ON THE SINGULARITIES OF FUNCTIONS DEFINED BY TAYLOR'S SERIES (*Remarks in addition to a former Paper*)

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1. The following note is a result of some remarks made to me by Prof. Bromwich. Its object is to indicate some extensions and further applications of the method which I used in my paper "A Method &c." (*Proceedings*, Vol. 3, pp. 381–9). This method is an adaptation of an idea due originally to Hadamard (*Journal de Math.*, 1893), who used it to establish certain general conclusions since made more precise by Le Roy.* In my paper I introduced certain loop integrals which enable us to obtain information in some respects more general and in others more precise than that given by the line integrals of the two writers just quoted.

2. Consider the series

$$f(x) = a_0 b_0 + a_1 b_1 x + a_2 b_2 x^2 + \dots$$

where a_ν is defined as in my former paper, and

$$F(x) = b_0 + b_1 x + b_2 x^2 + \dots$$

is a function of x, possibly many-valued, but having a branch (the *principal branch*) which is regular all over the plane with the exception of $x = 1$ and $x = \infty$, and one-valued inside the domain T formed by slitting the plane along the line $(1, \infty)$. Thus $F(x)$ might be

$$(1-x)^{-\delta}, \qquad x^{-1}\log(1-x), \qquad F(a, b, c, x).$$

Then, using the notation and arguments of my former paper, but writing for shortness

$$\Phi(u) = (\log u)^{\alpha-1}(u-1)^{\beta-1}u^{\gamma-1}\phi(u),$$

we obtain the equation

$$f(x) = \frac{1}{2i\sin(\alpha+\beta)\pi}\int_C \Phi(u)\,F(xu)\,du; \tag{1}$$

unless $\alpha+\beta$ is an integer k, in which case

$$f(x) = \frac{(-)^k}{2\pi i}\int_C \Phi(u)\,F(xu)\log(u-1)\,du. \tag{1a}$$

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* *Annales de la Faculté des Sciences de Toulouse*, 1900.

In particular, if $\alpha = \beta = \gamma = 1$, so that $\Phi(u) = \phi(u)$ is regular in a domain including (0, 1) in its interior,

$$(1b) \qquad f(x) = \frac{1}{2\pi i}\int_C \phi(u) F(xu) \log(u-1)\, du.$$

This expression gives the analytic continuation of $f(x)$ all over the region T bounded by a slit along $(1, \infty)$. The branch of $f(x)$ thus defined is called the *principal* branch.

By taking as our fundamental contour C, not a loop including the line (0, 1), but a loop including some other standard path from 0 to 1, we can replace T by another region T' bounded by a different slit from 1 to ∞. We infer that the only finite singularity of the principal branch of $f(x)$ is $x = 1$: or, as we may say, *$x = 1$ is the only finite principal singularity of $f(x)$.*

The argument is easily extended to meet the case in which $F(x)$ has any number of singular points. We define a uniform *principal branch* of $f(x)$ by appropriate slits from its singular points to ∞, and we infer that a corresponding principal branch of $f(x)$ exists, and that *the only principal singularities of $f(x)$ are those of $F(x)$.*

We have now to consider the *subsidiary* singular points of $f(x)$, *i.e.* those singular points which are not singular for the principal branch. Let us suppose first that $F(x)$ has a simple pole at $x = 1$. This is equivalent to taking

$$F(xu) = 1/(1-xu).$$

Le Roy considers this case in some detail. Applying to his fundamental line-integral Hermite's methods of dealing with *intégrales à coupures*, he proceeds to calculate the increment of $F(x)$ when x moves m times round $x = 1$, and gives the result

$$2m\pi i \frac{1}{x} \Phi\left(\frac{1}{x}\right).$$

This result is true only if $\Phi(u)$ is one-valued for circuits round $u = 1$, *i.e.*, if $\alpha+\beta$ is an integer. It then follows at once by applying the transformation of my former paper to the integral $(1a)$: we deduce that, if $\bar{f}(x)$ is the principal branch of $f(x)$,

$$\bar{f}(x) + \frac{(-)^k}{x} \Phi\left(\frac{1}{x}\right) \log\left(\frac{1}{x} - 1\right)$$

is regular near $x = 1$. In the more general case the increment is

$$\frac{2\pi i}{x}\left\{\Phi_1\left(\frac{1}{x}\right) + \ldots + \Phi_m\left(\frac{1}{x}\right)\right\}$$

where the suffices denote different values of Φ. The difference is most easily grasped by considering such a case as

$$f(x) = \tfrac{1}{2}\left\{\log\left(\frac{1}{1-x}\right)\right\}^2 = \sum_0^\infty \left(1+\tfrac{1}{2}+\ldots+\frac{1}{n}\right)\frac{x^{n+1}}{n+1}.$$

