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To cite this article: C.V. Drysdale D.Sc. (1905) XLIII. On the curvature method of teaching optics, Philosophical Magazine Series 6, 9:52, 467-491, DOI: [10.1080/14786440509463297](https://doi.org/10.1080/14786440509463297)

To link to this article: <http://dx.doi.org/10.1080/14786440509463297>



Published online: 08 Jun 2010.



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XLIII. *On the Curvature Method of Teaching Optics.*

By C. V. DRYSDALE, *D.Sc.**

NOTWITHSTANDING the fact that the wave theory of light has been employed to demonstrate some of the more simple problems in the domain of what is generally termed geometrical optics, and with manifest simplicity and convenience, this appears to have been done rather with the object of verifying the wave theory than of showing how the subject of optics can be completely dealt with from this standpoint; and few men of science or teachers of optics appear to have realized the advantages of physical methods both for practical work and teaching, and that they should entirely supersede the geometrical or ray methods. This is unquestionably due to a very large extent to unfamiliarity, and to the cramping effect of our university curricula and text-books; but a possible factor in the question is the impression which seems to be prevalent, even among optical specialists, that the physical methods have to be abandoned at a certain stage, and that the more complex problems relating to lens systems and aberrations must be treated by geometrical methods. Owing to the fact that British men of science have been actively engaged in extending the wave theory towards penetrating the more fascinating mysteries of interference, polarization, and electromagnetic theory, the practical applications of optics have passed for the last half century into the hands of the Germans, who took from us the geometrical methods then in vogue and have since extended them with such marked success as to give the impression that these geometrical methods are the most suitable for the purpose. At the present time a strong attempt is being made to revive the study of technical optics in this country, unfortunately with great difficulties, owing to the lack of satisfactory teaching; and the object of this paper is to show that not only are physical methods the most suitable at the outset, but that they are capable of being employed with the same increased simplicity in the whole domain of "geometrical optics."

A few words are desirable at the outset as to what has been done in the application of physical problems to reflexion and refraction. The first step in this direction appears to have been made by Herschel in 1827 †, who seems to have dealt with ordinary lens problems fairly completely and to have devised a very satisfactory curvature notation. According

* Communicated by the Physical Society: read February 24, 1905.

† Herschel, *Encyc. Metropolitana*, 1827.

to Mr. Cheshire *, in an article just published, Porro, in 1857 †, gave a very complete exposition of lenses and thin lens combinations from the physical standpoint, including suggestions for the study of spherical aberration and of thick lenses. In Preston's 'Light' ‡ a short account of the elementary treatment of lenses by this method appears; and Lord Rayleigh § has also devoted attention to the subject. It is, however, to Prof. S. P. Thompson || that we owe nearly everything that has been done in recent years towards the rational teaching of optics, and in 1889 he wrote a paper on the curvature method of treating lens problems which he had rediscovered and had used in teaching for about eight years; and he has followed this up by several valuable papers, and his two important works—'Optical Tables and Memoranda,' and the translation of Lummer's 'Photographic Optics.'

It will perhaps be well to mention here that the present writer has unfortunately, until the last few days, been absolutely unconscious of practically all that has been done in the application of wave methods by others. Although a student under Prof. Thompson in 1888–90, for some reason none of this work was done, and the only thing that directed his attention to the subject was a few lectures given by Dr. Sumpner at the Central Technical College in 1892, dealing with the matter as in Preston's 'Light.' The interest roused by the manifest superiority of the wave method, however, led him afterwards to make some simple applications in practical work, and later to start a course of lectures on technical optics at the Northampton Institute, entirely based on physical optics. It is a singular fact that although we owe to scientific men the wave theory of light and its applications to optical theory, the optical trade have, apparently quite independently, adopted a system of lens and prism nomenclature which harmonizes completely with it, and makes its application to reflexion and refraction problems simple and logical. It was due to the writer becoming acquainted with this notation that he was led to take up the subject so fully, and he ventures to think that few teachers of optical science would find a knowledge of optical trade methods detrimental.

Before dealing with the curvature method, it may be interesting to note that there are several alternative methods of

* F. J. Cheshire, *British Optical Journal*, Nov. 1904.

† Porro, *Société Française de Photographie*, vol. iii. pp. 211–222, 1857.

‡ Preston, 'Light,' pp. 99–106, Third edition, 1901.

§ Lord Rayleigh, *Encyclopædia Britannica*, 1884. Article on "Optics."

|| S. P. Thompson, *Phil. Mag.* 1889, vol. xxxiii., "Notes on Geometrical Optics," Part i. pp. 232–248.

attacking lens problems, of which one only, and that probably the least suitable, has been given a fair trial in this country. They may perhaps be enumerated as follows :—

- (a) The method of reckoning in conjugate distances—Geometrical or Gauss method.
- (b) The physical or curvature method.
- (c) The method of deviations—Von Seidel, Finsterwalter.
- (d) The use of the characteristic function, or principle of least time—Hamilton, Thiesen, and Chalmers.
- (e) { Thermodynamic method—Clausius.
Employment of the “eikonal”—Bruns.
- (f) Vector or Quaternion treatment.

Of these methods (b), (c), and (d) are the ones which the writer believes will be found most suitable, and it will be shown, as is almost self-evident, that they are essentially similar. Of the eikonal treatment he has no experience, but is inclined to think that its use is confined to problems of an advanced nature, and that it is unsuitable for a general elementary treatment. As to (f) it seems curious that no one has proposed the application of the modern vector calculus of Prof. Henrici, Mr. Oliver Heaviside, and Prof. Gibbs, to geometrical optics, as it should be capable of effecting considerable simplifications. For example, if \hat{a} is a unit vector representing the direction of an incident ray, \hat{b} the corresponding refracted ray, and \hat{n} the vector normal to the surface, both of the ordinary laws of refraction are summed up in the simple vector relation

$$\mu_1 [\hat{a} \hat{n}] = \mu_2 [\hat{b} \hat{n}],$$

or in Heaviside's notation

$$\mu_1 \mathbf{V} \hat{a} \hat{n} = \mu_2 \mathbf{V} \hat{b} \hat{n}.$$

The writer has deduced a few interesting consequences of this fundamental expression, but has not yet had an opportunity of following it up completely; moreover, as the method is a purely geometrical one, it could only have advantages in a possible simplification of ordinary procedure, and would not have any other physical signification.

Practical Optical Units.—The simple device of opticians for spectacle-lens notation was to adopt two units which are now fairly generally known. The first of these may be used to express either the curvature of a surface or the power of

a lens or mirror, and is termed a Dioptré*. This unit is a curvature corresponding to a radius of one metre; and we consequently say that a lens has a convergence of one dioptré when it has a focal length of one metre, or that a wave-front or surface has a curvature of one dioptré when it has a radius of curvature of one metre. The other unit adopted is one of angle or deviation and is termed the Prism Dioptré: it is defined as a deviation of one centimetre on a tangent line at a distance of one metre, and one prism dioptré therefore corresponds approximately to $\cdot 01$ of a radian. The prism dioptré and the radian thus serve as two units of angular measurement, and the former is very conveniently related to the dioptré, as the deviation in prism dioptrés produced by any thin lens at any zone is simply obtained by multiplying the convergence of the lens in dioptrés by the radius of the zone in centimetres.

