

XV.—*Expansion of Functions in terms of Linear, Cylindric, Spherical, and Allied Functions.* By P. ALEXANDER, M.A. Communicated by Dr T. MUIR.

(Read 20th December 1886.)

The expansion of  $\phi(x)$  in terms of  $G_0(x)$ ,  $G_1(x)$ ,  $G_2(x)$ , &c., connected by a given law, being of great importance in mathematico-physical investigation, every method of effecting this expansion must have some interest for scientists.

I therefore proceed to propose what I think to be a new method, in the hope that it may prove to be useful.

Many special expansions of this nature have been effected by FOURIER, LEGENDRE, and others.

After I had developed my method, my attention was called to two papers on this subject showing methods of development of great generality. The titles of the papers are—KÖNIG, J., “Ueber die Darstellung von Functionen u. s. w.,” *Mathematische Annalen*, v. pp. 310–340, 1871; and SONINE, N., “Recherches sur les fonctions cylindriques,” &c., *Mathematische Annalen*, xvi. pp. 1–80, 1879. KÖNIG, assuming that

$$\phi(x_0 + x) = F_0(x_0) \cdot G_0(x) + F_1(x_0) \cdot G_1(x) + F_2(x_0) \cdot G_2(x) + \&c.$$

where  $G_0$ ,  $G_1$ ,  $G_2$ , &c., are an infinite series of functions of  $x$ , connected by some given law, and also subject to the condition that when  $x$  is nearly equal to  $c$ , each of them is capable of expansion in ascending integral powers of  $(x - c)$ , beginning in the case of  $G_p(x)$  with  $(x - c)^p$ , proceeds to show that the coefficients  $F_0(x_0)$ ,  $F_1(x_0)$ , &c., are to be deduced from the following—

$$\begin{aligned} G_0(c) \cdot F_0(x_0) &= \phi(x_0 + c), \\ G_0(c) \cdot F_0'(x_0) &= F_0(x_0) \cdot G_0'(c) + F_1(x_0) G_1'(c), \\ G_0(c) \cdot F_0''(x_0) &= F_0(x_0) \cdot G_0''(c) + F_1(x_0) \cdot G_1''(c) + F_2(x_0) \cdot G_2''(c), \\ &\&c., \qquad \qquad \qquad \&c. \end{aligned}$$

SONINE shows that

$$S_0(a + x) = A_0(a) \cdot S_0(x) - 2\{A_1(a)S_1(x) - A_2(a)S_2(x) + \&c.\}$$

if the series is convergent, where  $S_0(x)$  and  $A_0(a)$  may be any functions what-

ever of  $x$  and  $a$  consistent with convergency, and  $A_0(a)$ ,  $A_1(a)$ ,  $A_2(a)$ , &c., and  $S_0(x)$ ,  $S_1(x)$ ,  $S_2(x)$ , &c., are connected by the following relations:—

$$\left. \begin{aligned} A_1(a) &= -\frac{d}{da}[A_0(a)], \\ A_{n+1} + 2\frac{dA_n}{da} - A_{n-1} &= 0, \end{aligned} \right\}$$

and

$$\left. \begin{aligned} S_1 &= -\frac{dS_0}{dx}, \\ S_{n+1} + 2\frac{dS_n}{dx} - S_{n-1} &= 0; \end{aligned} \right\}$$

and hence

$$A_n = (-i)^n \cos n \Delta_1 \cdot A_0,$$

and

$$S_n = (-i)^n \cos n \Delta \cdot S_0,$$

where

$$i = \sqrt{-1}$$

and  $\Delta_1$  and  $\Delta$  are operations defined by

$$i \cos \Delta_1 = \frac{d}{da},$$

and

$$i \cos \Delta = \frac{d}{dx}.$$

KÖNIG's method seems to be much more general than SONINE's, as KÖNIG's functions  $G_0$ ,  $G_1$ ,  $G_2$ , &c., may be connected by any law, while SONINE's functions  $A_0$ ,  $A_1$ ,  $A_2$ , &c., are connected by one law only. But on the other hand, KÖNIG's functions are limited by the condition that  $G_p(x)$  must, when  $x$  nearly equals  $c$ , be capable of expansion in ascending integral powers of  $(x-c)$  beginning with  $(x-c)^p$ , whereas SONINE's functions are subject to no such condition.

