

On the Stresses in the Earth's Crust before and after the Sinking of a Bore-hole

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XLVI. *On the Stresses in the Earth's Crust before and after the Sinking of a Bore-hole.* By C. CHREE, *Sc.D., LL.D., F.R.S.**

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§ 1. IN 'Nature,' October 20, 1904, p. 602, there appeared letters by Mr. G. Martin and the Hon. C. A. Parsons dealing with the size of the stresses in the Earth's crust and speculating as to what would happen if a hole were bored to a depth of twelve miles.

The letters indicate that some interest attaches to the problem, but its true character seems to stand in need of careful consideration. We know at present so little of the nature of the earth's material, even at such small depths as twelve miles, and have such scanty knowledge of the combined effects of high temperature and high pressure, that there are no data for making an *exact* calculation. Rocks as we know them at the earth's surface are not isotropic or even homogeneous solids, and they are not perfectly elastic for any considerable stresses †; but the crust of the earth seems to behave as a solid so far as can be inferred from earthquake-waves, and nothing that is positively known forbids the hypothesis that the material a few miles down is elastic for moderate changes of stress. Further, if any calculation is to be made of the internal stresses, there seems no alternative to the application

* Read March 10, 1905.

† Japanese Earthquake Investigation Committee, Reports No. 17.

of elastic solid theory. The following results based on the ordinary mathematical theory may thus be worth the consideration of engineers and others interested in the practical problem.

I propose first to present the results applicable to a series of mathematical problems, only one or two of which are absolutely novel, and then to consider what their bearing is on the problem now under discussion. Only the general character of the mathematical steps will be indicated.

§ 2. The two best known theories as to rupture, or more strictly as to the limits of application of the elastic solid theory, are :—

(i.) The maximum stress difference theory, according to which \bar{S} the greatest value of S , where S is the *difference* between the algebraically greatest and least of the three principal stresses at a point, must not exceed a certain experimental limit ;

(ii.) The greatest strain theory, according to which the limiting value attaches to the largest strain in any part of the structure, assuming that to be an extension.

Theory (i.) seems that least favourable to the permanence of the structure in such problems as those to be considered here, and attention is almost exclusively devoted to it. It has the recommendation that there seems considerable experimental evidence in favour of the view that it is the maximum stress difference on which depends the tendency to flow in solids under severe non-uniform pressure*.

Notation.

§ 3. In isotropic material m and n represent the two elastic constants in Thomson and Tait's notation. These are connected with Young's modulus E , and Poisson's ratio η , by the relations

$$m/1 = n/(1 - 2\eta) = E \div \{2(1 + \eta)(1 - 2\eta)\}. \quad (1)$$

In applications of spherical coordinates, r, θ, ϕ , the displacement along the radius vector is denoted by u ; in most of the

* Todhunter and Pearson's 'History of Elasticity,' vol. ii. art. 247, &c.

cases treated here the three principal stresses are the radial \widehat{rr} , and the two orthogonal stresses $\widehat{\theta\theta}$ and $\widehat{\phi\phi}$. Also $\widehat{\phi\phi}$ in the cases treated, with the exception of § 6, is equal to $\widehat{\theta\theta}$. The three principal strains are then du/dr , u/r , and u/r .

In cylindrical coordinates the displacements are u along r , the outward drawn perpendicular from the cylindrical axis, or axis of z , and w parallel to this axis. The principal stresses in the problems treated here are \widehat{rr} and \widehat{zz} , parallel respectively to r and z , and $\widehat{\phi\phi}$ perpendicular to these two directions. The corresponding principal strains are du/dr , dw/dz , and u/r .

The dilatation Δ is given in spherical and cylindrical coordinates respectively, in the cases of symmetry here considered, by

$$\Delta = du/dr + 2u/r, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\Delta = du/dr + u/r + dw/dz. \quad . \quad . \quad . \quad . \quad (3)$$

Homogeneous isotropic gravitating sphere.

§ 4. If a be the radius, ρ the density, and g 'gravity' at the surface, then, the centre being origin,

$$\left. \begin{aligned} u &= \frac{g\rho r}{10a(m+n)} \left\{ r^2 - a^2 \frac{5m+n}{3m-n} \right\}, \\ \widehat{rr} &= -g\rho(a^2 - r^2)(5m+n) \div \{10a(m+n)\}, \\ \widehat{\theta\theta} &= -g\rho\{a^2(5m+n) - r^2(5m-3n)\} \div \{10a(m+n)\}, \\ S &= 2g\rho r^2 n \div \{5a(m+n)\}, \\ \bar{S} \text{ (at surface)} &= 2g\rho a n \div \{5(m+n)\} \equiv \frac{1}{3}g\rho a(1-2\eta)/(1-\eta). \end{aligned} \right\} (4)$$

At points near the surface for which

$$h \equiv a - r \text{ is small,}$$

we have, retaining only the lowest power of h , as first approximations,

$$\left. \begin{aligned} \widehat{rr} &= -g\rho h(3-\eta) \div \{5(1-\eta)\}, \\ \widehat{\theta\theta} &= -S = -g\rho a(1-2\eta) \div \{5(1-\eta)\}. \end{aligned} \right\} \quad . \quad . \quad (5)$$

* Camb. Phil. Soc. Trans. vol. xiv. p. 281.

