On the Area of the Quadrangle formed by the Four Points of Intersection of Two Conics. By C. LEUDESDORF, M.A.

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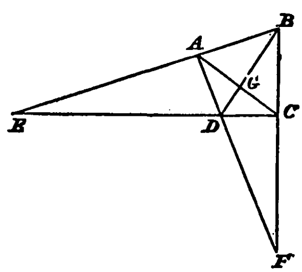
Let the conics be

$$S = (a, b, c, f, g, h \between x, y, 1)^2 = 0,$$

and $$S' = (a', b', c', f', g', h' \between x, y, 1)^2 = 0,$$

and write A, B, C, F, G, H for $bc-f^2$, $ca-g^2$, $ab-h^2$, $gh-af$, $hf-bg$, $fg-ch$, as usual. We suppose that the four points of intersection are all real: so that the discriminant of the cubic $\kappa^3\Delta+\kappa^2\Theta+\kappa\Theta'+\Delta'=0$ is negative. We suppose also that the quadrangle formed by these points is non-reentrant; the condition for this may be found from the consideration that no real ellipse or parabola can be drawn through four points which form a reentrant quadrangle, and is $\nu^2-4CC'>0$, where ν stands for $ab'+ba'-2hh'$. Before considering the general problem, we must investigate the simpler case in which $S=0$ and $S'=0$ each represent a pair of right lines.



Taking any non-reentrant quadrangle, as ABCD, whose opposite sides intersect in the points E, F, G; we see that, of these points, two will always lie without the quadrangle (as E and F), and the third within it (as G).

Considering first the quadrangle ABCD as formed by the points of intersection of the two pairs of straight lines AB, CD ($S=0$) and AD, BC ($S'=0$), whose vertices E, F both lie *without* it, the area ABCD may be found as the difference of the triangles cut off from the pair of lines $S=0$ by each of the lines of the pair $S'=0$; *i.e.*, as $\Delta EBC-\Delta EAD$. We require then, in the first place, an expression for the area of the triangle cut off from the pair of straight lines $S=0$ by another straight line, as $lx+my+n=0$.

Assume $$S = (px+qy+r)\left(\frac{a}{p}x+\frac{b}{q}y+\frac{c}{r}\right),$$

so that $$c\frac{q}{r}+b\frac{r}{q} = 2f, \text{ \&c. \&c.};$$

and therefore

$$c\frac{q}{r}-b\frac{r}{q} = 2\sqrt{-A} = 2\frac{\sqrt{AC}}{\sqrt{-C}} = \frac{2G}{\sqrt{-C}}, \text{ \&c. \&c.}$$

Then the area of the triangle required is

$$\begin{vmatrix} l & m & n \\ p & q & r \\ \frac{a}{p} & \frac{b}{q} & \frac{c}{r} \end{vmatrix}^2 \div 2 \begin{vmatrix} l & m \\ p & q \end{vmatrix} \begin{vmatrix} p & q \\ \frac{a}{p} & \frac{b}{q} \end{vmatrix} \begin{vmatrix} \frac{a}{p} & \frac{b}{q} \\ l & m \end{vmatrix}$$

$$= \frac{\left\{ l\left(c\frac{q}{r} - b\frac{r}{q}\right) + m\left(a\frac{r}{p} - c\frac{p}{r}\right) + n\left(b\frac{p}{q} - a\frac{q}{p}\right) \right\}^2}{2\left(b\frac{p}{q} - a\frac{q}{p}\right) \left\{ bl^2 + am^2 - lm\left(b\frac{p}{q} - a\frac{q}{p}\right) \right\}}$$

$$= \frac{1}{(-C)^{\frac{1}{2}}} \frac{(lG + mF + nC)^2}{bl^2 + am^2 - 2hlm} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Let now $\left(x - \frac{G'}{C'}\right)\sin\theta - \left(y - \frac{F'}{C'}\right)\cos\theta = 0$ be the equation of one of the straight lines composing S′; so that we have the condition $a'\cos^2\theta + b'\sin^2\theta + 2h'\cos\theta\sin\theta = 0$, or

$$b'\tan^2\theta + 2h'\tan\theta + a' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