In the prism dioptré and the radian we have two units of angular measurement which cover most requirements; but the curvature dioptré, although an exceedingly convenient unit for many purposes, is inconvenient to deal with when we are concerned with microscope lenses and others of short focal length. The writer has therefore recently proposed† the adoption of multiples and sub-multiples of this unit, using the ordinary prefixes as follows:—

| | |
|-------------------------------|---------------|
| Radius of curvature | Curvature. |
| Kilometre | Millidioptré. |
| Metre | Dioptré. |
| Centimetre | Hectodioptré. |
| Millimetre | Kilodioptré. |

The great advantages of such a system is, that to any particular length measurement there is a corresponding curvature measurement, which enables one to have a physical idea of the magnitudes involved and avoids the use of unwieldy fractions which frequently occur otherwise. The range of curvatures may be very great, the radius varying from a wave-length to infinity.

Notation.—The writer has also found it convenient to adopt a standard notation in optical work, using small letters for the distances, radii of curvature, focal lengths, and

* Various spelt, Dioptrie, Dioptré, and Diopter. The author has hitherto preferred the spelling Dioptré, harmonizing with metre, but modern usage favours meter and diopter. It appears, however, that Dioptrie was proposed by Monoyer in 1872, and adopted by the Brussels International Congress in 1875.

† "On some Points in the Design of Optical Instruments." Proc. Optical Soc., December 18th, 1902.

thicknesses ; large letters for the corresponding curvatures, and Greek letters for the angles *. The conjugate distances and focal lengths being denoted by u , v , and f respectively ; we have U , V , and F as the corresponding curvatures or convergences. If the distances u , v , &c. are expressed in metres, we have $U = \frac{1}{u}$, $V = \frac{1}{v}$, &c., the curvatures in dioptries, while if u and v are in mm. U and V are in kilodioptries.

In like manner, if we denote the radii of the surfaces of a lens by r_1 and r_2 , we should have R_1 and R_2 as the curvatures of the surfaces, and $C = R_1 - R_2$ the total curvature in the case of a thin lens. Curvatures of wave-fronts may be reckoned positive when they are convergent, and surfaces are said to have positive curvature when they are curved in the same direction as convergent emergent light. This harmonizes the formulæ for reflexion and refraction at curved surfaces.

PART I.—ELEMENTARY OPTICS.

In dealing with this subject it was the writer's first intention to give a fairly complete exposition of the methods he has adopted, but a subsequent perusal of Dr. Thompson's article in the *Phil. Mag.* has shown him that the procedure he has followed is so closely identical with that advocated by Dr. Thompson as to render this unnecessary. For the sake of completeness, however, the various steps in the development of an elementary optical course may be briefly given here.

Nature and mode of Propagation of Light.—The first important step is to familiarize the student with the notion of waves and their propagation. A beginner is easily convinced by a few simple illustrations that light must be either propagated by projectiles or undulations. The ripple tank can then be used to demonstrate the propagation of waves, and the effects of reflexion, &c. shown simultaneously by optical projection, and in the tank. The justification for the undulatory theory can be well shown by interference, ripples being excited from two simultaneously vibrating points, and the results compared with interference from a bi-prism. Even elementary students can quite appreciate that the wave theory is the only one which can satisfactorily account for the dark interference-bands. The finite rate of propagation of waves is then pointed out with references to determinations of the speed of light, and it is also shown that ripples of different frequencies can be excited and that the wave-lengths differ.

* This appears to be almost identical with the notation employed by Herschel and by Dr. Thompson.

Finally it is shown by analogy with the ripples that the wave-fronts diverge in expanding spheres, becoming less curved as they recede from the source, and ultimately plane, and that the direction of propagation is perpendicular to the wave-front; and the difference between plane, divergent, and convergent waves is illustrated. At the same time, it may be pointed out with advantage that if the medium were not isotropic, the speed of propagation would be different in different directions, the wave-front being ellipsoidal and not perpendicular to the lines of propagation. The writer has never found that elementary students have any difficulty over this, and it prepares the way for subsequent work in polarization, &c.

The next stage is to explain the formation of shadows and of an image by a pinhole, referring at the same time to diffraction. The formation of the penumbra may also be well illustrated by waves, and photometry explained by showing that any portion of the wave-front carrying a certain amount of light-energy expands proportionally to the square of the distance from the source.

The study of reflexion at plane surfaces comes next, first with plane and then with spherical waves, proving the laws of reflexion, application to various instruments, measurement of angles of a prism, &c., and formation of images by one or more plane mirrors.

Curvature and its Measurement.—At this stage the ideas and units of curvature are introduced, explaining the necessity for the measurement of the curvatures both of waves and of reflecting surfaces. Students will easily realize that a satisfactory definition of curvature is the reciprocal of the radius, but it may be proved from the ordinary definition of curvature if preferred. The diopetre with its multiples and submultiples may then be introduced and the relations :

$$\text{curvature in dioptries} = \frac{1}{r \text{ (metres)}} = \frac{100}{r \text{ (cm.)}} = \frac{39.37}{r''},$$

and the corresponding conversions from curvatures to radii.

Practical curvature measurement may then be taken up, the equality of the products of the segments of any two chords of a circle being demonstrated either by geometry or by simple measurement, and from this the ordinary spherometer formula

$$r = \frac{c^2 + h^2}{2h}, \text{ or } R = \frac{2h}{c^2 + h^2} = \frac{2}{c^2}h$$

when the curvature is small. When c and h are in millimetres

the resultant R must evidently be in kilodiotres, and the curvature in diotres is then $\frac{2000\ h}{c^2 + h^2} = \frac{2000}{c^2}h$ for small curvatures. It is then pointed out that if $c^2 = 2000$ or $c = 44.7$ mm. the sagitta or sag of the curve denotes directly the curvature in diotres, and Prof. Thompson's "dioptrie" spherometer and other forms are explained. At the same time attention is called to the facts, first that the curvatures of all circular curves having the same chord are proportional to their sagittas, and secondly, that this is only true for chords of small lengths compared with the radius. This assists the explanation of spherical aberration later.

The properties of spherical mirrors follow immediately. From the fact that light striking the surface travels back the same distance as it would have gone forward in the absence of the mirror, it follows immediately that the curvature of the mirror is the mean between the curvatures of the incident and reflected wave-fronts. Hence $R = \frac{V-U}{2}$ or $V-U=2R$, where V and U are the convergences of the incident and reflected light and $2R=F$ is shown to be the convergence of the mirror. By considering the image of an object formed by the mirror when the latter is stopped down to a pinhole at the vertex, we at once find the magnification $m = \frac{v}{u} = \frac{U}{V}$, and the whole of the properties of mirrors are deduced from the two equations

$$V-U=2R=F$$

$$m = \frac{v}{u} = \frac{U}{V}.$$

The error of considering the sagitta proportional to curvature for large apertures is again pointed out, with reference to parabolic mirrors, caustics, &c.