In the special case when the material is incompressible,

$$\text{i. e.} \quad n/m = 0, \quad \text{or} \quad \eta = 0.5,$$

we have throughout the mass of the sphere

$$\left. \begin{aligned} u &= 0, \\ \widehat{rr} = \widehat{\theta\theta} &= -g\rho(a^2 - r^2)/2a, \\ S = \bar{S} &= 0; \end{aligned} \right\} \quad \dots \quad (6)$$

and close to the surface

$$\widehat{rr} = \widehat{\theta\theta} = -g\rho h. \quad \dots \quad (7)$$

§ 5. Gravitating spherical "Earth," consisting of a core of radius b , density $\rho + \rho'$ and elastic constants m', n' , and of a crust or layer of density ρ , and elastic constants m, n , resting on the core and bounded externally by a spherical surface of radius a .

The expressions for the displacements and stresses are as follows:—

In the core:

$$\left. \begin{aligned} u &= \frac{1}{3}rA + \frac{2}{15}\pi \frac{(\rho + \rho')^2 r^3}{m' + n'}, \\ \widehat{rr} &= \frac{1}{3}(3m' - n')A + \frac{5m' + n'}{15(m' + n')} 2\pi(\rho + \rho')^2 r^2, \\ \widehat{\theta\theta} &= \frac{1}{3}(3m' - n')A + \frac{5m' - 3n'}{15(m' + n')} 2\pi(\rho + \rho')^2 r^2. \end{aligned} \right\} \quad \dots \quad (8)$$

In the layer:

$$\left. \begin{aligned} u &= \frac{1}{3}rB + r^{-2}C + \frac{2\pi\rho^2 r^3}{15(m+n)} - \frac{2\pi\rho\rho' b^3}{3(m+n)}, \\ \widehat{rr} &= \frac{1}{3}(3m-n)B - 4nr^{-3}C + \frac{5m+n}{15(m+n)} 2\pi\rho^2 r^2 - \frac{m-n}{m+n} \frac{4}{3} \frac{\pi\rho\rho' b^3}{r}, \\ \widehat{\theta\theta} &= \frac{1}{3}(3m-n)B + 2nr^{-3}C + \frac{5m-3n}{15(m+n)} 2\pi\rho^2 r^2 - \frac{m}{m+n} \frac{4}{3} \frac{\pi\rho\rho' b^3}{r}. \end{aligned} \right\} \quad (9)$$

Here A, B, C are three arbitrary constants to be determined by the surface conditions, and the gravitational force between two masses μ and μ' at distance R is taken as $(\mu\mu'/R^2) \times 1$.

We shall suppose the outer surface $r=a$ to be free from

force. If uniform normal pressure acts its effects are most easily obtained separately.

Supposing the outer surface $r=a$ free from force, \widehat{rr} must vanish over it, whence

$$\frac{1}{3}(3m-n)B - 4na^{-3}C = -\frac{5m+n}{15(m+n)} 2\pi\rho^2 a^2 + \frac{m-n}{m+n} \frac{4}{3} \pi\rho\rho' \frac{b^3}{a}. \quad (10)$$

Over the common surface $r=b$, the radial displacements must be the same for the core and layer, whence

$$B + 3b^{-3}C = A + \frac{2\pi(\rho+\rho')^2 b^2}{5 \frac{m'}{m} + n'} - \frac{2}{5} \frac{\pi\rho^2 b^2}{m+n} + \frac{2\pi\rho\rho' b^2}{m+n}. \quad (11)$$

Finally the values of \widehat{rr} in the core and layer must be equal over the common surface, and so

$$\begin{aligned} \frac{1}{3}(3m-n)B - 4nb^{-3}C = & \frac{1}{3}(3m'-n')A + \frac{5m'+n'}{15(\frac{m'}{m}+n')} 2\pi(\rho+\rho')^2 b^2 \\ & - \frac{5m+n}{15(m+n)} 2\pi\rho^2 b^2 + \frac{m-n}{m+n} \frac{4}{3} \pi\rho\rho' b^2. \quad (12) \end{aligned}$$

These three equations determine A, B, and C without ambiguity, and there is no difficulty, except in the length of the expressions, in obtaining the solution for the general case.

As we shall see, however, physical interest is mainly if not entirely restricted to the case when the material of the core is incompressible or very nearly so, and I shall thus limit myself to the case when n'/m' is negligible, n' being finite.

From (12) we see that when n'/m' is negligible A must vanish. For n' being finite, m' must be infinite, and so the one term $\frac{1}{3}(3m'-n')A$ unless A vanished would become infinite, whilst the other terms in the equation would remain finite. We must thus treat A as negligible, except when multiplied by m' , and regard (12) as simply determining the (finite) value of $(3m'-n')A/3$.

Our surface equations thus reduce to two, viz. (10) and

$$B + 3b^{-3}C = -\frac{2}{5} \frac{\pi\rho^2 b^2}{m+n} + \frac{2\pi\rho\rho' b^2}{m+n}, \quad . \quad . \quad (11')$$

this being the form taken by (11) when A is negligible and m' is infinite.