The area of the triangle cut off by this straight line from the pair $S=0$ is found, by writing in (1)

$$l = \sin\theta, \quad m = -\cos\theta, \quad n = -\frac{G'}{C'}\sin\theta + \frac{F'}{C'}\cos\theta,$$

to be

$$\frac{1}{(-C)^{\frac{1}{2}}} \frac{\left\{ G\sin\theta - F\cos\theta + \frac{C}{C'}(F'\cos\theta - G'\sin\theta) \right\}^2}{a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta}$$

or

$$\frac{1}{(-C)^{\frac{1}{2}}C'^2} \frac{\{(GC' - G'C)\tan\theta - (FC' - F'C)\}^2}{b\tan^2\theta + 2h\tan\theta + a}.$$

If then θ_1, θ_2 be the two values of θ which satisfy (2),

$$\text{Area ABCD} = \Delta\text{EBC} - \Delta\text{EAD}$$

$$= \frac{1}{(-C)^{\frac{1}{2}}C'^2} \left[\frac{\{(GC' - G'C)\tan\theta_1 - (FC' - F'C)\}^2}{b\tan^2\theta_1 + 2h\tan\theta_1 + a} - \frac{\{(GC' - G'C)\tan\theta_2 - (FC' - F'C)\}^2}{b\tan^2\theta_2 + 2h\tan\theta_2 + a} \right]$$

$$= \frac{1}{(-C)^{\frac{1}{2}}C'^2}(\tan\theta_1 - \tan\theta_2) \frac{\left\{ \begin{array}{l} (GC' - G'C)^2\{2h\tan\theta_1\tan\theta_2 + a(\tan\theta_1 + \tan\theta_2) \\ + (FC' - F'C)^2\{-b(\tan\theta_1 + \tan\theta_2) - 2h\} \\ + 2(GC' - G'C)(FC' - F'C)\{b\tan\theta_1\tan\theta_2 - a\} \end{array} \right.}{(b\tan^2\theta_1 + 2h\tan\theta_1 + a)(b\tan^2\theta_2 + 2h\tan\theta_2 + a)}$$

which, by means of equation (2), reduces to

$$\frac{4}{(CC')^{\frac{1}{2}}(\nu^2 - 4CC')} [(GC' - G'C)^2(ha' - h'a)$$
$$+ (FC' - F'C)^2(h'b - hb') + (GC' - G'C)(FC' - F'C)(a'b - ab')].$$

The expression within the brackets may be written

$$\{a'(GC'-G'C)+h'(FC'-F'C)\}\{h(GC'-G'C)+b(FC'-F'C)\}$$
$$-\{a(GC'-G'C)+h(FC'-F'C)\}\{h'(GC'-G'C)+b'(FC'-F'C)\}$$
$$=\{C'(a'G+h'F+g'C)-C(a'G'+h'F'+g'C')\}$$
$$\times\{C'(hG+bF+fC)-C(hG'+bF'+fC')\}$$
$$-\{C'(aG+hF+gC)-C(aG'+hF'+gC')\}$$
$$\times\{C'(h'G+b'F+f'C)-C(h'G'+b'F'+f'C')\}$$
$$=CC'\{(aG'+hF'+gC')(h'G+b'F+f'C)$$
$$-(a'G+h'F+g'C)(hG'+bF'+fC')\}\text{......}(3),$$

and this may be proved* to be equal to $\sqrt{(CC')^3(\nu\Theta\Theta'-C\Theta'^2-C'\Theta^2)}$; thus the area of the quadrangle ABCD

$$=\frac{4}{\nu^2-4CC'}\sqrt{\nu\Theta\Theta'-C\Theta'^2-C'\Theta^2}\ \text{..................}\ (4).$$