This point does not, however, affect the general conclusions which can be drawn. For, in any case, it follows that the only subsidiary singularities of $f(x)$ are given by

$$x = 0, \quad x = 1/\xi$$

where ξ is a singularity of Φ. Hence the singularities of $f(x)$ are 1, ∞ (*principal*) and 0, $1/\xi$ (*subsidiary*), ξ being now any singularity of ϕ.

When F has a variety of simple poles $x = X$, the singularities of f are X, ∞ and 0, X/ξ. The same conclusion holds when F has multiple poles or essential singularities near which it is one-valued. The treatment of such cases (as Le Roy shows) introduces no new difficulties of principle. When, however, $F(x)$ is many-valued (say at $x = 1$), the "increment" cannot be so expressed in so simple a form. In fact, as I shall show shortly, we can define another function $g(x)$ such that $f(x)+g(x)$ is one-valued near $x = 1$, but the complete determination of the singular points of $g(x)$ is, in general, a problem just as difficult as the original problem. And similarly if (as we may do) we calculate directly from Le Roy's line-integral the increment of $f(x)$ corresponding to a circuit round $x = 1$.

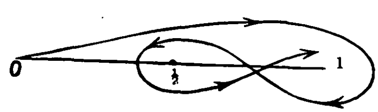
Le Roy, curiously enough, passes over this difficulty in silence, merely remarking that the result proved in the simpler case is true in general. I do not myself see how to effect the extension directly; but we can deduce the required result from Hadamard's "multiplication theorem," that the only singular points of $\Sigma a_n b_n x^n$ are 0, $\alpha\beta$ where α is a singular point of $\Sigma a_n x^n$ and β one of $\Sigma b_n x^n$ (0, of course, can only be a subsidiary singularity). We thus obtain Le Roy's theorem that the only singular points of $f(x)$ are

$$X, \infty; \quad 0, X/\xi.*$$

* It should be noted that the point $x = 1$ may be for the other branches of $f(x)$ a singularity of a kind quite different from what it is for the principal branch. For example, if

$$\alpha = \beta = \gamma = 1$$

and $$\Phi(u) = \exp\left\{1/[\sqrt{(2-u)}+1]\right\},$$



the principal branch of $f(x)$ has an ordinary logarithmic singularity at $x = 1$. The point $x = \frac{1}{2}$ is a subsidiary singularity. If we encircle first 1 and then $\frac{1}{2}$ (as in the figure), we return to 1 with a value of $f(x)$ which has an exponentially essential singularity there. A discussion of the difficulties which are involved in the extension of Hadamard's theorem to subsidiary singularities will be found in a recent paper by O. Faber (*Jahresbericht der Deutschen Math. Ver.*, 1907, Bd. XVI., p. 285).

3. The preceding discussion does not contain anything really novel. I have included it because (1) the use of loop integrals instead of line integrals enables us to reach somewhat more general conclusions (the reduction of a loop to a line integral being only possible when the constants α, β, ... satisfy certain quite irrelevant inequalities); (2) there are several points which Le Roy passes over rather hastily; and (3) some preliminary discussion is needed in order to make intelligible the later sections of this note and the longer paper which follows.

4. I now proceed to consider the application of the method of my former paper to the case in which $F(x)$ is many-valued. The typical case is that in which $F(x) = (1-x)^{-\delta}$ where δ is not an integer.

We introduce the contour C' of my former paper, and either the contour C_1 shown in Fig. 1 or the contour C_2 shown in Fig. 2.

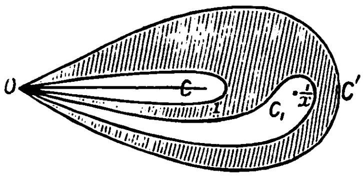


FIG. 1.