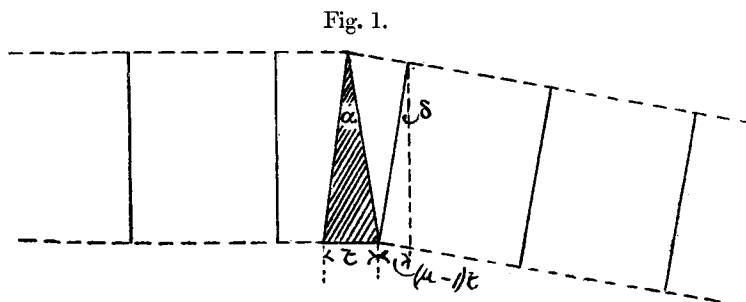
Refraction and Dispersion.—It is at this point that the value of the wave method begins to show itself most strikingly. Refraction at a simple plane surface may be shown by Huyghens' method. The writer has found it of the greatest assistance to students to illustrate the wave-fronts by parallel lines of men marching towards a river which they can only ford at some fraction of their marching speed. The swinging round of the line is easily realized; and if at the same time the idea is introduced of the lines being made up of men of different heights, with uniforms ranging from violet to red

according to size and strength, the reason for dispersion is at once grasped. It is at the same time explained that the more rapid violet light vibrations or shorter waves would naturally be more encumbered by dense or gross matter than the larger red waves. At the same time the general effects of refraction by parallel plates, prisms and lenses are illustrated by the crossing of rivers with parallel, oblique or curved banks. This paves the way for a complete study of refraction and of the properties of prisms and lenses. As it would take up too much space to give a detailed course here, it is proposed simply to give one or two examples as showing either original methods or ones which are less generally known and which the writer considers should be adopted.

It should be here mentioned that it has been found convenient when dealing with thin prisms and lenses, to consider the alteration of the wave-front by the prism or wave as a whole, and only to introduce the formulæ for single surfaces when thick lenses and prisms or refracting systems in general are considered.

Retardation by a Parallel Plate.—Since the refractive index $\mu = \frac{\text{velocity in air}}{\text{velocity in medium}}$, it follows that in passing through a thickness t of glass, the wave would have travelled forward a distance μt in air, and the retardation caused by the plate is $\mu t - t = (\mu - 1)t$.

Refraction by a Thin Prism.—In the case of a thin prism (fig. 1) having a thickness t at its base and of zero at its

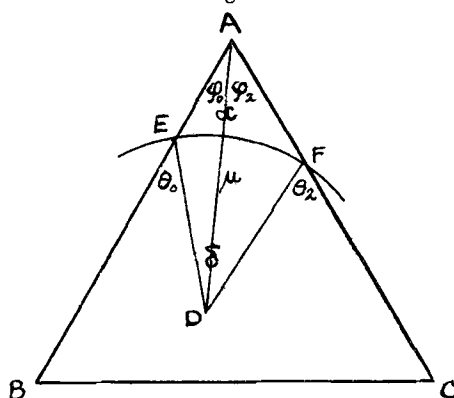


Refraction by Thin Prism.

apex, the retardation of the wave at the base being $(\mu - 1)t$, we have evidently $\delta = (\mu - 1)\alpha$. It is pointed out that this cannot hold for thick prisms, as in that case the distance traversed by the light depends on the direction of entry, and it is not justifiable to take the angles as being proportional to the bases.

Thick Prisms.—Here we have only to take a section of the prism ABC (fig. 2), and indicate the direction of the

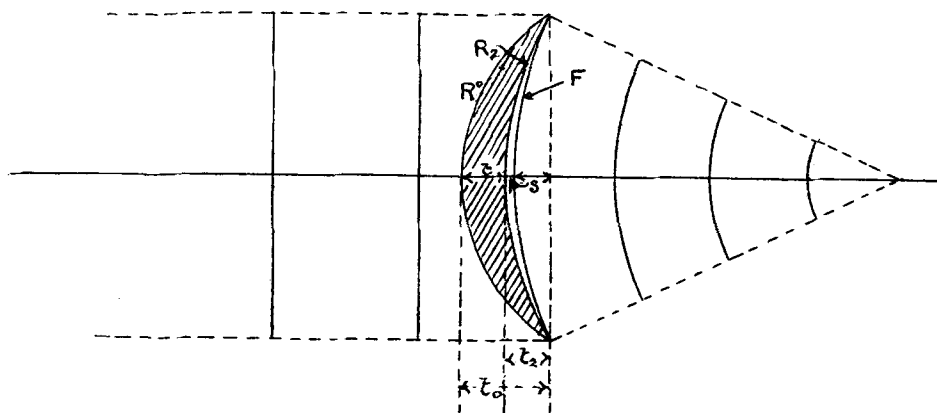
Fig. 2.



Refraction by Thick Prism.

three wave-fronts before incidence, in the glass, and after emergence by DE, DA, and DF respectively. The angles θ_0 and ϕ_0 are obviously the angles of incidence and of refraction with reference to the left face of the prism, while θ_2 and ϕ_2 are those relating to the right face. We thus have immediately $\alpha = \phi_0 + \phi_2$ and $\delta + \alpha = \theta_0 + \theta_2$, and since $\sin \theta_0 = \mu \sin \phi_0$ and $\sin \theta_2 = \mu \sin \phi_2$ we have the distances DE, DF, and DA

Fig. 3.



Refraction by Thin Lens.

in the ratio of $1 : 1 : \mu$, and we are immediately led to the ordinary construction for, and properties of thick prisms.

Thin Spherical Lenses.—In fig. 3 we have a thin lens, the

thickness at its centre being t , which is the sag corresponding to the total curvature of the lens. A wave passing through it will have its centre retarded by an amount $s=(\mu-1)t$ as with a thin prism. If the wave is initially plane it receives a curvature corresponding to the sag S , which we term the convergence F of the lens, and we therefore have immediately $F=(\mu-1)C$ corresponding to $\delta=(\mu-1)\alpha$ for a thin prism.

If the wave is initially curved or convergent by an amount U , the retardation increases its convergence by the amount

$$V-U=(\mu-1)C=F.$$

The thickness t of the lens can be made up of the two sags t_0 and t_2 to the curvatures R_0 and R_2 of the faces.

Hence $V-U=F=(\mu-1)C=(\mu-1)(R_0-R_2)$.

By considering a pinhole stop in contact with the lens we immediately have $m=\frac{v}{u}=\frac{U}{V}$ as with the mirror; and the properties of lenses and formation of images follow immediately.

Thin Cylindrical Lenses.—For a principal section of a cylindrical lens perpendicular to its axis, we have obviously the same relation as for a spherical lens. For an oblique section the retardation of the wave-front is obviously the same, but the breadth of the beam is increased (fig. 4). From the spherometer formula $R=\frac{2h}{c^2}$ we have

$$R'=R\left(\frac{c}{c'}\right)^2=R\sin^2\theta,$$

and hence also $F'=F\sin^2\theta$. In a meridian perpendicular to the former we must similarly have $F''=F\cos^2\theta$, hence $F'+F''=F$, or the sum of the convergences in two meridians at right angles is constant and equal to the convergence of the lens, analogous to a well-known theorem for curved surfaces.