Let us next consider the quadrangle ABCD as formed by the points of intersection of two pairs of straight lines whose vertices lie the one *without*, the other *within*, the quadrangle, such as AB, CD and AC, BD. Let the equation to the former pair be $S=0$, and to the latter $S'=0$. If we were to apply the formula (4), just found, to this case, it would clearly give us, not the area ABCD, but the difference of the triangles cut off from the pair of straight lines $S=0$ by each of the straight lines which compose the pair $S'=0$; *i.e.*, $\Delta EAC \sim \Delta EBD$, or $\Delta GCD \sim \Delta GAB$. Thus a different formula must be found for this case; and this may be done by adapting the formula (4) to the case of the two pairs of straight lines AB, CD ($S=0$), and AD, BC (whose equation is evidently $\Theta'S-\Theta S'=0$), which have both their vertices lying without the quadrangle, and to which therefore the investigation above applies. We have only to write, instead of a', b', c', f', g', h', $a'-\frac{\Theta'}{\Theta}a$, $b'-\frac{\Theta'}{\Theta}b$, &c. When this is done it is found that ν must be replaced by $\nu-2C\frac{\Theta'}{\Theta}$, and C' by $\frac{C\Theta'^2+C'\Theta^2-\nu\Theta\Theta'}{\Theta^2}$; for Θ' we must

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* The work of direct proof is omitted, being rather tedious. The identity stated may be seen, however, to be probably true, since the expression within the brackets in (3) and the quantity $\sqrt{CC'(\nu\Theta\Theta'-C\Theta'^2-C'\Theta^2)}$ denote, when equated to zero, exactly the same thing, viz., that one of the pairs of straight lines (E), (F), (G) should consist of parallel lines; and the dimensions of the two are the same. In the more general case, where $S=0$, $S'=0$ denote any two conics, the expression within the brackets in (3) is equal to the square root of the determinant

$$\begin{vmatrix} 0 & \Delta & \Theta & \Theta' & \Delta' \\ \Delta & \Theta & \Theta' & \Delta' & 0 \\ C & \nu & C' & 0 & 0 \\ 0 & C & \nu & C' & 0 \\ 0 & 0 & C & \nu & C' \end{vmatrix}$$

write $a\left[\frac{\Theta'^2}{\Theta^2}A+A'-\frac{\Theta'}{\Theta}(bc'+b'c-2ff')\right]+\&c.$, which reduces to $-\Theta'$; while C, Θ, ν^2-4CC' remain unchanged. Thus the formula (4) will be changed into

$$\frac{4}{\nu^2-4CC'}\sqrt{\left(-\nu+2C\frac{\Theta'}{\Theta}\right)\Theta\Theta'-C\Theta'^2-C'\Theta^2-C\Theta'^2+\nu\Theta\Theta'},$$

that is, into
$$\frac{4\Theta\sqrt{-C'}}{\nu^2-4CC'} \quad\ldots\ldots\ldots\ldots(5),$$

which is therefore the expression for the area ABCD in this case.

Lastly, considering the quadrangle ABCD as formed by the two pairs of straight lines AC, BD ($S=0$), and AD, BC ($S'=0$), where again one vertex lies without and the other within the quadrangle, it is clear from the foregoing that the area ABCD will in this case be represented by the expression

$$\frac{4\Theta'\sqrt{-C}}{\nu^2-4CC'} \quad\ldots\ldots\ldots\ldots(6).$$

If now, in this last case, we had written down the formula (4), it would have given us, not the area ABCD, but the difference of the triangles cut off from the pair of straight lines $S=0$ by each of the straight lines composing the pair AD, BC; *i. e.*, $\Delta GAD \sim \Delta GBC$. And the formula (5) would in this case have given us the difference of the triangles cut off from the pair of straight lines $S=0$ by each of those composing the third pair AB, CD; *i. e.*, $\Delta GCD \sim \Delta GAB$. Similar reasoning will apply to each of the other two cases; the roots of the cubic

$$\left(x-\frac{4\Theta\sqrt{-C}}{\nu^2-4CC'}\right)\left(x-\frac{4\Theta\sqrt{-C'}}{\nu^2-4CC'}\right)\left(x-\frac{4\sqrt{\nu\Theta\Theta'-C\Theta'^2-C'\Theta^2}}{\nu^2-4CC'}\right)=0\ldots(7),$$

being, in each case, the three quantities

area ABCD, $\Delta GAD \sim \Delta GBC$, $\Delta GCD \sim \Delta GAB$.