The values of B and C may be at once written down from (10) and (11'). Confining our attention to the case where the thickness $a-b=t$ of the crust is so small that $(t/a)^2$ is negligible, we find

$$\begin{aligned} B &= 2\pi\rho a^2\{-\rho+2\rho'(1-3t/a)\} \div \{3(m+n)\}, \\ C &= 2\pi\rho a^3\{\frac{2}{3}\rho+\rho'(1-3t/a)\} \div \{9(m+n)\}. \end{aligned} \quad (13)$$

Substituting these values of B and C in (9), then putting $r=a-h$, and neglecting $(h/a)^2$, we find after reduction

$$\left. \begin{aligned} \widehat{rr} &= -\frac{4}{3}\pi\rho ah\{\rho+\rho'-3(t/a)\rho'\}, \\ \widehat{\theta\theta} &= -\frac{4}{3}\pi\rho ah\frac{m-n}{m+n}\{\rho+\rho'-3(t/a)\rho'\}. \end{aligned} \right\} \quad (14)$$

But if g be the acceleration of gravity at the surface

$$\begin{aligned} g &= (4\pi/3)\{\rho a^3+\rho'b^3\}/a^2, \\ &= \frac{4}{3}\pi a\{\rho+\rho'-(3t/a)\rho'\}, \end{aligned}$$

when $(t/a)^2$ is neglected.

Thus we have in the crust :

$$\left. \begin{aligned} \widehat{rr} &= -g\rho h, \\ \widehat{\theta\theta} &= -g\rho h(m-n)/(m+n) \equiv -g\rho h\eta/(1-\eta), \\ S &= g\rho h(1-2\eta)/(1-\eta). \end{aligned} \right\} \quad (15)$$

§ 6. Slightly spheroidal, homogeneous, gravitating and rotating earth.

The surface values of the stresses, which are by no means very complicated, will suffice for our immediate object.

Let $2a$, $2c$ represent the equatorial and polar diameters, r the perpendicular from any point on the polar axis, p the perpendicular from the centre on the tangent plane, ρ the density, G the gravitational force between two unit masses at unit distance, and ω the angular velocity.

At the surface the principal stresses are \widehat{nn} along the normal, \widehat{tt} along the tangent in the meridian plane, and $\widehat{\phi\phi}$ perpendicular to the meridian plane.

Neglecting terms of order $(1-c^2/a^2)^2$ we find *

$$\begin{aligned}\widehat{nn} &= 0, \\ \widehat{tt} &= -\frac{4\pi G\rho^2 a^2}{15(1-\eta)} \left\{ \frac{c^2}{p^2}(1-2\eta) + \frac{a^2-c^2}{a^2} \frac{1-\eta-4\eta^2}{7+5\eta} \right\} \\ &\quad + \frac{\omega^2 \rho a^2 (c/p)^2}{5(1-\eta)(7+5\eta)^2} \left\{ (7+5\eta)(3-6\eta-5\eta^2) + 4 \frac{a^2-c^2}{a^2} (1+\eta)(6-5\eta-5\eta^2) \right\}, \\ \widehat{\psi\psi} &= -\frac{4\pi G\rho^2 a^2}{15(1-\eta)} \left\{ 1-2\eta + \frac{a^2-c^2}{a^2} \left(1-3\frac{\eta^2}{a^2} \right) \frac{1-\eta-4\eta^2}{7+5\eta} \right\} \\ &\quad + \frac{\omega^2 \rho a^2}{5(1-\eta)(7+5\eta)^2} \left[(\eta+5\eta)(3-6\eta-5\eta^2) + 4 \frac{a^2-c^2}{a^2} (1+\eta)(6-5\eta-5\eta^2) \right. \\ &\quad \left. + \frac{\eta^2}{a^2} \left\{ 5(7+5\eta)(1-\eta^2) - \frac{a^2-c^2}{a^2} (30-4\eta-82\eta^2-41\eta^3) \right\} \right].\end{aligned}$$

When $c=a$, $4\pi bGa/3=g$ the gravitational acceleration at the surface. Thus in a perfect sphere the contributions from the gravitational terms to \widehat{tt} and $\widehat{\phi\phi}$ are equal, and agree with the value $-g\rho a(1-2\eta) \div \{5(1-\eta)\}$ given in (5) of § 4 for $\widehat{\theta\theta}$; they vanish when the material is incompressible.

The contributions from the "centrifugal force" terms to the surface stresses are in a perfect sphere, λ denoting the latitude,

$$\left. \begin{aligned}\widehat{tt} &= \theta\theta = \omega^2 \rho a^2 (3-6\eta-5\eta^2) \div \{5(1-\eta)(7+5\eta)\}, \\ \widehat{\phi\phi} &= \frac{\omega^2 \rho a^2}{5(1-\eta)(7+5\eta)} \left\{ 3-6\eta-5\eta^2 + 5 \cos^2 \lambda (1-\eta^2) \right\}.\end{aligned} \right\} (17)$$

The value of $\widehat{\theta\theta}$ is thus constant over the surface. Its sign depends on that of $3-6\eta-5\eta^2$. This expression vanishes when $\eta = .3798$, or $n/m = .2404$. It also vanishes when $m/n = .2404$, but this latter value is physically impossible. Thus $\widehat{\theta\theta}$ is positive or negative, *i. e.* a tension or a pressure, according as η is less or greater than 0.38. $\widehat{\phi\phi}$ is always positive in the equator; at the poles it is equal to $\widehat{\theta\theta}$.

When $\widehat{\phi\phi}$ is positive and $\widehat{\theta\theta}$ negative (which implies η being $>.3798$) $\widehat{\phi\phi} - \widehat{\theta\theta}$ is the maximum stress difference, and the greatest value occurs in the equator where

$$S = \omega^2 \rho a^2 (1+\eta)/(7+5\eta). \quad . \quad . \quad . \quad (18)$$

* Roy. Soc. Proc. vol. lviii. eqns. (15), p. 43, and eqns. (23) to (26), pp. 44 and 45.