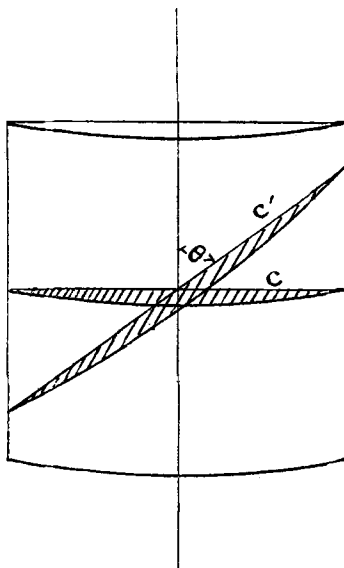
Deviation and Decentring of Lenses.—By multiplying the equation $V-U=F$ by the height of intersection of the pencil with the surface we have immediately

$$\tan\sigma_2-\tan\sigma_1=Fh \quad \text{or} \quad \sigma=\sigma_2-\sigma_1=Fh$$

if the angles are small. This gives us the valuable result that the deviation at any zone of a thin lens in prism-dioptres is given by the product of the convergence of the lens in dioptres

and the radius of the zone or decentration in cm. The more exact equation is also a valuable introduction to Von Seidel's

Fig. 4.



Cylindrical Lens.

method of deviations. From the equation $m = \frac{U}{V}$ we have immediately $m = \frac{\tan \sigma_1}{\tan \sigma_2}$ or Helmholtz's expression for the magnification.

Combinations of Thin Lenses in contact.—It is obvious from the summation of the thicknesses that the resultant convergence $F = \Sigma F_1$ for any number of thin lenses in contact, and that in the case of cylindrical or sphero-cylindrical lenses, the powers in any meridian may be added. It has been already shown that the convergence of a cylindrical lens in a meridian making an angle of θ with the axis is

$$F \sin^2 \theta = \frac{F}{2} (1 - \cos 2\theta).$$

Hence if we have any number of cylindrical lenses in contact of convergences $F_1, F_2, \&c.$, and whose axes make angles $\alpha_1, \alpha_2, \&c.$, with a reference meridian, we have

$$F = \frac{1}{2} \Sigma F_1 - \frac{1}{2} \Sigma F_1 \cos 2(\theta - \alpha_1)$$

in any meridian making an angle θ with the reference meridian. On expanding we have

$$F = \frac{1}{2} \Sigma F_1 - \frac{1}{2} \{ \cos 2\theta \Sigma F_1 \cos 2\alpha_1 - \sin 2\theta \Sigma F_1 \sin 2\alpha_1 \},$$

and this is a maximum or minimum for

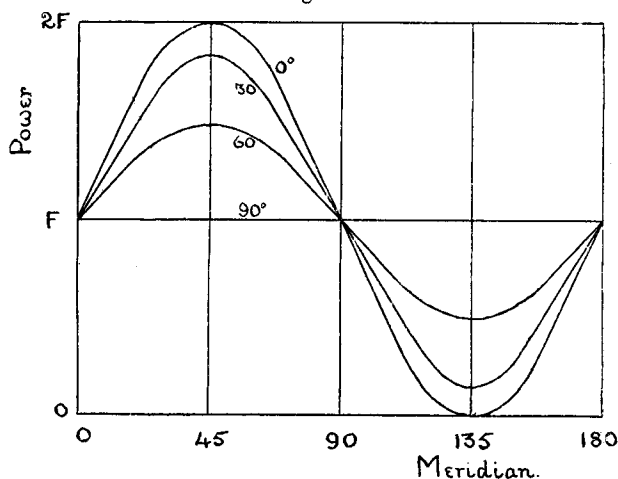
$$\tan 2\theta = \frac{\Sigma F_1 \sin 2\alpha_1}{\Sigma F_1 \cos 2\alpha_1},$$

giving the direction of the axes of the combination, while the maximum and minimum values are given by the expression

$$\frac{1}{2} \Sigma F_1 \pm \frac{1}{2} \sqrt{(\Sigma F_1 \sin 2\alpha_1)^2 + (\Sigma F_1 \cos 2\alpha_1)^2}$$

corresponding to the convergences in the two principal meridians. Fig. 5 shows curves exhibiting the relation

Fig. 5.



Power of Combination of two equal cylinders crossed at various angles.

between the convergence and angle for two cylinders crossed at various angles. The properties of such combinations have already been brought before this Society by Dr. Thompson *, but the writer had come to the same conclusions earlier. It is interesting to note that while the displacement of a prism corresponds to an ordinary vector, the curvature of a cylinder is what is called by Steinmetz a double frequency vector, and this can be well illustrated experimentally. In general the distinction which Prof. Thompson has drawn between unipolar

* S. P. Thompson, Proc. Phys. Soc.

and dipolar quantities has its origin in the distinction between simple and double frequency vectors.

Combinations of Two Thin Lenses.—These may be mentioned here before considering thick lenses, as they do not imply a knowledge of refraction at a single surface. Considering two thin lenses of convergences F_0 and F_2 separated by an interval d_1 , parallel light falling on the first emerges from it with a convergence F_0 which becomes $\frac{1}{1/F_0 - d}$ or $\frac{F_0}{1 - F_0 d_1}$ on reaching the second. To this is added the convergence of the second lens, giving us the emergent convergence as

$$\frac{F_0}{1 - F_0 d_1} + F_2 = \frac{F_0 + F_2 - F_0 F_2 d_1}{1 - F_0 d_1}.$$

This may be termed the “back” or “emergent” convergence if desired, corresponding to the “back focus,” and the other emergent convergence will evidently be $\frac{F_0 + F_2 - F_0 F_2 d_1}{1 - F_2 d_1}$.

Equivalent Convergence.—Introducing the equivalent lens as that giving the same size of image as the combination, we have $x = \frac{\tan \alpha}{F}$ or $F = \frac{\tan \alpha}{x}$, where α is the angular magnitude of the object and x the magnitude of the image. For the first lens we have $x_0 = \frac{\tan \alpha}{F_0}$, and the magnification by the second lens is

$$\frac{U_2}{V_2} = \frac{F_0}{1 - F_0 d_1} \frac{1 - F_0 d_1}{F_0 + F_2 - F_0 F_2 d_1} = \frac{F_0}{F_0 + F_2 - F_0 F_2 d_1}.$$

Hence

$$x = m x_0 = \frac{\tan \alpha}{x} \frac{F_0}{F_0 + F_2 - F_0 F_2 d_1} = \frac{\tan \alpha}{F_0 + F_2 - F_0 F_2 d_1},$$

and $F = \frac{\tan \alpha}{x} = F_0 + F_2 - F_0 F_2 d_1$.

