(ii.) Four lines in space can be regarded as two pairs in three ways. If the common normals of the two pairs are themselves normal in two of the ways, they are so in the third. This statement of the case was suggested by Mr. Richmond.

(iii.) Two rectangular pentagons can be normal to each other; that is, each side of the one can be normal to a side of the other.

There are in the configuration ten rectangular hexagons, five systems of four lines of the kind just mentioned, and six pairs of mutually normal rectangular pentagons.

Point-Groups in a Plane, and their effect in determining Algebraic Curves.

By F. S. MACAULAY, D.Sc. Read and received June 9th, 1898.

I. INTRODUCTION.

The following is a continuation of my former paper on "Point-Groups in relation to Curves" in Vol. XXVI. of the *Proceedings*, p. 519. It deals especially with the reduction of point-groups which supply a known number of conditions for an algebraic curve of any order.

The effect of a group of N points in determining an algebraic curve of order n (called hereafter a C_n) need not depend on N and n alone. It may, and often does, happen that the N points do not supply N independent conditions for a C_n, but only a smaller number $N-r_n$. In any case, if the point-group N is given, the number r_n has a definite positive* (integral or zero) value. The extreme case is that in which all the N points lie on a straight line; and we then have $r_n = N-(n+1)$ if $n \leqslant N-1$, and $r_n = 0$ if $n \geqslant N-1$.

For the case in which the N points form the complete intersection of two curves, the values of r_n for all values of n have long been known. Thus, if N consists of the complete intersection of a C_l and C_m, and if n is less than $l+m$, but not less than l or m, then

$$r_n = \tfrac{1}{2}(l+m-n-1)(l+m-n-2),$$

* In this paper the curves are subject to no other conditions than those of passing through points, *i.e.*, they are general algebraic curves through given point-groups in which the parameters or coefficients enter linearly. For linear systems of curves no value of r could be negative; this would not be true for non-linear systems.

by Cayley's theorem; and the value of r_n for *any* value of n may be written in the form

$$\left[\tfrac{1}{2}(l+m-n-1)(l+m-n-2)\right]-\left[\tfrac{1}{2}(m-n-1)(m-n-2)\right]$$
$$-\left[\tfrac{1}{2}(l-n-1)(l-n-2)\right],$$

each pair of square brackets indicating that the product enclosed is only to be retained so long as its individual factors are positive.

But the case in which N is the complete intersection of two curves is only a very special one. It is easy to give other examples. In general, if two point-groups N, N' together make up the complete intersection of two curves C_l, C_m, having no common factor, and if for N there are values of r_n which do not vanish, then for N' there are corresponding values of $r'_{n'}$ which do not vanish, where $n+n'=l+m-3$.

We name the number r_n the *n-ic excess* of the point-group N. It is the number of points which lie on each and every C_n drawn through $N-r_n$ of the N points, *provided these* $N-r_n$ *points are so chosen that their n-ic excess is zero.* Such a choice is evidently always possible,* although it may also be possible to choose $N-r_n$ of the N points whose n-ic excess is not zero.

Corresponding to r_n there is a complementary number q_n, viz., the degree of freedom of a C_n through N, which we name the *n-ic defect* of the point-group N.† This is the number of general points in the

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* Such a selection may be made with certainty by choosing the $N-r_n$ points one at a time, each new one being so chosen that a C_n through all those previously chosen does not necessarily pass through it. In this way we must arrive at $N-r_n$ points and no more, since the n-ic excess of N is r_n, neither more nor less.

† [*Note added October 13th.*—The terminology of the theory of point-groups is extensive. This is owing partly to the number of descriptive terms required to distinguish various kinds of point-groups and their special characteristics, and partly to the fact that there are several essentially different ways of approaching the subject. One branch of the subject has been confined almost entirely to Germany, another to Italy, but the branch to which the present paper belongs—that in which the point-groups themselves are subjected to direct operation—has been developed both by English and Continental mathematicians.

The term *regular* (*regolare*) has been applied by Professor G. Castelnuovo to a system of curves in reference to a given point-group when the point-group possesses no *excess* for the general curve of the system. By a slight inversion we may say that a point-group is regular with respect to a general curve of order n when $r_n = 0$. Point-groups which are regular with respect to all general algebraic curves which can be drawn through them I call *general* point-groups, thereby implying only that they are general in their effect in determining algebraic curves.