This increases slightly with η , varying from $0.155\omega^2\rho a^2$ when $\eta = .3798$ to $0.158\omega^2\rho a^2$ when $\eta = 0.5$.

Supposing the earth an incompressible sphere in which $\rho = 5.5$, $a = 3963$ miles, and $\omega^2 a = g/293$, we find for the maximum *surface* value of S

$$\bar{S} = 12.0 \text{ tons weight per square inch.} \quad . \quad . \quad (19)$$

Except for incompressible material, the principal terms depending on ω^2 in \widehat{tt} and $\widehat{\phi\phi}$ are small compared with the principal gravitational terms; and when the material is incompressible the secondary terms depending on ω^2 which contain $a^2 - c^2$ are small compared with the secondary gravitational terms. Thus we may in general neglect the secondary terms in ω^2 for a first approximation. We may also neglect the differences between a , c , and p in all terms depending on ω or containing $a^2 - c^2$ as a factor, and may replace $4\pi G\rho a/3$ by g , where g is the mean value of gravity over the surface.

Doing so, we find as first approximations to the principal stresses in the surface of an incompressible nearly spherical spheroid

$$\left. \begin{aligned} \widehat{nn} &= 0, \\ \widehat{tt} &= \frac{1}{19} g\rho a \left(\frac{4}{5} \frac{a-c}{a} - \frac{\omega^2 a}{g} \right), \\ \widehat{\phi\phi} &= \frac{1}{19} g\rho a \left(\frac{4}{5} \frac{a-c}{a} - \frac{\omega^2 a}{g} \right) \left(1 - 3 \frac{r^2}{a^2} \right). \end{aligned} \right\} \quad . \quad (20)$$

Thus \widehat{tt} and $\widehat{\phi\phi}$ would both vanish all over the surface if

$$\frac{4}{5} \frac{a-c}{a} = \frac{\omega^2 a}{g}.$$

In the actual earth

$$\omega^2 a/g = 1/290 \text{ approximately.}$$

Bessel's value for $(a-c)/a$ is $1/299$, while Clarke's is $1/293.5$. Taking the former value, and employing as before λ to denote the latitude, we find in the surface

$$\left. \begin{aligned} \widehat{tt} &= -0.90 \quad . \quad . \quad \text{tons weight per square inch.} \\ \widehat{\phi\phi} &= -0.90(1 - 3 \cos^2 \lambda) \quad . \quad . \quad . \\ \bar{S} &= 2.7 \quad . \quad . \quad . \end{aligned} \right\} \quad (21)$$

The following particular values of $\widehat{\phi\phi}$ (all in tons weight per square inch) should be noted:—

Equator.	Lat. $35^{\circ} 16'$.	Lat. $54^{\circ} 44'$.	Poles.
+1.8	+0.9	0.0	-0.9

The stress in the meridian plane is always a pressure when the material is incompressible; but that perpendicular to the meridian is a tension in latitudes below $54\frac{3}{4}^{\circ}$. The numerical values of these surface stresses would be reduced by assigning to the earth's compression $\overline{a-c}/a$ any value, such as Clarke's, which is larger than Bessel's.

§ 7. Right vertical prism of density ρ , acted on by gravity, the prismatic surface being exposed to normal pressure $p-Cz$, where p and C are constants, and the upper plane surface $z=h$ being free from force.

To give definiteness to the problem we shall suppose the vertical displacement nil at the C.G. of the base. The shape of the cross section does not matter.

The solution is obtainable from one on p. 545 of the Phil. Mag. for November 1901, by suitably altering the notation.

Taking rectangular axes of x, y, z , the axis of z being drawn vertically upwards, and the origin being at the C.G. of the base of the prism, the displacements α, β, γ are given by

$$\begin{aligned}\alpha/x &= \beta/y = (1/E)\{\eta g\rho(h-z) - (1-\eta)(p-Cz)\}, \\ \gamma &= -(z/E)\{g\rho(h-\tfrac{1}{2}z) - \eta(2p-Cz)\} + (1/2E)(x^2+y^2)\{g\rho\eta - (1-\eta)C\}\end{aligned}$$

The principal stresses and the stress-difference are given by

$$\left. \begin{aligned}\widehat{xx} &= \widehat{yy} = -(p-Cz) = -p_H, \\ \widehat{zz} &= -g\rho(h-z) = -p_V, \\ S &= p_V \sim p_H,\end{aligned}\right\} \quad \dots \quad (23)$$

where p_V and p_H are the pressures at the depths in question in the vertical and horizontal directions respectively.

Assuming $p-Cz$ everywhere positive, the stresses are all numerically greatest over the base $z=0$; also

$$S_{z=0} = g\rho h \sim p \quad \dots \quad (24)$$

§ 8. Hollow, vertical, circular cylinder of density ρ , acted on by gravity, the outer cylindrical surface $r=a$ being exposed to normal pressure $p-Cz$, where p and C are constants, the inner cylindrical surface $r=a'$ and the upper plane surface $z=h$ being free from force.