Nodal or Principal Points.—The Nodal points being defined as those points through which light passes undeviated, we have, since the deviation at any zone of a lens = Fh ,

$$F_0 h_0 + F_2 h_2 = 0,$$

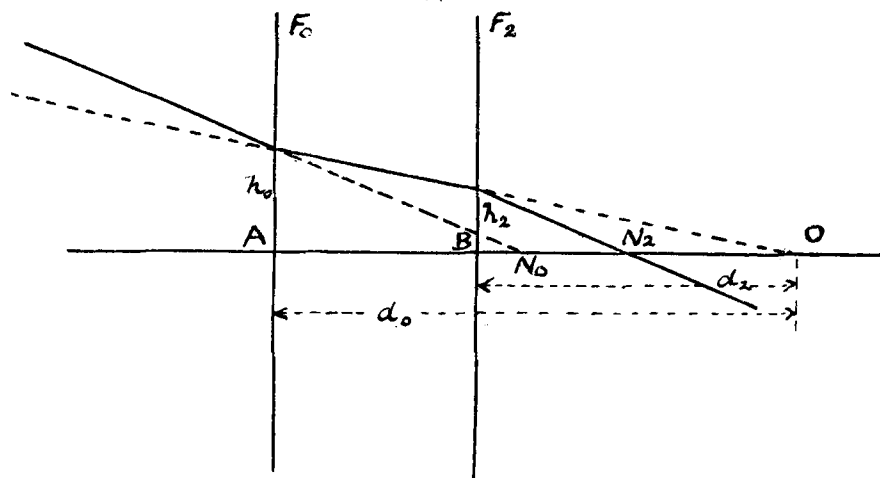
$$\text{or} \quad F_0 d_0 + F_2 d_2 = 0 \quad (\text{see fig. 6}),$$

but $d_0 - d_2 = d$ the distance between the lenses.

Hence $d_0 = \frac{F_2 d}{F_0 + F_2}$ giving the optical centre,

or $D_0 = \frac{F_0 + F_2}{F_2 d}$ the convergence of the light focussing at O.

Fig. 6.



Nodal points of a Two-lens Combination.

Similarly $D_2 = -\frac{F_0 + F_2}{F_0 d}$.

Hence $D_0' = D_0 - F_0 = \frac{F_0 + F_2 - F_0 F_2 d}{F_2 d}$,

the convergence of the light proceeding to N_0 ,

and $d_0' = \frac{F_2 d}{F_0 + F_2 - F_0 F_2 d}$ the distance AN_0 ,

while $d_2' = -\frac{F_0 d}{F_0 + F_2 - F_0 F_2 d}$ the distance BN_2 .

We have seen above that the emergent convergence at the second lens $F' = \frac{F_0 + F_2 - F_0 F_2 d_1}{1 - F_0 d_1}$, and now that the convergence at any distance $d_2 = \frac{F'}{1 - F' d_2}$.

But $d_2 = -\frac{F_1 d}{F_1 + F_2 - F_1 F_2 d}$; $\therefore 1 + F' d_2 = \frac{1}{1 + F_1 d}$,

and therefore convergence at Nodal point =

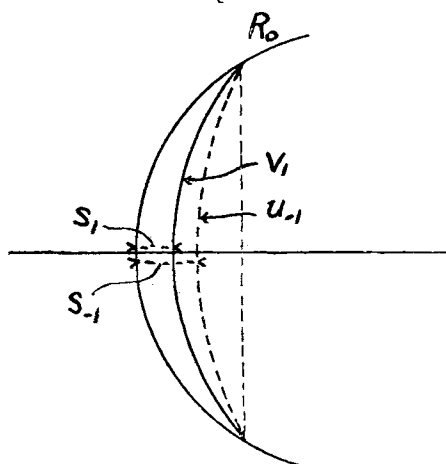
$$\frac{F_0 + F_2 - F_0 F_2 d_1}{1 - F_0 d_1} (1 - F_0 d_1) = F_0 + F_2 - F_0 F_2 d_1 = F;$$

or the convergence in the plane through the nodal point of emergence is the equivalent convergence; or the distance between the nodal and focal points is the equivalent focal length. This latter might of course have been directly obtained by taking the back focal length and distance of nodal point.

Thick Lenses and Refracting Systems.

Refraction at Single Surface.—Up to the present we have not needed to consider refraction at a single curved surface, but it is necessary to do so before studying thick lenses and lens systems. If light passes through a surface (fig. 7)

Fig. 1.



Refraction at a Single Spherical Surface.

separating two media of refractive indices μ_{-1} and μ_1 , we have always as our fundamental equation $\mu_{-1}s_{-1} = \mu_1s_1$, where s_{-1} and s_1 are the departures or sags of corresponding parts of the incident and refracted wave-fronts from the surface.

In the case of a normal homocentric pencil of convergence U and a spherical surface of curvature R_0 this gives us immediately

$$\mu_{-1}(R_0 - U_{-1}) = \mu_1(R_0 - V_1)$$

or

$$\mu_1V_1 - \mu_{-1}U_{-1} = (\mu_1 - \mu_{-1})R_0 = F_0.$$

In the interspace between two surfaces we have

$$U_1 = \frac{1}{\frac{1}{V_1} - t_1} = \frac{V_1}{1 - V_1t_1},$$

where U_1 is the convergence on meeting the second surface and t is the thickness.

This equation can, however, be written

$$\mu_1 U_1 = \frac{1}{\frac{1}{\mu_1 V_1} - \frac{t_1}{\mu_1}}.$$

Nomenclature for Thick Lenses and Lens Systems.—We can greatly reduce the labour and simplify the working of problems relating to lens systems by adopting a suitable nomenclature and notation. This was done by Gauss*, who introduced the idea of “absolute” and “reduced” distances and thicknesses. Following this we may introduce the term “reduced convergence” of a pencil as the product of its actual or “absolute” convergence and the refractive index of the material; while the “reduced thickness,” as with Gauss, is defined as the negative of the “absolute thickness” divided by the refractive index of the medium.

Denoting these quantities by accented letters, we have

$$U' = \mu U, \quad V' = \mu V, \quad t' = -\frac{t}{\mu};$$

and the equations for the surface and interspace are

$$V_1' - U_{-1}' = (\mu_1 - \mu_{-1})R_0 = F_0,$$

and

$$U_1' = \frac{1}{\frac{1}{t_1' + V_1'}} = \frac{V_1'}{V_1' t_1' + 1}.$$

It is worthy of note that each of the reduced quantities has a simple physical meaning. For in the first equation, if the surface is plane

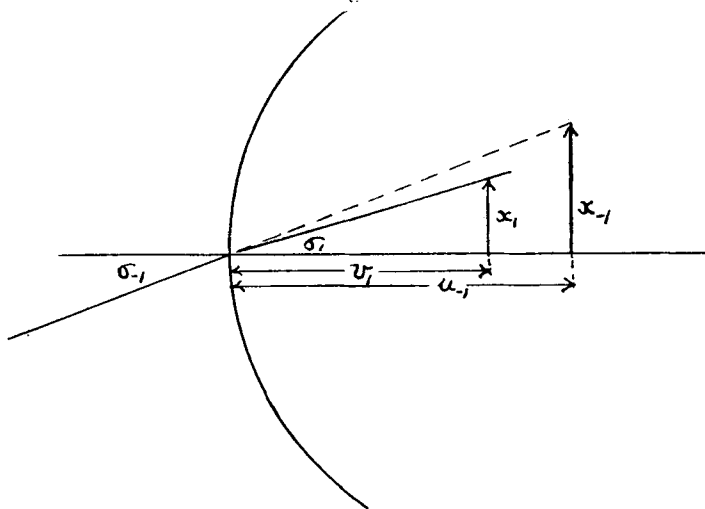
$$V_1' - U_{-1}' = 0, \quad \text{or} \quad V_1' = U_{-1}'.$$

Consequently the reduced convergence of a pencil does not change in passing through a plane surface separating any two media. But if the second medium is air $\mu_1 = 1$ and $V = U_{-1}'$. Hence the reduced convergence of a wave-front in any medium is the convergence it would have if it emerged from that medium into air through a plane surface. The term “equivalent” instead of reduced convergence may therefore be employed if preferred, but is preferably kept for the equivalent convergence of the combination. Similarly $t_1' = -\frac{t}{\mu}$ is the “apparent thickness” of a parallel plate of the medium when viewed in air.