A convenient English term for point-groups which are not general in the above sense is desirable. The term *constructional* (instead of *point-group of special form*, used in my former paper) seems not inappropriate, since such point-groups can be constructed as the partial (or total) intersection of algebraic curves. The old use of the term *special* was applied to a point-group on a base-curve C_m for which $R-r<p-1$ (p being the deficiency of C_m, R the number of points, and r the multiplicity, of the point-group). The modern use of *special* is, however, applied to any

plane through which a C_n can be drawn which already passes through the N points. Hence, since the N points supply exactly $N-r_n$ conditions for a C_n, and the q_n points, being general, q_n more conditions, we have

$$N-r_n+q_n = \tfrac{1}{2}n\,(n+3),$$

or

$$N-r_n+q_n+1 = \tfrac{1}{2}\,(n+1)\,(n+2) \qquad \text{(1)}$$

Hence also

$$r_{n-1}-r_n = n+1-(q_n-q_{n-1}). \qquad \text{(2)}$$

The effect of a point-group N with respect to algebraic curves of all orders is not known unless all the excesses, or all the defects, are known. As we shall find it convenient to regard the subject chiefly from the point of view of the defects, we may suppose at once that the defects q are all given. If $q_n = 0$, one, and only one, C_n can be drawn through N. If no C_n can be drawn through N, we assign the value -1 to q_n, the reason for which will be given later (p. 677). It follows that q_n and r_n have definite values for *all* values of n, which always satisfy formula (1) above. It is also shown later that, if all the defects are given, the value of the number N can be deduced.

We may say then that a point-group is fully characterized if all the defects, or all the excesses, are given; and that the point-group has a partially or completely assigned *characterization* according as some or all of the defects, or excesses, are assigned. Two of the most general questions that suggest themselves are:—(i.) What is, and what is not, a possible characterization for a point-group? (ii.) What is the quickest method of arriving at or constructing a point-group with an assigned characterization? These and other questions are answered in the following paper.

The above remarks refer to groups of points which coincide at most in pairs. In the first sections of the paper I have dealt only with ordinary point-groups of this type; but in the footnotes on "Multiple Points,"*

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point-group for which $R-r \leqslant p-1$, that is, to any point-group on C_m which lies on an adjoined C_{m-3}. (*Cf.* Professor Charlotte Angas Scott, "Intersections of Plane Curves," *Bulletin of the American Mathematical Society*, 2nd series, Vol. IV., March, 1898, p. 267). Miss Scott suggests the convenient term *intraspecial* in place of the old term *special*. The connexion between *intraspecial* and *constructional* is expressed as follows:—"An intraspecial point-group on a C_m to which is added an $(i-1)$-point at each i-fold point of C_m is a constructional point-base through which a C_{m-3} can be drawn."

For the meaning of the terms *i-point* and *point-base*, see note on "Multiple Points. I.," below.]

* [*Multiple Points. I.*—It is well known that the number of conditions supplied to a curve by an i-fold point is $\tfrac{1}{2}i\,(i+1)$, and that these conditions, when combined with others, may quite possibly not be independent. It is very convenient in the geometrical theory of point-groups to replace, if possible, these

and in the last section of the paper, I have shown how the whole question is capable of generalization.

II. Summary and Deductions.

We use the letter ρ to denote the difference of two successive defects. Thus $\rho_n = q_n - q_{n-1}$. If ρ_l is the first ρ which does not vanish, we have $q_l + 1 = \rho_l$ and $q_p + 1 = \rho_l + \rho_{l+1} + \ldots + \rho_p$ ($p \geqslant l$). The differences of the ρ's, or second differences of the defects, are denoted by δ, so that $\delta_l = \rho_l$, $\delta_p = \rho_p - \rho_{p-1}$.