But of these the area ABCD must always be the greatest. Therefore, when $S=0$, $S'=0$ denote each a pair of straight lines, the area of the quadrangle formed by their four points of intersection is the greatest root of the cubic (7).

Let us now proceed to consider the general case, in which $S=0$, $S'=0$ denote each a conic of any kind. Let κ_1, κ_2, κ_3 be the three values of κ for which $\kappa S+S'=0$ breaks up into straight lines; and suppose κ_3 that value for which the vertex of the pair of lines $\kappa S+S'=0$ lies *within* the quadrangle formed by the four points of intersection of S and S'. Then the area of this quadrangle may be deduced from the formula (4), applied to the case of the two pairs $\kappa_1 S+S'=0$, $\kappa_2 S+S'=0$. We must, in order to do this, write instead of a, b, &c., $\kappa_1 a+a'$, $\kappa_1 b+b'$, &c.; and instead of a', b', &c., $\kappa_2 a+a'$ $\kappa_2 b+b'$, &c. We must therefore write, for A, $(\kappa_1 b+b')(\kappa_1 c+c')-(\kappa_1 f+f')^2$, &c. &c.; for F, $(\kappa_1 g+g')(\kappa_1 h+h')-(\kappa_1 a+a')(\kappa_1 f+f')$, &c. &c.; *i. e.*, if we put

$$\lambda = bc'+b'c-2ff', \qquad \mu = ca'+c'a-2gg', \qquad \nu = ab'+a'b-2hh',$$
$$\lambda' = gh'+g'h-af'-a'f, \quad \mu' = hf'+h'f-bg'-b'g, \quad \nu' = fg'+f'g-ch'-c'h$$

we must write, instead of A, B, C, F, G, H respectively,

$$\kappa_1^2 A+\kappa_1\lambda+A', \quad \kappa_1^2 B+\kappa_1\mu+B', \quad \kappa_1^2 C+\kappa_1\nu+C',$$
$$\kappa_1^2 F+\kappa_1\lambda'+F', \quad \kappa_1^2 G+\kappa_1\mu'+G', \quad \kappa_1^2 H+\kappa_1\nu'+H',$$

and for A′, B′, C′, F′, G′, H′ precisely similar quantities, κ_2 being substituted for κ_1. It is easily seen that ν will be replaced by $2\kappa_1\kappa_2 C+(\kappa_1+\kappa_2)\nu+2C'$, while ν^2-4CC' becomes $(\kappa_1-\kappa_2)^2(\nu^2-4CC')$. For Θ we must put

$$\Sigma(\kappa_2 a+a')(\kappa_1^2 A+\kappa_1\lambda+A')$$
$$= \kappa_1^2\kappa_2\Sigma aA+\kappa_1\kappa_2\Sigma a\lambda+\kappa_2\Sigma aA'+\kappa_1^2\Sigma a'A+\kappa_1\Sigma a'\lambda+\Sigma a'A'$$
$$= 3\kappa_1^2\kappa_2\Delta+2\kappa_1\kappa_2\Theta+\kappa_2\Theta'+\kappa_1^2\Theta+2\kappa_1\Theta'+3\Delta'$$
$$= \Delta\{3\kappa_1^2\kappa_2-(2\kappa_1\kappa_2+\kappa_1^2)(\kappa_1+\kappa_2+\kappa_3)+(\kappa_2+2\kappa_1)(\kappa_1\kappa_2+\kappa_2\kappa_3+\kappa_3\kappa_1) -3\kappa_1\kappa_2\kappa_3\}$$