* Pendlebury, ‘Lenses and Systems of Lenses.’

The magnification produced by a single surface is readily seen by the same device of a pinhole stop at the vertex, fig. 8.

Fig. 8.



Magnification by Single Spherical Surface.

We then have

$$x_{-1} = \mu_1 \tan \sigma_{-1} = \frac{\tan \sigma_{-1}}{U_{-1}}$$

and

$$x_1 = \frac{\tan \sigma_1}{V_1}$$

Hence

$$\begin{aligned} m = \frac{x_1}{x_{-1}} &= \frac{U_{-1}}{V_1} \frac{\tan \sigma_1}{\tan \sigma_{-1}} \\ &= \frac{U_{-1}}{V_1} \frac{\sin \sigma_1}{\sin \sigma_{-1}} \text{ approx. } = \frac{U'_{-1}}{V'_1}. \end{aligned}$$

It will be unnecessary to go here more fully into the properties of the single surface, its two principal convergences of the object and image space, principal and nodal points, &c. We will immediately pass to the case of the thick lens.

Simple Thick Lens.—Denoting the curvatures of the surfaces by R_0 and R_2 respectively, and the refractive indices of the media by μ_{-1} , μ_1 , and μ_3 , we have :—

At first surface, $V_1' - U_{-1}' = F_0$ or $V_1' = E_0 + U_{-1}'$

Interspace $U_1' = \frac{1}{t_1' + \frac{1}{V_1'}} = \frac{V_1'}{V_1' t_1' + 1}$

Second surface $V_3' = F_2 + U_1'$

Combining these equations we have immediately

$$V_3' = F_2 + \frac{F_0 + U'_{-1}}{(F_0 + U'_{-1})t_1' + 1} = \frac{F_0 + F_2 + F_0 F_2 F_1' + (F_2 t_1' + 1)U'_{-1}}{F_0 t_1' + 1 + t_1' U'_{-1}} \\ = \frac{A + BU'_{-1}}{C + DU'_{-1}},$$

$$\text{where } \begin{vmatrix} A & B \\ C & D \end{vmatrix} = -1.$$

To find the size of the image produced we have

$$\frac{1}{m} = \frac{1}{m_0} \frac{1}{m_2},$$

where m_0 and m_2 are the successive magnifications at the surfaces.

$$\text{But } \frac{1}{m_0} = \frac{V_1'}{U'_{-1}} \text{ and } \frac{1}{m_2} = \frac{V_3'}{U_1'} = \frac{A + BU'_{-1}}{C + DU'_{-1}} \frac{V_1' t_1' + 1}{V_1'}$$

and $C + DU'_{-1} = V_1 t_1' + 1$ by the above.

$$\text{Hence } \frac{1}{m_2} = \frac{A + BU'_{-1}}{V_1'} \text{ and } \frac{1}{m} = \frac{A + BU'_{-1}}{U'_{-1}}.$$

This re-establishes the three Gauss relations in the convergence form. It should be noted that the quantity A is the reduced equivalent convergence of the lens.

It will not be necessary to show how the properties of thick lenses may be deduced from these equations as methods similar to those of Gauss can be followed. We may therefore pass on to the consideration of a complete system, but before doing so reference may be made to the deviation or Von Seidel method.

If we take our equation for a single surface

$$\mu_1 V_1 - \mu_{-1} U_{-1} = F_0$$

and multiply it by h_0 , the lateral distance of intersection of a pencil, we have

$$\mu_1 \tan \sigma_1 - \mu_{-1} \tan \sigma_2 = F_0 h_0.$$

If we now denote the product of the tangent of an angle by the refractive index of the medium, by the term "reduced angle" and represent these reduced angles by accented letters, we have

$$\sigma_1' - \sigma'_{-1} = F_0 h_0.$$

In the interspace between two refracting surfaces we have clearly

$$h_2 - h_0 = -t_1 \tan \sigma_1' = -\frac{t_1}{\mu_1} \mu_1 \tan \sigma_1',$$

or

$$h_2 - h_0 = \sigma_1' t_1'.$$

Hence, for tracing the course of a pencil through any system, we have simply

$$\text{Surface 0. } \sigma_1' - \sigma'_{-1} = F_0 h_1 \text{ or } \sigma_1' = F_0 h_0 + \sigma'_{-1};$$

$$\text{Interval 1. } h_2 - h_0 = \sigma_1' t_1' \text{ or } h_2 = t_1' \sigma_1' + h_0;$$

$$\text{Surface 2. } \sigma_3' - \sigma'_{-1} = F_2 h_2 \text{ or } \sigma_3' = F_2 h_2 + \sigma'_{-1};$$

and this may be extended to any number of surfaces by putting down the convergences and reduced thicknesses, and the corresponding lateral intersections and reduced angles in the order

$$\begin{array}{ccccccc} & F_0 & t_1' & F_2 & t_3' & \&c. \\ \sigma'_{-1} & h_0 & \sigma_1' & h_2 & \sigma_3' & \&c. \end{array}$$

each member of the lower series being derived by multiplying the penultimate member by the corresponding value above it, and adding the antepenultimate. This is the extremely valuable method due to Von Seidel*.

For a single thick lens we have to take simply the three equations given, and we find

$$\text{and } h_2 = t_1' \sigma_1' + h_0 = (F_0 t_1' + 1) h_0 + t_1' \sigma'_{-1} = C h_0 + D \sigma'_{-1}$$

$$\sigma_3' = F_2 h_2 + \sigma_1' = (F_0 + F_2 + F_0 F_2 t_1') h_0 + (F_2 t_1' + 1) \sigma'_{-1} = A h_0 + B \sigma'_{-1}.$$

We may express this either in the ordinary Gauss or in the convergence form by taking

$$v_3' = \frac{h_2}{\sigma_3'}, \text{ or } V_3' = \frac{\sigma_3'}{h_2}.$$

In the latter case

$$V_3' = \frac{A h_0 + B \sigma'_{-1}}{C h_0 + D \sigma'_{-1}} = \frac{A + B U'_{-1}}{C + D U'_{-1}} \text{ as before.}$$

Extension of Convergence Theory to any System.

Our equations for the convergence method were

$$V_1' - U'_{-1} = F_0;$$

$$U_1' = \frac{1}{t_1' + \frac{1}{V_1'}};$$

$$V_3' - U_1' = F_2, \text{ \&c.}$$

* Dr. S. P. Thompson's translation of Lummer's *Photographic Optics*, Appendix II.