(i.) Writing the general equation of a C_n, referred to two coordinate axes Ox, Oy, in the form

$$C_n \equiv u_0 + u_1 + \ldots + u_n = 0,$$

where u_p is a homogeneous function of x, y of order p, with $p+1$ coefficients, and substituting the coordinates of each of the N points in $C_n = 0$, we have a set of N linear equations for the coefficients of C_n. These equations have always a solution, whether a C_n can be

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$\frac{1}{2}i(i+1)$ conditions by those of passage through $\frac{1}{2}i(i+1)$ points. (*Cf.* pp. 508, 509 of the *Proceedings*, Vol. xxvi.). The theorem which renders such an interchange of conditions practicable is as follows:—"Given a curve C_n, with any number and kind of multiple points, it is always possible to find a curve $C'_{n'}$ (n' being either equal to or greater than n) whose coefficients differ only to an infinitely small extent from those of C_n, and such that corresponding to each and every i-fold point A of C_n the curve $C'_{n'}$ passes through $\frac{1}{2}i(i+1)$ points chosen arbitrarily and generally about and infinitely near to A." All that is necessary then to effect the change required is to place at each i-fold point ($i > 1$) a general set of $\frac{1}{2}i(i+1)$ points on an infinitely small scale, and to consider in the place of any curve C_n, with the given multiple points, a proximate curve $C'_{n'}$, which passes through all the sets of $\frac{1}{2}i(i+1)$ points, and of which C_n is the limit.

Such a set or *cluster* of $\frac{1}{2}i(i+1)$ points may itself be called a point of order i; we shall therefore call it an *i-point*, reserving the term *i-fold point* for a multiple point of order i on a curve. Thus i-point and i-fold point are practical equivalents; but the one refers to an element of a point-group, and the other to an element of a curve. It should be noticed that an ordinary point is a point of order 1, that is, a 1-point. An i-point is equivalent to $\frac{1}{2}i(i+1)$ simple points. The degree N of a point-group, that is, the total number of simple points to which it is equivalent, is given by $N = \frac{1}{2}\Sigma i(i+1)$, the summation extending to all the points of the group, including those for which $i = 1$.

We also give the name *point-base* (meaning "base of points" = *gruppo base*, Castelnuovo) to a point-group made up of points of assigned orders. In Section vi. we generalize the meaning of this term, and distinguish the point-base here defined by the added epithet *simple*. We shall then, hereafter, only use the term *point-group* in the restricted sense of a group of points which are all of order 1. This is the sense in which it is used throughout the text. The term *point-base* must be distinguished from *base-point;* the latter, however, we shall have no need to use.

The chief importance of this method of dealing with multiple points is that any two curves drawn through an i-point must each have there an i-fold point, and intersect there again in an $(i-1)$-point. Consequently, in the reduction of a point-base, each step leads to a reduction throughout in the orders of the points, while the new points introduced are all of order 1.—*October* 13*th*.]

drawn through N or not. If the N points lie on a C_n, the general solution of the N equations simply determines a certain number of the coefficients of C_n in terms of the rest, which are left arbitrary. The number so determined is $N-r_n$; and the number left arbitrary is q_n+1, since a C_n through N has still a degree of freedom q_n. If N does not lie on a C_n the solution requires all the coefficients to vanish, and the number determined is $N-r_n = \frac{1}{2}(n+1)(n+2)$, and the number left arbitrary is $q_n+1=0$. *Hence we say that the n-ic defect of a point-group N which does not lie on a C_n is always* -1.

Returning to the case in which a C_n can be drawn through N, the number of arbitrary coefficients in $u_0+u_1+\ldots+u_p$ $(p \leqslant n)$ is q_p+1, by the same reasoning as before; and the number in $u_0+u_1+\ldots+u_{p-1}$ is $q_{p-1}+1$. *Hence the number of arbitrary coefficients in u_p is q_p-q_{p-1}, i.e., ρ_p.* The number ρ_p has therefore a precise analytical interpretation; and, consequently, the properties mentioned below can easily be interpreted analytically. It follows that one limitation to the value of ρ_p is given by $\rho_p \leqslant p+1$, since u_p contains only $p+1$ coefficients in all; but it will be seen later that this limitation disappears when we regard the ρ's from a slightly different point of view.

(ii.) *In order that a point-group N may be a possible one it is necessary and sufficient that the values of the ρ's, after ceasing to be zeros, should consist of continually increasing positive integers, subject to the limitations* $\rho_p \leqslant p+1$ *and* $N+q_p+1 \geqslant \frac{1}{2}(p+1)(p+2)$.