Both methods give the expansion of  $\phi(x)$  in terms of  $J_0(x)$ ,  $J_1(x)$ ,  $J_2(x)$ , &c., BESSEL's functions. But neither of them give the expansion

$$\phi(x) = A_0 J_n(k_0 x) + A_1 J_n(k_1 x) + A_2 J_n(k_2 x) + \&c.,$$

where  $k_0$ ,  $k_1$ ,  $k_2$ , &c., are the roots of some equation of condition.

The most general method of expansion I have seen is that of expansion in normal co-ordinates employed by RAYLEIGH throughout his *Theory of Sound*, which is so satisfactory that had I become acquainted with it somewhat earlier, I would probably not have sought after the following method:—

The general problem is to determine  $A_0$ ,  $A_1$ ,  $A_2$ , &c., so that when possible

$$\phi(x) = A_0 G_0(x) + A_1 G_1(x) + A_2 G_2(x) + \&c., \quad (1)$$

where  $G_0$ ,  $G_1$ ,  $G_2$ , &c., are connected by some given law.

The solution of this in all its generality has not yet been obtained, but in most of the particular cases which have been solved the method seems to be to operate on (1) with an operator  $O_n$  such that

$$O_n \cdot G_m = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

except  $m=n$ .

And, therefore,

$$O_n \cdot \phi(x) = A_n O_n \cdot G_n(x)$$

$$\therefore A_n = \frac{O_n \cdot \phi(x)}{O_n \cdot G_n(x)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Following this lead, I have found an operator of this nature in the case where  $G_0, G_1, G_2, \&c.$ , are elementary solutions of the equation,

$$(\delta + g)G = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

when  $g$  has the values  $g_0, g_1, g_2, \&c.$ , derived from the condition

$$\{\sigma \cdot G\}_{x=a} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where  $\delta$  and  $\sigma$  are operations which may have the forms

$$\delta = X_0 + X_1 \left( \frac{d}{dx} \right) + X_2 \left( \frac{d}{dx} \right)^2 + \&c., \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\sigma = P_0 + P_1 \left( \frac{d}{dx} \right) + P_2 \left( \frac{d}{dx} \right)^2 + \&c., \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where  $X_0, X_1, X_2, \&c.$ , and  $P_0, P_1, P_2, \&c.$ , may be either constants or functions of  $x$ .

The operator I have discovered for the solution of this problem is—

$$O_n = \{\sigma(\delta + g_n)^{-1}\}_{x=a} \quad (8)$$

The proof is as follows :—

$$\sigma(\delta + g_n)^{-1} G_m = \sigma \{ [g_n^{-1} - g_n^{-2} \delta + g_n^{-3} \delta^2 - \&c.] G_m + G_n \}. \quad . \quad . \quad . \quad (9)$$

But from (4),

$$\therefore \left. \begin{aligned} \delta \cdot G_m &= -g_m \cdot G_m \\ \delta^2 \cdot G_m &= -g_m \delta G_m = g_m^2 \cdot G_m \\ \delta^3 \cdot G_m &= g_m^2 \delta G_m = -g_m^3 G_m \\ &\&c. \quad \&c. \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (10)$$

$$\begin{aligned}
\therefore \quad \sigma(\delta + g_n)^{-1}G_m &= \sigma \{ [g_n^{-1} + g_n^{-2}g_m + g_n^{-3}g_m^2 + \&c.] G_m + G_n \} \\
&= (g_n - g_m)^{-1} \sigma G_m + \sigma G_n \\
&= \frac{\sigma G_m}{g_n - g_m} + \sigma G_n \quad \dots \dots \dots (11)^*
\end{aligned}$$

But from (5),

$$[\sigma G_n]_{x=a} = 0 = [\sigma G_m]_{x=a}$$

Hence (11) gives—

$$\left. \begin{aligned}
\{ \sigma(\delta + g_n)^{-1}G_m \}_{x=a} &= \frac{0}{g_n - g_m} + 0 \\
&= 0 \text{ if } m \text{ is not } = n \\
&= \frac{0}{0} \text{ if } m = n
\end{aligned} \right\} \quad \dots \dots \dots (12)$$