(from the equation $\kappa^3\Delta+\kappa^2\Theta+\kappa\Theta'+\Delta'=0$, satisfied by $\kappa_1, \kappa_2, \kappa_3$)

$$= \Delta(\kappa_1-\kappa_2)^2(\kappa_3-\kappa_1).$$

Similarly, Θ' will be replaced by $\Delta(\kappa_1-\kappa_2)^2(\kappa_3-\kappa_2)$. Making these substitutions in (4), we find for the required area

$$\frac{4\Delta}{\nu^2-4CC'}\left[-(\kappa_1^2 C+\kappa_1\nu+C')(\kappa_3-\kappa_2)^2-(\kappa_2^2 C+\kappa_2\nu+C')(\kappa_3-\kappa_1)^2 + (2\kappa_1\kappa_2 C+\overline{\kappa_1+\kappa_2}\,\nu+2C')(\kappa_3-\kappa_2)(\kappa_3-\kappa_1)\right]^{\frac{1}{2}},$$

which reduces to

$$\frac{4\Delta}{\nu^2-4CC'}(\kappa_1-\kappa_2)\sqrt{-(C\kappa_3^2+\nu\kappa_3+C')} \quad \ldots\ldots\ldots\ldots\ldots (8).$$

Writing x for the area of the quadrangle, we have

$$\left(\frac{\nu^2-4CC'}{4}x\right)^2 = -\Delta^2(\kappa_1-\kappa_2)^2(C\kappa_3^2+\nu\kappa_3+C')$$
$$= -\left(\Theta^2-2\Delta\Theta'-\Delta^2\kappa_3^2+\frac{2\Delta\Delta'}{\kappa_3}\right)(C\kappa_3^2+\nu\kappa_3+C'),$$

or

$$\kappa_3^4\Delta\Theta C+\kappa_3^3\{C(\Theta^2-2\Delta\Theta')+\nu\Delta\Theta\}+\kappa_3^2\{3\Delta\Delta'C+\Delta\Theta C'+\nu(\Theta^2-\Delta\Theta')\} +\kappa_3\left\{3\nu\Theta\Theta'+C'(\Theta^2-\Delta\Theta')+\left(\frac{\nu^2-4CC'}{4}x\right)^2\right\}+3C'\Delta\Delta'=0;$$

and by eliminating κ_3 between this equation and $\kappa_3^3\Delta+\kappa_3^2\Theta+\kappa_3\Theta'+\Delta'=0$, we obtain a cubic in x^2; and, just as above, it is seen that the greatest root of this cubic is the square of the area of the quadrangle ABCD; the other roots being $(\Delta GAB \sim \Delta GCD)^2$ and $(\Delta GAD \sim \Delta GBC)^2$. We may form this cubic more conveniently thus:—If x_1^2, x_2^2, x_3^2 be its roots, and if δ stand for $CC'-\frac{\nu^2}{4}$,

$$-\delta^2(x_1^2+x_2^2+x_3^2) = \Delta^2(\kappa_1-\kappa_2)^2(C\kappa_3^2+\nu\kappa_3+C')+\ldots+\ldots$$
$$= \Delta^2\{C\Sigma\kappa_3^2(\kappa_1-\kappa_2)^2+\nu\Sigma\kappa_3(\kappa_1-\kappa_2)^2+C'\Sigma(\kappa_1-\kappa_2)^2\},$$

which, when the symmetric functions are calculated out, is found to be

$$2C(\Theta'^2-3\Delta'\Theta)-\nu(\Theta\Theta'-9\Delta\Delta')+2C'(\Theta^2-3\Delta\Theta').$$

This is one of the fundamental invariants of the cubic $\kappa^3\Delta+\kappa^2\Theta+\kappa\Theta'+\Delta'$ and the quadratic $C\kappa^2+\nu\kappa+C'=0$. If it be denoted by I, we have $\quad x_1^2+x_2^2+x_3^2 = -\frac{I}{\delta^2}$ ....(9).