Combining these equations we have for any system

$$V'_{2n+1} = F_{2n} + \frac{1}{t_{2n-1} + \frac{1}{F_{2n-2} + \cdots + \frac{1}{t'_1 + F_0 + U'_{-1}}}}$$

corresponding to the ordinary continued fraction of Gauss.

We could derive the properties of complete lens systems by the theory of continued fractions as was done by Gauss; but it is much more simple and satisfactory to have recourse to a proof by induction, as the writer believes was first done by Dr. Sumpner, for the ordinary Gauss method.

Let it be granted that for a given system the equations

$$V'_{2n-1} = \frac{A + BU'_{-1}}{C + DU'_{-1}} \quad \frac{1}{m_{2n-1}} = \frac{A' + BU'_{-1}}{U'_{-1}} \quad \text{and} \quad \begin{vmatrix} A & C \\ B & D \end{vmatrix} = -1$$

hold.

Now let the light pass through a space t'_{2n+1} and another surface of convergence F_{2n} . We shall have

$$\begin{aligned} V'_{2n-1} &= \frac{A + BU'_{-1}}{C + DU'_{-1}}; \\ U_{2n-1} &= \frac{V'_{2n-1}}{V'_{2n-1}t'_{2n-1} + 1}; \\ V_{2n+1} &= U'_{2n-1} + F_{2n}. \end{aligned}$$

We then have

$$\begin{aligned} V_{2n+1} &= \frac{A(F_{2n}t'_{2n-1} + 1) + CF_{2n} + \{B(F_{2n}t'_{2n-1} + 1) + DF_{2n}\}V'_{-1}}{At'_{2n-1} + (Bt'_{2n-1} + D)U'_{-1}} \\ &= \frac{A' + B'U'_{-1}}{C' + D'U'_{-1}}, \end{aligned}$$

where

$$\begin{vmatrix} A' & C' \\ B' & D' \end{vmatrix} = \begin{vmatrix} A(F_{2n}t'_{2n-1} + 1) + CF_{2n} & At'_{2n-1} + C \\ B(F_{2n}t'_{2n-1} + 1) + DF_{2n} & Bt'_{2n-1} + D \end{vmatrix} = \begin{vmatrix} A & C \\ B & D \end{vmatrix} = -1$$

by the ordinary properties of determinants.

Also

$$\begin{aligned} \frac{1}{m_{2n+1}} &= \frac{1}{m_{2n}} \frac{1}{m_{2n-1}} \\ \frac{1}{m_{2n-1}} &= \frac{A + BU'_{-1}}{U'_{-1}}; \\ \frac{1}{m_{2n}} &= \frac{V'_{2n+1}}{U'_{2n-1}} = \frac{A' + B'U'_{-1}}{C' + D'U'_{-1}} \frac{At'_{2n-1} + C + (Bt'_{2n-1} + D)U'_{-1}}{A + BU'_{-1}} \\ &= \frac{A' + B'U'_{-1}}{A + BU'_{-1}}. \end{aligned}$$

Hence
$$\frac{1}{m_{2n+1}} = \frac{A' + B'U'_{-1}}{U'_{-1}}.$$

Consequently, since these equations were proved to hold for two surfaces, and have now been shown to hold for an additional surface, they must hold generally.

It may be of interest as a conclusion to this section to exhibit the three methods in a comparative form.

Comparative Table of Formulæ for Refracting Systems.

| | GAUSS METHOD. | DEVIATION METHOD. | CURVATURE METHOD. |
|--|--|---|--|
| Surface 0 ... | $\frac{1}{v'_1} - \frac{1}{u'_{-1}} = F_0$ | $\sigma'_1 - \sigma'_{-1} = F_0 h_0$ | $V'_1 - U'_{-1} = F_0$ |
| Interval 1 ... | $u'_1 = v'_1 + t'_1$ | $h_2 - h_0 = t'_1 \sigma'_1$ | $U'_1 = \frac{V'_1}{t'_1 V'_1 + 1}$ |
| Surface 2 ... | $\frac{1}{v'_3} - \frac{1}{u'_1} = F_2$ | $\sigma'_3 - \sigma'_1 = F_2 h_2$ | $V'_3 - U'_1 = F_2$ |
| Interval 3 ... | $u'_3 = v'_3 + t'_3$ | $h_4 - h_2 = t'_3 \sigma'_3$ | $U'_3 = \frac{V'_3}{t'_3 V'_3 + 1}$ |
| &c. | &c. | &c. | &c. |
| Results referred to first and last surfaces. | $v'_{2n+1} = \frac{Cu + D}{Au + B}$ $\frac{1}{m} = Au + B$ $AD - BC = -1$ | $h_{2n} = Ch_0 + D\sigma'_{-1}$ $\sigma'_{2n+1} = Ah_0 + B\sigma'_{-1}$ $\frac{1}{m} = \frac{Ah_0 + B\sigma'_{-1}}{\sigma'_{-1}}$ $AD - BC = -1$ | $V_{2n+1} = \frac{A + BU'_{-1}}{C + DU'_{-1}}$ $\frac{1}{m} = \frac{A + BU'_{-1}}{U'_{-1}}$ $AD - BC = -1$ |
| Results referred to principal planes. | $f' = \frac{1}{A}$ $\frac{1}{v'} - \frac{1}{u'} = \frac{1}{A}$ $m = \frac{v'}{u'}$ | $\sigma'_{2n+1} - \sigma'_{-1} = Ah^*$ $m = \frac{\sigma'_{-1}}{\sigma'_{2n+1}}$ | $F' = A$ $V' - U' = A$ $m = \frac{U'}{V'}$ |

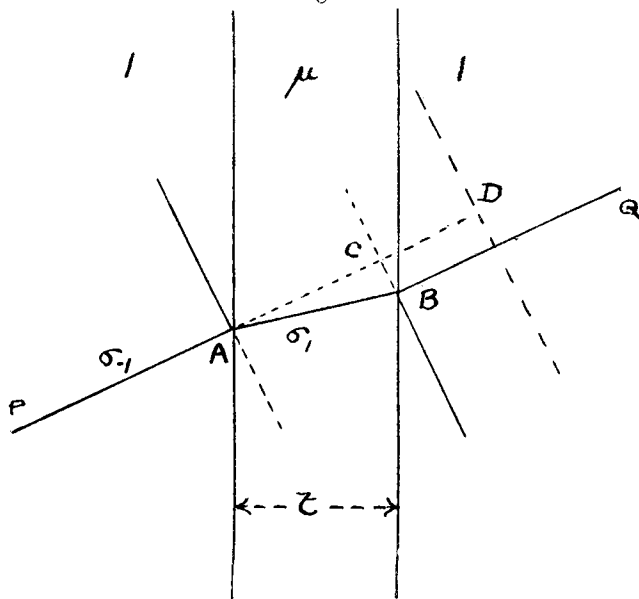
* h is here the height of intersection with the principal planes.

Aberrations.

The subject of aberrations will have to be treated in a separate paper; but to show how the curvature method lends itself to aberration problems, we will here take the case of oblique refraction in thin lenses. Just as we started the ordinary theory of thin lenses by considering the retardation produced by a plate, we may here commence with oblique refraction by a similar plate.

Retardation by Oblique Parallel Plate.—In fig. 9, PABQ is the course of a parallel beam of light, AB being the

Fig. 9.