Hence from (3),

$$\begin{aligned}
A_n &= \frac{O_n \cdot \phi(x)}{O_n \cdot G_n} \\
&= \left\{ \frac{\sigma(\delta + g_n)^{-1} \phi(x)}{\sigma(\delta + g_n)^{-1} G_n} \right\}_{x=a} \quad \dots \dots \dots (13)
\end{aligned}$$

By the method of vanishing fractions,

$$\begin{aligned}
\left\{ (\sigma + g_n)^{-1} G_n \right\}_{x=a} &= \left\{ \frac{\sigma \cdot G_m}{g_n - g_m} \right\}_{(m=n, x=a)} \\
&= \left\{ \frac{\frac{d}{dg_m} (\sigma \cdot G_m)}{\frac{d}{dg_m} (g_n - g_m)} \right\}_{(m=n, x=)} \\
&= \left\{ \frac{d}{dg_n} (\sigma \cdot G_n) \right\}_{x=}
\end{aligned}$$

Hence (13) becomes

$$A_n = - \left\{ \frac{\sigma(\delta + g_n)^{-1} \phi(x)}{\frac{d}{dg_n} (\sigma \cdot G_n)} \right\}_{x=a} \quad \dots \dots \dots (14)$$

which is the required solution.

As new results, especially when very general, are liable to suspicion, I proceed to test this by a particular example whose solution can be otherwise found.

Let

$$\delta = \left( \frac{d}{dx} \right)^2 + \frac{\lambda}{x} \left( \frac{d}{dx} \right) - \frac{p^2 + (\lambda - 1)p}{x^2} \quad \dots \dots \dots (15)$$

\* In the proof of (11) it has been assumed that  $g_m$  is less than  $g_n$ . The same may be proved for  $g_m$  greater than  $g_n$  by expanding  $(\delta + g_n)^{-1}$  in the reverse order.

and

$$\sigma = h + \frac{d}{dx} \quad (16)$$

An elementary solution of (4) is in this case

$$\left. \begin{aligned} G_n &= \left( x \sqrt{g_n} \right)^{-\frac{\lambda-1}{2}} J_{p+\frac{\lambda-1}{2}} \left( x \sqrt{g_n} \right) \\ &\quad \left( x \sqrt{g_n} \right)^{\frac{\lambda-1}{2}} K_{p+\frac{\lambda-1}{2}} \left( x \sqrt{g_n} \right) \end{aligned} \right\} \dots \dots \dots (17)^*$$

or

where  $J$  and  $K$  are BESSEL's functions of the first and second orders.

Supposing then that it is possible to expand  $\phi(x)$  in terms of the first of these and operating on (1) with the operator

$$O_n = \int_0^a dx \, x^\lambda G_n$$

we have

$$\int_0^a \phi(x) x^\lambda G_n dx = A_0 \int_0^a G_0 G_n x^\lambda dx + A_1 \int_0^a G_1 G_n x^\lambda dx + \&c. \quad (18)$$

But

$$\int_0^a G_m G_n x^{\lambda} dx = a^{\lambda} \cdot \frac{\left\{ G_m \frac{dG_n}{dx} - G_n \frac{dG_m}{dx} \right\}_{x=a}}{g_m - g_n} \quad (19)$$

$$= -a^\lambda \cdot \frac{\left\{ G_n \left( h + \frac{d}{dx} \right) G_m \right\}}{g_m - g_n} \Big|_{x=a} \text{ from (5) and (16)}$$

$$\left. \begin{aligned} &= -\alpha^\lambda \left\{ \frac{\mathbf{G}_n \sigma \cdot \mathbf{G}_m}{g_m - g_n} \right\}_{z=a} \text{ from (16)} \\ &= 0 \text{ if } m \text{ is not } = n \\ &= \frac{0}{0} \text{ if } m = n \end{aligned} \right\} \text{ from (5)} \quad (20)$$