The solution is obtainable from one on p. 550 of the Phil. Mag. for November 1901 by suitably altering the notation. It is, in cylindrical coordinates,

$$\left. \begin{aligned} u &= (r/E) \left[g\rho\eta(h-z) - (1-\eta) \frac{a^2(p-Cz)}{a^2-a'^2} \right] - \frac{a^2a'^2(p-Cz)}{2nr(a^2-a'^2)}, \\ w &= (z/E) \left[-gp h + \frac{2\eta a^2 p}{a^2-a'^2} \right] + \frac{1}{2} \frac{g\rho}{E} (z^2 + \eta r^2) \\ &\quad - \frac{Ca^2}{2(a^2-a'^2)} \left\{ \frac{(1-\eta)r^2 + 2\eta z^2}{E} + \frac{a'^2}{n} \log(r/r_1) \right\}. \end{aligned} \right\} \quad (25)$$

Here r_1 is a constant which can be determined so that w has any assigned value at any given point.

For the stresses we have

$$\left. \begin{aligned} \widehat{rr} &= -\frac{a^2}{a^2-a'^2} \left(1 - \frac{a'^2}{r^2} \right) (p-Cz), \\ \widehat{\phi\phi} &= -\frac{a^2}{a^2-a'^2} \left(1 + \frac{a'^2}{r^2} \right) (p-Cz), \\ \widehat{zz} &= -g\rho(h-z). \end{aligned} \right\} \quad \dots \quad (26)$$

Assuming p , C , and $p-Ch$ positive, the stresses are all negative and numerically greatest at the base of the cylinder, where z vanishes; and the greatest value of the maximum stress difference—which is met with over the inner surface—is the greater of the two quantities:—

$$\left. \begin{aligned} \bar{S}_1 &= 2pa^2/(a^2-a'^2), \\ \bar{S}_2 &= g\rho h. \end{aligned} \right\} \quad \dots \quad (27)$$

When a'/a is very small, then very approximately

$$\bar{S}_1 = 2p. \quad \dots \quad (28)$$

§ 9. Right prism exposed to uniform pressures P and p over its flat ends and prismatic surface respectively. The

axis of z being along the prismatic axis, the stresses and stress-difference are given by

$$\left. \begin{aligned} \widehat{xx} = \widehat{yy} &= -p, \\ \widehat{zz} &= -P, \\ S = \bar{S} &= P \sim p. \end{aligned} \right\} \dots \dots \dots (29)$$

§ 10. Hollow right circular cylinder, outer radius a , inner a' , exposed to pressures p and P over its external cylindrical surface and flat ends respectively.

The stresses are

$$\left. \begin{aligned} \widehat{rr} &= -pa^2(1 - a'^2/r^2)/(a^2 - a'^2), \\ \widehat{\phi\phi} &= -pa^2(1 + a'^2/r^2)/(a^2 - a'^2), \\ \widehat{zz} &= -P. \end{aligned} \right\} \dots \dots (30)$$

At a given distance r from the axis the maximum stress difference may according to circumstances be any one of the three following:—

$$\left. \begin{aligned} S_1 = \widehat{rr} - \widehat{\phi\phi} &= 2pa^2a'^2 \div r^2(a^2 - a'^2), \\ S_2 = \widehat{rr} - \widehat{zz} &= P - pa'^2(1 - a'^2/r^2)/(a^2 - a'^2), \\ S_3 = \widehat{zz} - \widehat{\phi\phi} &= -P + pa^2(1 + a'^2/r^2)/(a^2 - a'^2). \end{aligned} \right\} \dots (31)$$

But for the greatest value of S there are only two options, both referring to the inner surface of the cylinder, viz.:—

$$\left. \begin{aligned} \bar{S}_1 &= 2pa^2/(a^2 - a'^2), \\ \bar{S}_2 &= P. \end{aligned} \right\} \dots \dots \dots (32)$$

When a'/a is very small, a close approximation is

$$\bar{S}_1 = 2p. \dots \dots \dots (33)$$

§ 11. Solid sphere of radius a , exposed to uniform surface-pressure p .

The solution is

$$\left. \begin{aligned} u/r &= du/dr = -p/(3m - n), \\ \widehat{rr} = \widehat{\theta\theta} &= -p, \\ S &= 0. \end{aligned} \right\} \dots \dots (34)$$

§ 12. Spherical shell outer radius a , inner a' , exposed to uniform pressure p over the outer surface.

The stresses and stress-difference are given by

$$\left. \begin{aligned} \widehat{rr} &= -pa^3(1-a'^3/r^3)/(a^3-a'^3), \\ \widehat{\theta\theta} &= -pa^3(1+a'^3/2r^3)/(a^3-a'^3), \\ S &= \frac{3}{2}p(aa')^3 \div \{r^3(a^3-a'^3)\}, \\ \bar{S} \text{ (at inner surface)} &= \frac{3}{2}pa^3/(a^3-a'^3). \end{aligned} \right\} \quad (35)$$

When a'/a is very small a closely approximate value is

$$\bar{S} = 3p/2 \quad (36)$$

§ 13. Material bounded by an infinite plane,—on one side of which it extends to infinity,—acted on by a surface-tension uniformly distributed over a circular area of radius a' .

For clearness suppose the material to be on the lower side of the plane $z=0$, the tension T , per unit area, acting vertically upwards. Let z be measured positively downwards from an origin at the centre of the stressed area, and let R denote the distance between an element $d\sigma$ of the stressed area and a point r, z in the solid, r being the perpendicular on the axis of z .