Also, by (8),

$$-\delta^6 x_1^2x_2^2x_3^2 = \Delta^6\,\Pi(\kappa_1-\kappa_2)^2\,\Pi(C\kappa_3^2+\nu\kappa_3+C').$$

Now $\Pi(\kappa_1-\kappa_2)^2$ is equal to $-\frac{D}{\Delta^4}$, if D be the discriminant of the cubic $\kappa^3\Delta+\kappa^2\Theta+\kappa\Theta'+\Delta'$; also $\Pi(C\kappa_3^2+\nu\kappa_3+C')$ is equal to $\frac{1}{\Delta^2}$ times the resultant (R) of this cubic and the quadratic $C\kappa^2+\nu\kappa+C'$; so that

$$x_1^2x_2^2x_3^2 = \frac{DR}{\delta^6} \quad \text{..............................(10).}$$

Again, by (8),

$$\delta^4(x_1^4+x_2^4+x_3^4) = \Delta^4(\kappa_1-\kappa_2)^4(C\kappa_3^2+\nu\kappa_3+C')^2+\ldots+\ldots,$$

which may be reduced, by the calculation of symmetric functions, to

$$2C^2(\Theta'^2-3\Delta'\Theta)^2+2C'^2(\Theta^2-3\Delta\Theta')$$
$$+(\nu^2+2CC')(\Theta^2\Theta'^2+27\Delta^2\Delta'^2-2\Delta\Theta'^3-2\Delta'\Theta^3)$$
$$-2C\nu(\Theta'^2-3\Delta'\Theta)(\Theta\Theta'-9\Delta\Delta')-2C'\nu(\Theta^2-3\Delta\Theta')(\Theta\Theta'-9\Delta\Delta'),$$

which may be written as

$$\tfrac{1}{2}\{2C(\Theta'^2-3\Delta'\Theta)-\nu(\Theta\Theta'-9\Delta\Delta')+2C'(\Theta^2-3\Delta\Theta')^2\}$$
$$-\left(\frac{\nu^2}{2}-2CC'\right)(27\Delta^2\Delta'^2-\Theta^2\Theta'^2+4\Delta\Theta'^3+4\Delta'\Theta^3-18\Delta\Delta'\Theta\Theta'),$$

that is, as $\frac{I^2}{2}+2\delta D$; so that

$$x_1^2x_2^2+x_2^2x_3^2+x_3^2x_1^2 = \tfrac{1}{2}\{(x_1^2+x_2^2+x_3^2)^2-(x_1^4+x_2^4+x_3^4)\}$$
$$= \tfrac{1}{2}\left\{\frac{I^2}{\delta^4}-\frac{1}{\delta^4}\left(\frac{I^2}{2}+2\delta D\right)\right\}, \text{ by (9)},$$
$$= \frac{1}{\delta^4}\left(\frac{I^2}{4}-\delta D\right) \quad \text{..............................(11).}$$

From (9), (10), and (11), the required cubic is seen to be

$$x^6+\frac{I}{\delta^2}x^4+\frac{\frac{I^2}{4}-\delta D}{\delta^4}-\frac{DR}{\delta^6}=0,$$

or

$$x^2\left(x^2+\frac{I}{2\delta^2}\right)^2-\frac{D}{\delta^3}\left(x^2+\frac{R}{\delta^3}\right)=0\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

and the greatest root of this equation is the square of the area of the quadrangle ABCD formed by the four points of intersection of the two conics S and S′; the other roots being $(\Delta GAB \sim \Delta GCD)^2$ and $(\Delta GAD \sim \Delta GBC)^2$.

With regard to the four quantities δ, D, R, I, in terms of which this area can therefore be found, these are seen to be the four invariants of the system formed by the cubic $\kappa^3\Delta+\kappa^2\Theta+k\Theta'+\Delta'$ and the quadratic $C\kappa^2+\nu\kappa+C'$; viz., the discriminant of the quadratic (δ), that of the cubic (D), their resultant (R), and the intermediate invariant which arises from combining the quadratic with the Hessian of the cubic (I). The cubic and the quadratic are themselves discriminants, the former of $\kappa S+S'$, the latter of $\kappa\,(a, h, b \between x, y)^2+(a', h', b' \between x, y)^2$.