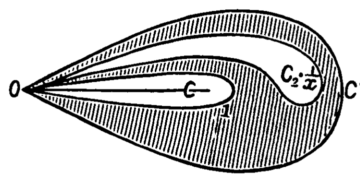


FIG. 2.

In either case the subject of integration is one-valued and regular within the area bounded by the the three loops (and shaded). Also $\int_{C'}$ is regular near $x = 1$. Hence (Fig. 1)

$$\int_C = \int_{C'} - \int_{C_1}$$

and
$$f(x) + \frac{1}{2i\sin(\alpha+\beta)\pi}\int_{C_1}\Phi(x)\frac{du}{(1-xu)^{\delta}}$$

is regular near $x = 1$. When $\delta = 1$, we can calculate the last integral in finite terms; and so we arrive at the results of my former paper.

It may be observed that the initial and final values near the origin of the factors $(\log u)^{\alpha-1}$, $(u-1)^{\beta-1}$ are not the same as in the original integral. It is easy to see that

$$f(x)+g(x)$$

is regular where

$$g(x) = \frac{e^{(\delta-\alpha-\beta)\pi i}}{2i\sin(\alpha+\beta)\pi}\int_{C_1}\left(\log\frac{1}{u}\right)^{\alpha-1}(1-u)^{\beta-1}\frac{u^{\gamma-1}\phi(u)\,du}{(xu-1)^{\delta}}.$$

Here the initial and final values of $(1-u)^{\beta-1}$ are each unity, and

$$\left(\log\frac{1}{u}\right)^{\alpha-1} = e^{(\alpha-1)\log\log 1/u}$$
$$= \exp(\alpha-1)\left[\log\sqrt{\left\{\left(\log\frac{1}{r}\right)^2+\theta^2\right\}}+i\tan^{-1}\left\{\theta\Big/\left(\log\frac{1}{r}\right)\right\}\right],$$

the inverse tangent being initially and finally very small ; also $(xu-1)^\delta$ is $\exp\{\delta\log(xu-1)\}$, $\log(xu-1)$ being real at the point of C_1 for which $xu > 1$.

Similarly, from Fig 2, we obtain

$$g(x) = \frac{e^{-(\delta-\alpha-\beta)\pi i}}{2i\sin(\alpha+\beta)\pi}\int_{C_2}\left(\log\frac{1}{u}\right)^{\alpha-1}(1-u)^{\beta-1}\frac{u^{\gamma-1}\phi(u)\,du}{(xu-1)^\delta}.$$

It is convenient to take the first or second form of $g(x)$ according as $R(x/i) \lesseqgtr 0$: we can then suppose C_1 (or C_2) to be a loop closely surrounding the line $(0, 1/x)$.

If $F(x) = \psi(x)/(1-x)^\delta$, ψ being regular near $x = 1$, we have only to include an additional term $\psi(xu)$ under the sign of integration in either form of $g(x)$.

5. Suppose, *e.g.*, that $\alpha = 1$. We may write the above equations in the form

$$g(x) = K_1 x^{-\gamma}\int_{\Gamma_1}\left(1-\frac{u}{x}\right)^{\beta-1}\phi\left(\frac{u}{x}\right)\frac{u^{\gamma-1}\psi(u)\,du}{(u-1)^\delta}$$

(or a similar equation with K_2, Γ_2), the contour now being a loop surrounding (0, 1), and the choice being determined by the relative situation of the points 0, 1, x.

Then, if $|x| > 1$, we can expand $g(x)$ in the form

$$Kx^{-\gamma}\sum\frac{c_n d_n}{x^n}$$

where c_n is the coefficient of ξ^ν in the expansion of

$$(1-\xi)^{\beta-1}\phi(\xi)$$

and
$$d_n = \int\frac{u^{\gamma+n-1}\psi(u)\,du}{(u-1)^\delta}.$$

We thus define the nature of the singularity of $f(x)$, but not as before in finite terms, the result being that

$$f(x)+x^{-\gamma}\chi\left(\frac{1}{x}\right)$$

is regular near $x = 1$. For example, if ϕ and ψ are each identically unity, we obtain a relation of this kind connecting two hypergeometric series. Relations of this kind are of common occurrence in the theory of many-valued functions defined by Taylor's series.