Both limitations disappear when we regard the number N and the orders of the curves as being given by the values of the ρ's. A still simpler way of enunciating the theorem is :—

A point-group is possible if the second differences of the defects, after once ceasing to be zeros, are positive integers, not including zero; otherwise a point-group is impossible.

It follows from this theorem that if only a C_l can be drawn through N for which the excess of N is not zero, then must $q_l+r_l \leqslant l$. For, in such a case, we have

$$q_{l-1} = -1, \quad r_{l+1} = 0;$$

therefore $$\rho_l = q_l - q_{l-1} = q_l+1;$$

and $$r_l - r_{l+1} = l+2-(q_{l+1}-q_l), \quad \text{by (2), p. 675};$$

therefore $$\rho_{l+1} = l+2-r_l,$$

and $$\rho_{l+1}-\rho_l-1 = l-q_l-r_l \geqslant 0.$$

If $q_l + r_l > l$, *and* $q_l \geq 0$, *N must have excess for more than a* C_l *through N.**

(iii.) *If* $\delta_p \equiv \rho_p - \rho_{p-1} = 1$, *and* $p > \rho_{p-1} > 0$, *then any* C_p *through N must contain a fixed constituent curve of order* $p - \rho_{p-1}$.

Not only the order $p - \rho_{p-1}$ of the fixed curve, but the number of points N' on it, is known, and also the full characterization of N' and that of the remainder N'' of the N points.

Thus a point-group N, fully characterized, can be separated into as many constituent point-groups as there are sets of one or more successive δ's equal to 1, each set being preceded and succeeded by one or more δ's greater than 1, together with a remainder.

The points of each constituent (including the remainder when it is not a general point-group) must have a certain number of interconnexions among themselves; but, so far as the characterization of N affects the result, there will not be any connexions between any two of the constituents. Each constituent has its own characterization, and, when its construction has been found, can be placed in any position, without reference to the positions of the other constituents. The sum thus obtained forms the most general point-group N with the assigned characterization. This property of the independence of the constituents, which is not easily apparent by intuition, is here emphasized, since it evidently results in a considerable simplification.†

(iv.) Two theorems suffice for the quickest reduction of a point-group. The first is the theorem mentioned in (iii.), which, in terms of the second differences of the defects, may be expressed as follows:—

(α) *If* $\delta_a = 1$, *then*

$$(\delta_1, \delta_2, \ldots, \delta_a, \ldots, \delta_b) = (\delta_1, \delta_2, \ldots, \delta_{a-1}) + (1^{\delta_1+\delta_2+\ldots+\delta_a}, \delta_{a+1}, \delta_{a+2}, \ldots, \delta_b).$$

The other theorem is

$$(\beta)\ (1^a, \delta_b + 1, \delta_{b+1} + 1, \ldots, \delta_c + 1) + (\delta_c, \delta_{c-1}, \ldots, \delta_b) = I(\Sigma\delta, a + \Sigma\delta),$$

where $I(l, m)$ *denotes the complete intersection of a* C_l *and* C_m.

The notation is explained in the next section; but the following remarks will perhaps be intelligible. The characterization of N is fully represented by the numbers

$$\ldots, 0, 0, \delta_1, \delta_2, \ldots, \delta_a, \ldots, \delta_{b-1}, \delta_b, 1, 1, 1, \ldots, ad\ inf.$$

* In the same way it can be proved that if $q_l \geq 0$, $r_{n-2} > 0$, $l < n-2$, then $q_{l-1} + 1$ cannot be zero unless $q_{l+1} + 1 > 2(q_l + 1)$, and r_{n-1} cannot be zero unless $r_{n-3} \geq 2r_{n-2}$.

† This property does not appear to hold in general for a point-base.

Any δ, or any set of successive equal δ's, preceded and succeeded by δ's of higher value is a minimum; and any minimum which reaches its lowest possible value 1 may be called a breaking-point. Theorem (α) exhibits the result after breaking. If there is no breaking-point, theorem (β) shows that from N we can derive a point-group N' such that the δ's of N' are simply the δ's of N each diminished by 1, in reversed order. It is thus evident that all the minima reach breaking-point, some time or other, before the reduction of N has been completed. If the δ's never decrease until the greatest value is reached, and after that never increase, there will be no minimum, and no breaking-point during the whole reduction.