The solution, as obtained by Boussinesq and Cerruti *, is

$$\left. \begin{aligned} u &= (T/4\pi n) \frac{d}{dr} \left\{ \frac{d}{dz} \iint R d\sigma + (1-2\eta) \iint \log(z+R) d\sigma \right\}, \\ w &= (T/4\pi n) \left\{ \frac{d^2}{dz^2} \iint R d\sigma - (3-2\eta) \iint \frac{1}{R} d\sigma \right\}, \\ \Delta &= -(T/2\pi n)(1-2\eta) \frac{d}{dz} \iint \frac{1}{R} d\sigma. \end{aligned} \right\} \quad (37)$$

The integrals extend over the whole of the stressed area, *i. e.* over the area enclosed by the circle $r=a'$.

The above integrals in their general form are somewhat unmanageable. At a point whose distance from the stressed area is such that a'/R is always small, closely approximate values for the displacements are

$$\left. \begin{aligned} u &= -\frac{Ta'^2r}{4nR} \left\{ \frac{z}{R^2} - \frac{(1-2\eta)}{R+z} \right\}, \\ w &= -\frac{Ta'^2}{4nR} \left\{ \left(\frac{z}{R} \right)^2 + 2(1-\eta) \right\} \end{aligned} \right\}; \quad (38)$$

where R now denotes $\sqrt{r^2+z^2}$.

* Todhunter and Pearson's 'History of Elasticity,' vol. ii. art. 1492, equations (ix.).

Along a given radius the above displacements vary inversely as the distance from the origin, *i. e.* from the centre of the stressed area; consequently the strains and stresses vary inversely as the square of this distance. For instance, at a point at a distance z vertically below the centre of the stressed area, a'/z being small,

$$\left. \begin{aligned} \widehat{rr} = \widehat{\phi\phi} &= -(1-2\eta)Ta'^2/(4z^2), \\ \widehat{zz} &= 3Ta'^2/(2z^2) \end{aligned} \right\}; \dots (39)$$

while in the plane of the loaded area, but outside it, a'/r being small,

$$\left. \begin{aligned} -\widehat{rr} = \widehat{\phi\phi} &= (1-2\eta)Ta'^2/2r^2, \\ \widehat{zz} &= 0. \end{aligned} \right\}. \dots (40)$$

Thus the effects diminish very rapidly as the distance from the loaded area increases, and so far as rupture is concerned interest centres in the material close to the loaded area. The determination of what happens close to the loaded area is in general rather a delicate operation, especially at points situated close to its boundary.

At any point on the axis of z

$$R^2 = z^2 + r'^2,$$

where r' is the distance of the element $d\sigma$ from the centre of the loaded area, and so

$$dR/dz = z/R.$$

Thus from the last of equations (37)

$$\begin{aligned} \frac{2\pi n\Delta}{(1-2\eta)T} &= \iint \frac{z}{R^3} d\sigma = \int_0^{a'} \frac{z \cdot 2\pi r' dr'}{(z^2 + r'^2)^{\frac{3}{2}}} \\ &= 2\pi \{1 - z/(a'^2 + z^2)^{\frac{1}{2}}\}. \end{aligned}$$

Thus in the limit when $z=0$, *i. e.* at the centre of the loaded area,

$$\Delta = T(1-2\eta)/n. \dots (41)$$

Now by symmetry it is clear that \widehat{rr} , $\widehat{\phi\phi}$, \widehat{zz} are the principal stresses at the centre, and also that

$$\widehat{rr} = \widehat{\phi\phi}.$$

Thus

$$\begin{aligned} 2\widehat{rr} + \widehat{zz} &= \widehat{rr} + \widehat{\phi\phi} + \widehat{zz} \\ &= (3m-n)\Delta = 2(1+\eta)T. \end{aligned}$$

But by the surface conditions

$$\widehat{zz} = T.$$

Hence
$$\left. \begin{aligned} \widehat{rr} &= (1+2\eta)T/2, \\ \text{and } S &= \widehat{zz} - \widehat{rr} = (1-2\eta)T/2. \end{aligned} \right\} \quad \dots \dots (42)$$

As this method of obtaining the stresses is very artificial, I may add that I have deduced directly from the second of equations (37) for points in the axis of z

$$w = (T/2n)[(1-2\eta)z - 2(1-\eta)(a'^2 + z^2)^{\frac{3}{2}} + z^2(a'^2 + z^2)^{-\frac{1}{2}}].$$

Thus in the limit when $z=0$, *i. e.* at the centre of the loaded area,

$$dw/dz = T(1-2\eta)/2n.$$

I have also succeeded in deducing from the first of equations (37) as limiting values when $z=0$

$$\alpha/x = \beta/y = u/r = du/dr = T(1-2\eta)/4n.$$

These values for the strains are in harmony with the values given above for the stresses.

In the case of tension or pressure over a very small area it is practically immaterial whether the bounding surface is a plane or a sphere of large radius. Thus in practice we can regard (15) and (37) as applicable to an elastic "Earth" so long as h is a very small fraction of the radius. Superposing the two stress systems, and replacing T by $g\rho h$, we have a small area $\pi a'^2$ at a depth h from the surface free from pressure, whilst other elements at the same depth experience a pressure $g\rho h$. The solution thus answers closely to the conditions existing below a bore-hole of depth h . At the base of the hole, at its centre, we find combining (15) and (42), using the notation of (15),

$$\left. \begin{aligned} \widehat{rr} &= 0, \\ S &= \widehat{\theta\theta} = g\rho h \frac{(1-2\eta)}{1-\eta} \frac{1+\eta}{2} \end{aligned} \right\} \quad \dots \dots (43)$$

This value of S it will be noticed bears to that in (15) the ratio

$$1 + \eta : 2.$$

Applications to the Earth.