But one must conclude that relations as simple as those found in my former paper for series such as

$$\Sigma \frac{x^n}{(\gamma+n)^a}, \quad \Sigma \frac{x^n}{1+(n+1)^2}, \quad \dots$$

do not exist for series such as

$$\Sigma \frac{\delta(\delta+1)\dots(\delta+n-1)}{1.2\dots n} \cdot \frac{x^n}{(\gamma+n)^a}$$

(or even series as simple as the ordinary hypergeometric series). All that can be done is to investigate relations such as those indicated in this paragraph, which will in general involve a second transcendent of at least equal complexity; or to obtain *asymptotic* formulæ valid near $x=1$. This is much easier: for example, we can prove that

$$\Sigma \frac{\Gamma(\delta_1+n)\,\Gamma(\delta_2+n)\dots\Gamma(\delta_k+n)}{\Gamma(\epsilon_1+n)\,\Gamma(\epsilon_2+n)\dots\Gamma(\epsilon_k+n)} \frac{x^n}{(\gamma_1+n)^{a_1}(\gamma_2+n)^{a_2}\dots(\gamma_l+n)^{a_l}}$$
$$= \frac{\Gamma(1+\Sigma\delta-\Sigma\epsilon-\Sigma a)}{(1-x)^{1+\Sigma\delta-\Sigma\epsilon-\Sigma a}}(1+\eta_x),$$

where $\lim \eta_x = 0$ for any manner of approach of x to $x=1$. In so far as modes of approach from *within* the circle of convergence are concerned this follows as a mere corollary from Pringsheim's generalisations of Appell's theorem (*Acta Math.*, t. XXVIII., p. 1). To show that it is valid all round the singular point, some method depending on a formula giving the analytic continuation of the function is of course essential. A modification of the method of my former paper leads readily to the result, which I need only mention at present.

6. The method of my former paper is capable of interesting extensions in a different direction. Let us consider the more general contour integral

$$f(x) = \frac{1}{2i\sin(a+\beta)\pi}\int_C \Phi(u)\frac{du}{1-x\psi(u)}$$

where $\psi(u)$ is regular in a domain including (0, 1) in its interior. For sufficiently small values of x,

$$f(x) = \Sigma a_n x^n$$

where
$$a_n = \frac{1}{2i\sin(a+\beta)\pi}\int_C \Phi(u)\{\psi(u)\}^n\,du.$$

The contour integral gives the analytical continuation of the series over the domain bounded by a certain cut. This cut contains those values of x given by

$$x = 1/\psi(u) \quad (0 < u < 1).$$

By varying the path from 0 to 1 included in the loop we vary the cut. The singular points of $f(x)$ can only be found among points excluded from the domain of regularity of $f(x)$ by *all possible* positions of the cut—as, *e.g.*, $x = 1$, $x = \infty$, when $\psi(u) \equiv u$.

Suppose, to take a definite case,

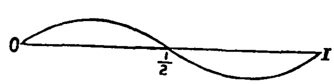
$$\psi(u) \equiv u(1-u).$$

When the path from 0 to 1 is along the real axis the cut begins at $+\infty$, goes along the real axis to $x = 4$, and returns to $+\infty$.

If $u = re^{i\theta}$, $1/u(1-u) = \xi + i\eta$, where

$$\xi = \frac{\cos\theta - r\cos 2\theta}{r(1-2r\cos\theta+r^2)}, \qquad \eta = -\frac{\sin\theta - r\sin 2\theta}{r(1-2r\cos\theta+r^2)}.$$

Thus $\eta = 0$ (i.) if u is real; and (ii.) if $r\cos\theta = \frac{1}{2}$, *i.e.*, when the path from 0 to 1 crosses the line $\mathbf{R}(u) = \frac{1}{2}$. When this is so, $\xi = 4\cos^2\theta < 4$. Hence the cut always passes from ∞ to ∞, passing between 0 and 4, *i.e., all possible positions of the cut exclude the point* $x = 4$. For example, if the path from 0 to 1 is the semicircle described on (0, 1) as diameter, the cut is the line $R(x) = 2$. Again, if we take the path from 0 to 1 to be as shown in the figure (the two curved portions being congruent) the



cut is a curve from 4 to ∞ described twice in opposite directions, and nowhere meeting the real axis except at 4. Thus $x = 4$ is the only principal singular point.