The second differences of the excesses of N are simply δ_1-1, δ_2-1, ..., δ_a-1, ..., δ_b-1, and exhibit the same properties as the second differences of the defects; but the breaking-point is at the value 0 instead of 1.

(v.) *If a C_l is the lowest curve through a point-group N, and a C_{n-2} the highest curve for which the excess of N does not vanish, then the number of the independent interconnexions of the N points, due to the characterization, is*

$$\rho_l(\rho_{l+2}-\rho_{l+1}-1)+\rho_{l+1}(\rho_{l+3}-\rho_{l+2}-1)+\ldots+\rho_{n-2}(\rho_n-\rho_{n-1}-1).*$$

If N breaks up into constituents, *i.e.*, if, for one or more values of p between l and n, $\rho_p-\rho_{p-1}-1=0$, then there are zero terms in the above series, which divide the whole into shorter series. These give

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* This result can be compared with the formula $(q+1)r$ given by Brill and Nöther as the number of conditional equations for the existence of a point-group G_R^r on a C_n. (*Cf.* Benoist's translation of Clebsch, *Leçons sur la Géométrie*, Vol. III., pp. 53 ff.) In the notation adopted above this formula would be written $(q_{n-3}+1)r_{n-3}$. The validity of the formula $(q+1)r$ really rests, however, on the hypothesis that the point-group G_R^r has excess only for a C_{n-3} adjoined to C_n, which requires, as we have seen in (ii.) above, $q+r\leqslant n-3$. If we suppose the excess of the point-group N to be similarly restricted, the number of its interconnexions reduces to

$$\begin{aligned}\rho_{n-3}(\rho_{n-1}-\rho_{n-2}-1) &= (q_{n-3}+1)(n-1-\overline{q_{n-2}-q_{n-3}}) \\ &= (q_{n-3}+1)(r_{n-3}-r_{n-2}) = (q_{n-3}+1)r_{n-3},\end{aligned}$$

so that, on this supposition, the two results agree. If $q+r>n-3$, the formula $(q+1)r$ is not in general valid.

It is easy to show that on a non-hyperelliptic curve in which $n>p>1$ the condition $q+r\leqslant n-3$ is necessarily satisfied. For, by the Riemann-Roch equations, $q+r=p-1-(R-2r)$, and, by Bertini's addition to Clifford's theorem, $R-2r>0$. Hence $q+r\leqslant p-2$; and $n\geqslant p+1$; therefore $n-q-r\geqslant 3$, *i.e.*, $q+r\geqslant n-3$. (*Cf.* Miss F. Hardcastle, p. 133 of this volume of the *Proceedings*, and references there given.)

the numbers of the interconnexions of the points of the several constituents, the first series corresponding to what has been called the remainder. As already remarked, the different constituents are unconnected with each other.

There is no reason why a certain number of the points should not coincide in pairs. A point-pair determines a direction, and is expressible in terms of three parameters, viz., the direction and two coordinates. If there are D point-pairs, the $2N$ coordinates of the N points thus reduce to $2N-D$ parameters, and the formula of the theorem then gives the number of independent conditional equations satisfied by these $2N-D$ parameters. A C_n through N will in general have fixed tangents at the point-pairs; but one additional condition applied to C_n at any point-pair will cause it to have there a double point, leaving the directions of its tangents free. It is not so clear as to what interpretation should be put upon the formula if some of the N points coincide in threes or more.

III. Preliminary Theorems and Notation.

(i.) *The n-ic excess r_n of any point-group N diminishes as n increases, until it becomes zero.*

For a C_n can be drawn through any $N-r_n-1$ of the N points without passing through (all) the remainder, and a straight line C_1 can be drawn through any one of the N points without passing through any more. Hence a C_{n+1}, viz., $C_n C_1$, can be drawn through any $N-r_n$ of the N points without passing through (all) the remainder; not so a C_n (Note, p. 674). Hence $r_{n+1} < r_n$.