§ 14. The first question to consider is the state of matters prior to the existence of the bore-hole. The Earth is spheroidal and rotates about its axis of figure, circumstances which introduce variety into the conditions in different latitudes. Results such as (17), (18), and (19) show that an elastic sphere of the Earth's size rotating in a day must suffer stresses from the "centrifugal forces" which are very considerable even for material such as steel. The results (20) and (21) show, however, that the stresses due to centrifugal force in a homogeneous Earth are at least in large measure neutralized by the compensating action of the spheroidal form on the gravitational stresses. In our ignorance of the distribution of density and elasticity throughout the Earth, there is necessarily uncertainty as to the extent of this compensating action. The compensation may be less exact than according to (21), but it is at least as likely to be more exact. Further, whilst the pure centrifugal stresses are very considerable at the surface, their rate of variation with the depth is small. There is thus every reason to believe that whatever may be the combined effect of rotation and ellipticity, it will not produce throughout a bore-hole only ten or twelve miles deep effects differing in any essential respect from those observed in ordinary mines. Owing to centrifugal forces there may be a slightly greater tendency for a bore-hole to collapse in a north-south than in an east-west direction. Rotation may slightly facilitate collapse; but everything points to the conclusion that its effects, whether direct or indirect, are only of secondary importance.

§ 15. A more serious source of uncertainty arises in connexion with the value (5) of the horizontal stress $\widehat{\theta\theta}$ in a homogeneous gravitating sphere. If, for example, we suppose η equal 0.25, which is approximately true of glass or iron, the value of $-\widehat{\theta\theta}$ is $(2/15)g\rho a$; this means a pressure answering to the weight of a column of density ρ some 500 miles high. Near the Earth's surface this enormous horizontal pressure would exist in company with only a relatively small vertical pressure; and under such conditions no known material would continue an elastic solid. There is further

the observed fact that near the Earth's surface large horizontal stresses are normally at least non-existent. If they occurred, mines could not be constructed. No theory, in short, of the Earth's condition can well be entertained which is incompatible with horizontal stresses remaining small at moderate depths.

The most obvious way out of the difficulty is to assume the homogeneous "Earth" incompressible. We then, as appears from (6) and (7), have the horizontal stresses vanishing at the surface equally with the vertical stresses. To this hypothesis there is, however, the objection that no known material is incompressible, and that the materials of which the Earth's crust is composed do not, so far as is known, show any close approach to such a condition.

The hypothesis § 5, which limits the incompressibility to the deep-seated material, is less open to criticism. At great depths, according to any view yet suggested, the material must be exposed to very severe pressure; and it is difficult to imagine material exposed to pressure of hundreds of tons weight on the square inch being anything but nearly incompressible under variations of that pressure. The hypothesis of an incompressible nucleus and compressible crust is of course at best only an approximation to the truth. A theory which treated the material as varying continuously with the depth, or as consisting of a large number of homogeneous layers, would presumably be more exact. Its complication, however, would be great; and in the absence of data the simplest consistent hypothesis seems the best.

§ 16. The above considerations indicate that in the absence of a bore-hole the stresses at depths of a few miles in reasonably homogeneous strata are probably not widely different from those appearing in (15). According to this equation the pressure on a horizontal plane equals the weight of the material between it and the surface. This pressure, in short, is the same as if the superincumbent material were liquid. The result implies that the superincumbent material is so much dead weight, and does not act to an appreciable extent as a protecting arch. In reality, some slight action of this kind is likely to exist. In not a few cases, the bending visible in rocks is suggestive of large horizontal thrusts. The

condition, however, of the material when the bending occurred may have been plastic.

An idea of the probable diminution in the vertical pressure due to the horizontal thrust in superincumbent material may be derived from (5). If in this case we suppose η equal $1/4$ the vertical pressure, $-\widehat{rr}$, is only $11/15$ of its value when the material is incompressible; but the horizontal or arching pressures to which this reduction is due (*i. e.* the value of $-\widehat{\theta\theta}$ in (5)) if we suppose the density to be 5.5 , amounts at the surface to nearly 3000 tons weight on the square inch. The natural inference is that whilst gph is a maximum estimate for the vertical pressure on a horizontal plane, it is unlikely to be many per cent. in error.

As regards horizontal pressure, the largest value given by (15) for $-\widehat{\theta\theta}$ is gph , answering to $\eta=0.5$. It follows from our previous considerations that if shallow borings show no considerable horizontal pressure, the value of this pressure at depth h is unlikely to be much in excess of gph . On the other hand, according to (15), $-\widehat{\theta\theta}$ vanishes when $\eta=0$, and is only $gph/3$ when $\eta=\frac{1}{4}$. It is thus by no means improbable that the horizontal pressure may be very considerably less than gph , and so less than the vertical pressure.

Noticing that \widehat{rr} in (15) is really a vertical stress, it is obvious that the stress systems (15) and (23) are of the same type. The system (23) is the more general, in so far as it does not assume any relationship between the magnitudes of the vertical and horizontal pressures, and does not make the latter necessarily vanish at the upper surface. The material in the Earth's crust thus presents the same elastic conditions as a vertical prism of similar density acted on by gravity, provided the latter be exposed to suitable horizontal pressures whose intensity depends only on the level.

