A Case of Complex Multiplication with Imaginary Modulus arising out of the Cubic Transformation in Elliptic Functions.

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The case in question is referred to in my "Note on the Theory of Elliptic Integrals" (*Math. Ann.*, XII. (1877), pp. 143—146); but I here work it out directly.

In the cubic transformation the modular equation is

$$u^4-v^4+2uv(1-u^2v^2) = 0,$$

and we have

$$y = \frac{\left(1+\frac{2u^3}{v}\right)x+\frac{u^6}{v^2}x^3}{1+vu^2(v+2u^3)x^2}, \text{ giving } \frac{dy}{\sqrt{1-y^2.1-v^8y^2}} = \frac{\left(1+\frac{2u^3}{v}\right)dx}{\sqrt{1-x^2.1-u^8x^2}}.$$

We thus have a case of complex multiplication if $v^8 = u^8$, or say $v = \gamma u$, where $\gamma^8 = 1$, or γ denotes an eighth root of unity. Substituting in the modular equation, this becomes

$$u^4(1-\gamma^4)+2\gamma u^2(1-\gamma^2u^4) = 0,$$

or, throwing out the factor u^2 and reducing,

$$u^4-\tfrac{1}{2}u^2(\gamma^5-\gamma)-\gamma^6 = 0,$$

that is,
$$\frac{u^2}{\gamma} = \tfrac{1}{4}(\gamma^4-1 \pm \sqrt{\gamma^8+14\gamma^4+1}),$$

or, what is the same thing,

$$= \tfrac{1}{4}\{\gamma^4-1 \pm \sqrt{14\gamma^4+2}\}$$

We have $\gamma^8 = 1$, that is, $\gamma^4 = \pm 1$. Considering first the case $\gamma^4 = 1$, here

$$\frac{u^2}{\gamma} = \pm 1,$$

and thence
$$1+\frac{2u^3}{v} = 1+\frac{2u^2}{\gamma}, =1\pm 2, =3 \text{ or } -1;$$

moreover, $u^8 = v^8 = 1$. We have thus only the non-elliptic formulæ

$$\frac{dy}{1-y^2} = \frac{-dx}{1-x^2}, \text{ satisfied by } y = -x,$$

and
$$\frac{dy}{1-y^2} = \frac{3dx}{1-x^2}, \qquad \text{by } y = \frac{3x+x^3}{1+3x^2}.$$

* [This Note was written by way of illustration of Mr. Greenhill's paper, and is printed here at Prof. Cayley's suggestion.]

If however, $\gamma^4 = -1$, then

$$\frac{u^2}{\gamma} = \tfrac{1}{4}(-2 \pm \sqrt{-12}),$$

viz., this is
$$\frac{u^2}{\gamma} = \tfrac{1}{2}(-1 \pm i\sqrt{3}) = \omega$$

if ω be an imaginary cube root of unity $(\omega^2+\omega+1 = 0)$; hence

$$u^8 = (\gamma\omega)^4 = -\omega.$$

Moreover,
$$1+\frac{2u^8}{v} = 1+\frac{2u^2}{\gamma}, \;= 1+2\omega,$$

or say, $\quad = \omega-\omega^2, \quad [= \sqrt{-3} \;\text{ if } \omega = \tfrac{1}{2}(-1+i\sqrt{3})]$;

and we thus have, as in the above-mentioned Note,

$$y = \frac{(\omega-\omega^2)\,x+\omega^2x^3}{1-\omega^2(\omega-\omega^2)\,x^2}, \text{ giving } \frac{dy}{\sqrt{1-y^2\,.\,1+\omega y^2}} = \frac{(\omega-\omega^2)\,dx}{\sqrt{1-x^2\,.\,1+\omega x^2}};$$

or, what is the same thing, for the modulus $k^2 = -\omega$, we have

$$\operatorname{sn}(\omega-\omega^2)\,\theta = \frac{(\omega-\omega^2)\operatorname{sn}\theta+\omega^2\operatorname{sn}^3\theta}{1-\omega^2(\omega-\omega^2)\operatorname{sn}^2\theta};$$

the values of $\operatorname{cn}(\omega-\omega^2)\,\theta$ and $\operatorname{dn}(\omega-\omega^2)\,\theta$ are thence found to be

$$\operatorname{cn}(\omega-\omega^2)\,\theta = \frac{\operatorname{cn}\theta\,(1-\omega^2\operatorname{sn}^2\theta)}{1-\omega^2(\omega-\omega^2)\operatorname{sn}^2\theta};$$

and
$$\operatorname{dn}(\omega-\omega^2)\,\theta = \frac{\operatorname{dn}\theta\,(1+\omega^2\operatorname{sn}^2\theta)}{1-\omega^2(\omega-\omega^2)\operatorname{sn}^2\theta};$$

which are the formulæ of transformation for the elliptic functions.

Complex Multiplication Moduli of Elliptic Functions.

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The problem of the Complex Multiplication of Elliptic Functions is the determination of the elliptic functions of the complex argument $(a+b\sqrt{\Delta}\,i)\,u$, in terms of the elliptic functions of the argument u, where the ratio of the periods $K'/K = \sqrt{\Delta}$, and Δ is a prime number;