We may notice the following particular cases:—

1. When S and S′ both break up into pairs of straight lines, Δ and Δ' vanish, so that

$$D=-\Theta^2\Theta'^2,\quad R=CC'(C\Theta'^2-\nu\Theta\Theta'+C'\Theta^2),\quad I=2C\Theta'^2-\nu\Theta\Theta'+2C'\Theta^2;$$

and (12) reduces to the form (7), as it should do.

2. When $D=0$, one root of (12) is zero, the other two being each equal to $-\frac{I}{2\delta^2}$; *i. e.*, when the conics S, S′ touch, the square root of this is the value of the area of the triangle formed by the three points of intersection. From the figure it is seen that, when two of the four points A, B, C, D (say C, D) coincide, the three roots of (12), $(ABCD)^2$, $(\Delta GAB \sim \Delta GCD)^2$, $(\Delta GAD \sim \Delta GBC)^2$ reduce to $(\Delta ABC)^2$, $(\Delta ABC)^2$, 0 respectively.

3. When $R=0$, one of the roots of (12) is zero, the greater of the other two roots being equal to $\sqrt{\frac{D}{\delta^3}}-\frac{I}{2\delta^2}$. But $R=0$ is the condition that one of the pairs of chords of intersection of the two conics should consist of parallel straight lines; in this case, then, the square root of the above expression gives the area ABCD. From the figure it is seen that, if AB, CD be parallel, $\Delta GAD=\Delta GBC$, so that one of the roots of (12) ought to vanish.

4. If $S=0$ denote a conic, and $S'=0$ a pair of straight lines whose vertex lies *within* the quadrangle formed by the four points of intersec-

tion of S and S′ (*e.g.*, if S be an ellipse or a parabola, S′ may be any pair of straight lines which meet within the conic), then Δ' vanishes, and one value of κ is zero, the other two being given by the equation $\Delta\kappa^2+\Theta\kappa+\Theta'=0$, so that their difference is $\frac{\sqrt{\Theta^2-4\Delta\Theta'}}{\Delta}$, and, by (8), the area of the quadrangle formed by the points of intersection is

$$\frac{4\sqrt{-C'}}{\nu^2-4CC'}\sqrt{\Theta^2-4\Delta\Theta'}.$$

5. If, in this last case (4), the pair of straight lines S′ be supposed to meet on S, then also $\Theta'=0$, and the area of the triangle formed is $-\frac{\Theta}{\delta}\sqrt{-C'}$.

6. If $S=0$, $S'=0$ each represent a pair of parallel straight lines, then C, C′ vanish as well as Δ, Δ'; so that, by (7), the area of the parallelogram formed is $\sqrt{\frac{\Theta\Theta'}{\nu^3}}$.

Geometrical Illustration of a Theorem relating to an Irrational Function of an Imaginary Variable. By Prof. Cayley.

[*Read May 11th*, 1876.]*

If we have v, a function of u, determined by an equation $f(u, v)=0$, then to any given imaginary value $x+iy$ of u there belong two or more values, in general imaginary, $x'+iy'$ of v: and for the complete understanding of the relation between the two imaginary variables, we require to know the series of values $x'+iy'$ which correspond to a given series of values $x+iy$, of v, u respectively. We must for this purpose take x, y as the coordinates of a point P in a plane Π, and x', y' as the coordinates of a corresponding point P′ in another plane Π'. The series of values $x+iy$ of u is then represented by means of a curve in the first plane, and the series of values $x'+iy'$ of v by means of a corresponding curve in the second plane. The correspondence between the two points P and P′ is of course established by the two equations into which the given equation $f(x+iy,\ x'+iy')=0$ breaks up, on the assumption that x, y, x', y' are all of them real. If we assume that the coefficients in the equation are real, then the two equations are

$$f(x+iy,\ x'+iy')+f(x-iy,\ x'-iy')=0,$$
$$f(x+iy,\ x'+iy')-f(x-iy,\ x'-iy')=0;$$

viz., if in these equations we regard either set of coordinates, say (x, y),

* Not received in time to be placed earlier in the Proceedings.