Suppose x is real and a little less than 4. The roots of

$$1 - xu(1-u) = 0$$

are

$$u = \tfrac{1}{2} \pm i\sqrt{\{(4-x)/4x\}};$$

as x approaches 4 these poles of the subject of integration approach $\frac{1}{2}$ from opposite sides, nipping the loop C between them. Introducing a contour C' which encloses the poles, we obtain

$$\int_C = \int_{C'} + \frac{2\pi}{\sqrt{\{x(4-x)\}}}\left\{\Phi[\tfrac{1}{2}+i\sqrt{\{(4-x)/4x\}}] - \Phi[\tfrac{1}{2}-i\sqrt{\{(4-x)/4x\}}]\right\}.$$

We thus determine the behaviour of $f(x)$ near $x = 4$.

Suppose, *e.g.*, $a = 1$, $\phi \equiv 1$. Then

$$a_n = \int_0^1 (1-u)^{\beta-1+n} u^{\gamma-1+n}\, du = \frac{\Gamma(\beta+n)\,\Gamma(\gamma+n)}{\Gamma(\beta+\gamma+2n)}.$$

Thus the only principal singular points of

$$f(x) = \Sigma \frac{\Gamma(\beta+n)\,\Gamma(\gamma+n)}{\Gamma(\beta+\gamma+2n)} x^n$$

are $x = 4$, ∞. Near $x = 4$, $f(x)$ behaves like

$$\frac{i\pi}{\sin\beta\pi}\,\frac{1}{\sqrt{\{x(4-x)\}}}\,\{(-\tfrac{1}{2}+iX)^{\beta-1}(\tfrac{1}{2}+iX)^{\gamma-1}-(-\tfrac{1}{2}-iX)^{\beta-1}(\tfrac{1}{2}-iX)^{\gamma-1}\},$$

where $X = \sqrt{\{(4-x)/4x\}}$. We may verify the result by supposing $\beta = \gamma = 1$, when we must use the degenerate integral

$$\frac{1}{2\pi i}\int \frac{\log(u-1)}{1-xu(1-u)}\, du.$$

We find for the irregular part

$$-\frac{2}{\sqrt{\{x(4-x)\}}}\tan^{-1}\sqrt{\left(\frac{4-x}{x}\right)}.$$

As a matter of fact

$$f(x) = 1+\frac{2}{3}\left(\frac{x}{4}\right)+\frac{2.4}{3.5}\left(\frac{x}{4}\right)^2+\ldots = \frac{2}{\sqrt{\{x(4-x)\}}}\tan^{-1}\sqrt{\left(\frac{x}{4-x}\right)}.$$

In this (and in the more general case above) 4, ∞, 0 are the only singularities, the last being subsidiary.

Another interesting set of series is given by supposing

$$\psi(u) = u\log\left(\frac{1}{u}\right).$$

If $\beta = 1$, $\phi \equiv 1$, this gives rise to the series

$$\Sigma \frac{\Gamma(n+a)}{(n+\gamma)^{n+a}} x^n.$$

The nature of the cross cuts may be seen to be much the same as in the last example: the one finite principal singularity is in this case $x = e$. As x approaches e the contour C is nipped by two poles which both approach the point $u = 1/e$. The expression of the irregular part involves two transcendental functions, viz., two roots of

$$1+xu\log u = 0,$$

or $u^u = \xi$, where $\xi = e^{-1/x}$. These functions have been (to a limited extent) studied by Eisenstein and Seidel.

Further examples are given by

$$\psi(u) = u^m(1-u)^n, \quad u^m\left(\log\frac{1}{u}\right)^n, \quad u(1+u), \quad \frac{u}{1+u}, \quad \dots.$$

7. Every admissible form of $\psi(u)$ (§ 6, beginning) gives rise to a family of series as extensive as that considered in my former paper. The preceding examples will sufficiently show the course of the argument in each case. I have purposely chosen an example—$\psi(u) = u(1-u)$—in which the details of the analysis differ from those necessary in the simplest case, viz., $\psi(u) = u$. The whole method is obviously capable of numerous generalisations which it is difficult and hardly worth while to attempt to classify. It is better to allow the generalisations to be suggested by the particular classes of series with which they enable us to deal.