	44.20	0.68	0.66
40	39.20	0.76	0.73	100	44.54	0.67	0.65
50	40.96	0.73	0.70

The greatest discrepancy between the real and assumed values of m occurs in the first 20 feet of fall, and if greater accuracy be desired, this portion can be calculated separately; a comparison of the velocities in the two Tables, however, shows that this is unnecessary for practical purposes.

In considering these two extreme cases the question naturally arises as to whether the kinetic energy of the water in the first case is too great and too concentrated to be reduced by moderate works, such as a pool, at the foot of the slope, and whether the engineer is therefore justified in adopting a watercourse nearly three times the width, although of less depth, as in the second case. In the second case the energy is only $\left(\frac{44.54}{60.93}\right)^2 = 0.53$, say half that in the first case, but this reduction is only gained by

trebling the width of the watercourse, and therefore at considerable expense.

It may be asked why, in this investigation, m and c have not been considered as variables, as they undoubtedly are, depending on v . The Author, however, doubts the possibility of integrating an expression including m and c as variables, and believes that, even if a result could be obtained, it would be too cumbrous to be of service.

In conclusion, the Author considers that it is not economical to restrict the velocity, whether by roughing or by steps, and on this assumption an equation is given for calculating the sectional area at any point in the channel.