§ 17. As a numerical example, let us take Mr. Martin's case, where h is twelve miles and ρ thrice the density of water. Employing these values in (15) or (23), we find for the vertical pressure at this depth

$$-\widehat{zz}=36.8 \text{ tons weight per square inch.}$$

Mr. Martin himself speaks of 440 tons, which suggests the

omission of 12 in some divisor. Mr. Parsons gives 40 tons as a rough approximation, which agrees substantially with the above estimate. If the prism existed as an isolated pillar, it would, neglecting atmospheric pressures, be free from horizontal pressure, and the maximum stress-difference at its base would be simply the above value of $-\widehat{zz}$, or 36.8 tons weight on the square inch. But when the prism forms part of the Earth's crust, horizontal pressures exist; and so long as they are less than gph , the larger they are the smaller is S . If, for example, we supposed the horizontal pressure to be given by $-\widehat{\theta\theta}$ in (15) with $\eta=1/4$ we should find S reduced to 24.4 tons weight per square inch.

Comparing (15) and (23) with (29), we see that our reasoning points to the conclusion that the material at any given depth in the crust, like the material in a gravitating prism, is under the same condition as a short non-gravitating prism over whose flat ends and curved surface there act *uniform* pressures, equal respectively to the vertical and horizontal pressures actually existent at the point in the crust. Thus experiments on a solid prism might supply definite information as to the elastic state of the Earth's crust if rocks were available of the same composition as the more deep-seated material, and the temperature at which the experiments were made was sufficiently high.

Effects of a Bore-hole.

§ 18. The existence of a bore-hole may profoundly influence the conditions in its immediate neighbourhood; but clearly it will not produce an appreciable effect at distances from the hole which are a large multiple of its diameter. Over the cylindrical surface of the hole the normal stress must vanish, *i. e.* the horizontal pressure must be *nil*, supposing the hole vertical. Thus if we suppose an imaginary cylindrical surface coaxial with the bore-hole, but of relatively large diameter, the material within it will be practically in the state of the hollow cylinder of § 8 when we suppose a'/a very small. From this conclusion we must exclude the material immediately surrounding the extreme foot of the hole, whose condition will be somewhat modified. Excluding

this very limited volume, we infer from (27) and (28) that the maximum stress-difference S at depth h , which was $gph \sim p$ before the hole was bored, is, subsequent to the boring, the greater of the two quantities $2p$ and gph . Here p denotes the horizontal pressure at depth h at a distance from the hole.

If p vanishes, *i. e.* if the material is in the condition of an isolated pillar, the presence of the bore-hole makes no difference.

On the hypothesis of a compressible crust on an incompressible nucleus, after the hole is formed S equals gph so long as η does not exceed $1/3$; thereafter it increases with η to a maximum of $2gph$ when η equals 0.5 . Particulars on this hypothesis are as follows:—

		Value of S/gph .							
		$\eta = 0$	0.2	.25	.3	.3	.4	.45	.5
Before hole made	1	0.75	0.6	0.57	0.5	0.3	0.18	.00
After	„ „	1	1	1	1	1	1.3	1.64	2.00

When a bore-hole is formed the engineer may thus anticipate over its surface a stress-difference varying from p to $2p$ according to circumstances, where p represents the hydrostatic pressure at the same depth in a liquid of the same mean density as the superincumbent material.

From the identity in type of the results in §§ 8 and 10, the probable behaviour of the tube-walls may be deduced from experiments in which a circular cylinder of the material with a coaxial hole of small bore is exposed to the combined effects of uniform pressures over its flat ends and curved surface. The former should equal the pressure in a liquid, whose density equals that of the crust, at a depth equal to that of the projected boring; the latter may be as large as the former or have any smaller value. Under the conditions supposed by Mr. Martin the pressures, in tons weight on the square inch, would be

36.8 on the flat ends,

from 0 to 36.8 on the curved surface.

§ 19. Near the bottom of the hole the conditions are complicated. An approximate idea of the conditions immediately

below the hole may be derived from § 13. At the centre of the bottom of the hole the tendency to rupture according to (43) is actually less than before the hole was bored, unless the material be incompressible when it vanishes in either case. This diminution is almost certainly confined to the central part of the base of the hole. When a core is hollowed out of a cylinder exposed to uniform pressure p the stress-difference—as shown in § 10—rises to $2p$. Under the same circumstances, when a core is taken out of a sphere the stress-difference—as shown in (36)—becomes $3p/2$. The conditions at the edge of the base of the hole seem more likely to approach the conditions in the sphere than those in the cylinder. Thus while uncertainty prevails as to the conditions in the base—a good deal depending in practice on the borer—this seems on the whole not so weak a spot as the walls of the boring a few diameters of the bore above the base.

The above considerations unite in indicating that the largest value to be anticipated for the maximum stress-difference anywhere over the surface of the bore-hole is $2p$, where p is the hydrostatic pressure that would exist at the bottom of the hole if filled with a liquid whose density equals the mean density of the core extracted.

§ 20. There are, however, two contingencies to be borne in mind. Whilst large horizontal pressures of the order of hundreds of tons weight on the square inch may be regarded as practically impossible near the surface, it is quite conceivable that smaller but still large horizontal pressures may exist in strata which differ notably from adjacent material. If a bore-hole should pierce such a stratum a nipping action may well ensue. Again, whilst no one is likely to select the immediate neighbourhood of an active volcano as a site for a boring, it might be well in any case to be prepared for the possibility of piercing material in so unstable a condition that sudden relief of pressure may lead to semi-volcanic action.