By the method of vanishing fractions,

$$\begin{aligned} \int_0^a G_n^2 x^\lambda dx &= -\alpha^\lambda \left\{ \frac{G_n \sigma \cdot G_m}{g_m - g_n} \right\}_{(m=n, x=a)} \\ &= -\alpha^\lambda \left\{ G_n \frac{d}{dg_n} (\sigma G_n) \right\}_{x=a} \quad (21) \end{aligned}$$

\* Or  $G_n = (x \sqrt{g_n})^{-\frac{\lambda-1}{2}} J_{p+\frac{\lambda-1}{2}}(x \sqrt{g_n}) + B_n (x \sqrt{g_n})^{\frac{\lambda-1}{2}} K_{p+\frac{\lambda-1}{2}}(x \sqrt{g_n}).$

Again,

$$\begin{aligned}
 x^\lambda [G_n \delta \phi - \phi \delta G_n] &= x^\lambda \left\{ G_n \left[ \left( \frac{d}{dx} \right)^2 + \frac{\lambda}{x} \left( \frac{d}{dx} \right) - \frac{p^2 + (\lambda-1)p}{x^2} \right] \phi \right. \\
 &\quad \left. - \phi \left[ \left( \frac{d}{dx} \right)^2 + \frac{\lambda}{x} \left( \frac{d}{dx} \right) - \frac{p^2 + (\lambda-1)p}{x^2} \right] G_n \right\}, \\
 &= x^\lambda \left\{ G_n \left[ \left( \frac{d}{dx} \right)^2 + \frac{\lambda}{x} \left( \frac{d}{dx} \right) \right] \phi - \phi \left[ \left( \frac{d}{dx} \right)^2 + \frac{\lambda}{x} \left( \frac{d}{dx} \right) \right] G_n \right\}, \\
 &= G_n \frac{d}{dx} \left[ x^\lambda \frac{d\phi}{dx} \right] - \phi \frac{d}{dx} \left[ x^\lambda \frac{dG_n}{dx} \right], \\
 &= \frac{d}{dx} \left\{ x^\lambda \left[ G_n \frac{d\phi}{dx} - \phi \frac{dG_n}{dx} \right] \right\}.
 \end{aligned}$$

$$\therefore -x^\lambda \phi \delta G_n = \frac{d}{dx} \left\{ x^\lambda \left[ G_n \frac{d\phi}{dx} - \phi \frac{dG_n}{dx} \right] \right\} - x^\lambda G_n \delta \phi;$$

$$\therefore \int_0^a \phi G_n x^\lambda dx = -\frac{1}{g_n} \int_0^a x^\lambda \phi \delta G_n dx, \text{ from (4)}$$

$$= \left[ \frac{x^\lambda \left( G_n \frac{d\phi}{dx} - \phi \frac{dG_n}{dx} \right)}{g_n} \right]_{x=a} - \frac{1}{g_n} \int_0^a x^\lambda G_n \delta \phi dx$$

or

$$\begin{aligned}
 \int_0^a \phi \cdot G_n \cdot x^\lambda dx &= \frac{\alpha^\lambda \left\{ G_n \left[ \frac{d\phi}{dx} + h\phi \right] \right\}}{g_n} \Big|_{x=a} - \frac{1}{g_n} \int_0^a \delta \phi \cdot G_n \cdot x^\lambda dx \\
 &= \frac{1}{g_n} \left\{ \alpha^\lambda \left[ G_n \cdot \sigma \phi \right]_{x=a} - \int_0^a \delta \phi \cdot G_n \cdot x^\lambda dx \right\}.
 \end{aligned}$$

Similarly,

$$\int_0^a \delta \phi \cdot G_n \cdot x^\lambda dx = \frac{1}{g_n} \left\{ \alpha^\lambda \left[ G_n \sigma \delta \phi \right]_{x=a} - \int_0^a \delta^2 \phi \cdot G_n \cdot x^\lambda dx \right\},$$

and so on.