Oblique Refraction by a Parallel Plate.

distance travelled through the plate ; we have D as the position the wave-front would have reached in the absence of the plate such that

$$AD = \mu \cdot AB$$

and retardation

$$s = \mu \cdot AB - AC.$$

But $AB = \frac{t}{\cos \sigma_1}$ and $AC = AB \cos (\sigma_{-1} - \sigma_1).$

Hence $s = \frac{\mu - \cos (\sigma_{-1} - \sigma_1)}{\cos \sigma_1} t.$

Putting $\sin \sigma_{-1} = \mu \sin \sigma_1$

and simplifying we have

$$s = \{ \sqrt{\mu^2 - \sin^2 \sigma_{-1}} - \cos \sigma_1 \} t$$

accurately, or to a second approximation

$$s = (\mu - 1) \left(1 + \frac{\sigma_{-1}^2}{2\mu} \right) t.$$

Oblique Refraction through Thin Lens.—In the case of a thin lens we may consider the centre of the lens as a parallel plate, and the retardation of the centre of the wave-front is therefore as given above. Hence, if we consider two meridians of the lens, the primary or meridional plane containing the axis of the lens and beam, and the secondary or sagittal plane lying in the axis of the beam perpendicular to the other plane,—

We have in the sagittal plane therefore if ϕ is the obliquity

$$\mathcal{H}_2 = F \left(1 + \frac{\phi^2}{2\mu} \right).$$

In the meridional plane the retardation of the centre of the beam is the same, but the effective aperture of the lens is reduced in the ratio of $\cos \phi$, and the curvature therefore increased in the ratio $\frac{1}{\cos^2 \phi} = 1 + \phi^2$ approx. and hence the meridional convergence

$$\mathcal{H}_1 = F \left(1 + \frac{\phi^2}{2\mu} \right) (1 + \phi^2).$$

The lens therefore acts as a spherocylinder of spherical power $F \left(1 + \frac{\phi^2}{2\mu} \right)$ and cylindrical power $F\phi^2$, or when ϕ is in degrees

$$\text{Spherical power} = F \left(1 + \frac{\phi^2}{5622\mu} \right).$$

$$\text{Cylindrical } ,, = \frac{F\phi^2}{2861}.$$

The axis of the cylinder is of course in the sagittal plane.

The wave-front also evidently reaches one edge of the lens before the other, giving rise to asymmetrical refraction or coma, but this will not be further considered here.

Primary and Secondary Focal Lines.—We have evidently

$$\text{Primary convergence} = \mathcal{H}_1 = F \left\{ 1 + (2\mu + 1) \frac{\phi^2}{2\mu} \right\}$$

$$\text{Mean } ,, = \mathcal{H}_0 = F \left\{ 1 + (\mu + 1) \frac{\phi^2}{2\mu} \right\}$$

$$\text{Secondary } ,, = \mathcal{H}_2 = F \left\{ 1 + \frac{\phi^2}{2\mu} \right\}.$$

The mean convergence corresponds to the circle of least
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confusion. Or in focal lengths we have

$$\text{Primary focal length } f_1 = f' \left\{ 1 - (2\mu + 1) \frac{\phi^2}{2\mu} \right\}$$

$$\text{Mean } \quad \quad \quad f_0 = f' \left\{ 1 - (\mu + 1) \frac{\phi^2}{2\mu} \right\}$$

$$\text{Secondary } \quad \quad \quad f_2 = f' \left\{ 1 - \frac{\phi^2}{2\mu} \right\}$$

Curvature of Focal Surfaces.—In order that the oblique foci should lie on a plane through the principal focus we have

$$\mathcal{C} = F \cos \phi = F \left(1 - \frac{\phi^2}{2} \right) \text{ approx.}$$

If $\Delta \mathcal{C}$ is the excess of the actual convergence over that necessary to focus on the plane, we have

$$\text{sag } \Delta f = \frac{\Delta \mathcal{C}}{F^2},$$

and lateral distance of focus from principal focus

$$w = \frac{\tan \phi}{F} = \frac{\phi}{F} \text{ approx.}$$

$$\text{Hence curvature} = \frac{2\Delta f}{x^2} = \frac{2\Delta \mathcal{C}}{\phi^2} \text{ approx.}$$

Applying this to the above we have :

$$\begin{aligned} \text{Curvature of primary focal surface,} \quad R_1 &= \frac{3\mu + 1}{\mu} F = 3F + \frac{F}{\mu} \\ \text{,, of surface of least confusion, } R_0 &= \frac{2\mu + 1}{\mu} F = 2F + \frac{F}{\mu} \\ \text{,, of secondary focal surface, } R_2 &= \frac{\mu + 1}{\mu} F = F + \frac{F}{\mu} \end{aligned}$$

The corresponding radii of curvature can be at once written down and are in agreement with those given by Mr. Dennis Taylor and others*. It is also obvious from the method of proof that the curvature of the image surfaces is independent of the position of the object plane.

Focal Surfaces for Thin Lenses in contact.—Since the curvature is given by $\frac{\Delta \mathcal{C}}{\phi^2}$, and the convergence of any

* H. Dennis Taylor, "Design of Telescope Object Glasses," Proceedings Royal Astron. Soc.

number of thin lenses in contact is the sum of their individual convergences, we immediately have the interesting result that for combinations of thin lenses in contact, without diaphragm separated from the combination, the curvatures of the fields may be added together just as with convergences. We thus have

$$R_1 = 3\mathbf{F} + \Sigma \frac{\mathbf{F}}{\mu}$$

$$R_0 = 2\mathbf{F} + \Sigma \frac{\mathbf{F}}{\mu}$$

$$R_2 = \mathbf{F} + \Sigma \frac{\mathbf{F}}{\mu}$$

where

$$\mathbf{F} = \Sigma F.$$

For such a combination the mean curvature therefore is $2\mathbf{F} + \Sigma \frac{\mathbf{F}}{\mu}$, or $2\mathbf{F}$ if the Petzval condition is satisfied, and under these circumstances R_1 , R_2 , and $R_3 = \mathbf{F}$, $2\mathbf{F}$, and $3\mathbf{F}$ respectively, while the curvature corresponding to the "astigmatic difference" is always $2\mathbf{F}$ independently of the materials of the lenses.

This example will serve to show how easily and directly aberration problems may be solved by physical methods, and the writer proposes to show in another paper that the whole theory of aberrations in refracting systems may be similarly treated with advantage. He ventures to hope, however, that enough has been said in this paper to convince everyone that not only is there no necessity for the abandonment of curvature methods at any stage in optical work, but that there is every advantage in retaining them throughout. It will also be obvious that they lend themselves to combining the study of diffraction and image formation, which should lead to valuable new results.

XLIV. *On Maxwell's Stress Theory.* By V. BJERKNES*.

MAXWELL considered his theory of the stress in the dielectric medium as very important. But, on the other hand, he did not regard it as complete. His own words plainly prove both assertions † :—

"It must be carefully borne in mind that we have made only one step in the theory of the action of the medium.

* Communicated by the Author.

† Maxwell, 'Electricity and Magnetism,' 2nd edition, vol. i. p. 154.