Hence

$$\begin{aligned}
 \int_0^a \phi \cdot G_n \cdot x^\lambda dx &= \alpha^\lambda \left\{ G_n \cdot \sigma \left[ \frac{\phi}{g_n} - \frac{\delta \phi}{g_n^2} + \frac{\delta^2 \phi}{g_n^3} - \&c. \right] \right\} \Big|_{x=a} \\
 &= \alpha^\lambda \{ G_n \sigma (g_n + \delta)^{-1} \phi \}_{x=a} \quad \quad \quad (22)
 \end{aligned}$$

But

$$\begin{aligned} A_n &= \frac{O_n \cdot \phi}{O_n \cdot G_n} \\ &= \frac{\int_0^a \phi \cdot G_n \cdot x^\lambda dx}{\int_0^a G_n^2 x^\lambda dx} \end{aligned} \quad (23)^*$$

Hence from (21) and (22) this becomes

$$\begin{aligned} A_n &= \frac{\alpha^\lambda [G_n \cdot \sigma(\delta + g_n)^{-1} \phi]_{x=a}}{-\alpha^\lambda [G_n \frac{d}{dg_n} (\sigma G_n)]_{x=a}} \\ &= - \left\{ \frac{\sigma(\delta + g_n)^{-1} \phi}{\frac{d}{dg_n} (\sigma G_n)} \right\}_{x=a} \end{aligned} \quad (24)$$

which verifies (14) for this case.

If  $\lambda = 0$  and  $p = 0$  (FOURIER'S *Heat*, ch. vii. and viii.; RAYLEIGH'S *Sound*, § 135), then (23) or (24) will give an expansion of  $\phi(x)$  in linear functions (trigonometric),

$$\phi(x) = A_0 \cdot \sqrt{\frac{2}{\pi}} \cos(x \sqrt{g_0}) + A_1 \sqrt{\frac{2}{\pi}} \cos(x \sqrt{g_1}) + \&c.$$

where  $g_0, g_1, \&c.$ , are the roots of

$$a \sqrt{g} \cdot \tan(a \sqrt{g}) = ah.$$

If  $\lambda = 1$  (FOURIER'S *Heat*, ch. vi.; RAYLEIGH'S *Sound*, § 201),

$$\phi(x) = A_0 \cdot J_p(x \sqrt{g_0}) + A_1 \cdot J_p(x \sqrt{g_1}) + \&c.,$$

where  $g_0, g_1, \&c.$ , are the roots of

$$\sqrt{g} J'_p(a \sqrt{g}) + h J_p(a \sqrt{g}) = 0,$$

which gives an expansion of  $\phi(x)$  in cylindric or BESSEL'S functions.

\* Since writing this I have proved that if  $\delta = X_2 \left( \frac{d}{dx} \right) + X_1 \frac{d}{dx} + X_0$

$$A_n = \frac{\int_0^a \phi \cdot G_n \cdot \frac{e^{\int \frac{X_1}{X_2} dx}}{X_2} dx}{\int_0^a G_n^2 \cdot \frac{e^{\int \frac{X_1}{X_2} dx}}{X_2} dx};$$

but I find that Sturm and Liouville have anticipated me (Liouville's *Journal de Mathématiques*, vol. i., 1836).

If  $\lambda=2$  (FOURIER'S *Heat*, ch. v.; RAYLEIGH'S *Sound*, ch. xvii.),

$$G_n = (x \sqrt{g_n})^{-\frac{1}{2}} J_{p+\frac{1}{2}}(x \sqrt{g_n}) ,$$

$$\therefore \quad \phi(x) = A_0 (x \sqrt{g_0})^{-\frac{1}{2}} J_{p+\frac{1}{2}}(x \sqrt{g_0}) + A_1 (x \sqrt{g_1})^{-\frac{1}{2}} J_{p+\frac{1}{2}}(x \sqrt{g_1}) + \&c.,$$

where  $g_0, g_1, g_2, \&c.$ , are the roots of

$$2a \sqrt{g} J'_{p+\frac{1}{2}}(a \sqrt{g}) + (2ah-1) J_{p+\frac{1}{2}}(a \sqrt{g}) = 0 ,$$

which gives an expansion of  $\phi(x)$  in terms of spherical functions (Kugelfunctionen).