(ii.) (*a*) *If a C_p through N is necessarily degenerate, then one constituent of C_p must be fixed.*

Choose any q_p general points in the plane. Then there is one and only one C_p through the $N+q_p$ points. Suppose that this C_p breaks up into $C_{p'} C_{p''}$, and let N' of the N points and q' of the q_p points lie on $C_{p'}$, and the remaining N'' of the N points and q'' of the q_p points lie on $C_{p''}$. Then, since the $q'+q''$ points are general, q' must be the p'-ic defect of N', and q'' the p''-ic defect of N''. Hence, if neither q' nor q'' is zero, a $C_{p'}$ can be drawn through N' and any q' of the $q'+q''$ points, and a $C_{p''}$ through N'' and the remaining q'' of the $q'+q''$ points; and this $C_{p'}$ and $C_{p''}$ would make up a second C_p

through the $N+q_p$ points, which is impossible.* Hence q' or q'' is zero; let q' be zero; then q_p is the p''-ic defect of N''. Hence, in whatever new position we choose q_p general points, the C_p through them and the N points will consist of the $C_{p''}$ through the $N''+q_p$ points, and a fixed $C_{p'}$ through the N' points (whose p'-ic defect is zero).

Also any curve lower than a C_p through N must have the same fixed constituent $C_{p'}$.

(β) *If a C_{n-2} is the highest curve for which the excess of N does not vanish, then a proper C_n can be drawn through N.*

For, if C_n is necessarily degenerate, it must have a fixed constituent $C_{n'}$. Also a C_{n-1} can be drawn through all the N points except one (chosen on $C_{n'}$) without passing through the last, since $r_{n-1}=0$. This C_{n-1} cannot have $C_{n'}$ for a constituent, for, if it had, it would pass through the last point. Also a C_1 can be drawn through the last point in any arbitrary direction. Then $C_{n-1}C_1$ is a C_n through N, not having $C_{n'}$ for a constituent. Hence a proper C_n can be drawn through N.

It may be that a C_{n-1} through N is necessarily degenerate, or that there is no C_{n-1} through N; but a C_{n-1} can be drawn through any $N-1$ of the N points.

(iii.) We express the orders of curves in terms of l, m, n, p, the first three being generally fixed, and the last, p, having any value. A C_l is the lowest curve through N, and a C_{n-2} the highest curve for which N has excess; and we suppose, in general, that N lies on a C_{n-2}. A C_m is the lowest curve through N which has not any fixed constituent, and is not fixed as a whole. We always have $l \leqslant m \leqslant n$.

If the suffixes of q, r, ρ, δ are expressed in terms of l, m, n, p, they are to be understood as having explicit reference to the orders of curves; thus q_p is the p-ic defect of N, and $\rho_p = q_p - q_{p-1}$, $\delta_p = \rho_p - \rho_{p-1}$.

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* [*Multiple Points. II.*—This reasoning fails in a special case when we are dealing with a point-base N. This happens when the constituents of the second C_p are a simple rearrangement of those of the first, so that the C_p itself is not changed. One constituent of C_p, say $C_{p'}$, may be assumed to be a proper curve. If then $q' \geqslant 2$, we could interchange 1 of the q' points with 1 of the q'' points, thus obtaining a second set of q' points, and a second $C_{p'}$, which is certainly not a constituent of the original C_p; in this case therefore the theorem will hold, as also when $q'=0$. But if $q'=1$, it might happen that any second $C_{p'}$, was necessarily a constituent of the original C_p. In this case every proper constituent of the general C_p through N must have defect 0 or 1, and the q_p constituents with defect 1 must belong to a fixed pencil of curves. The conclusion is that, *when a C_p through a point-base N is necessarily degenerate, either* (i.) *one constituent of C_p is fixed, and may break up into several parts, while the remaining constituent is a proper curve with defect q_p; or* (ii.) *C_p breaks up into q_p constituents belonging to a fixed pencil of curves, and the remaining constituents of C_p, if any, are fixed absolutely.—October* 13*th.*]

But if the suffixes are 1, 2, 3, ..., a, $a+1$, ..., b, ..., they refer only implicitly to the orders of curves; q_a is not the a-ic defect of N, but we still have $\rho_a = q_a - q_{a-1}$, $\delta_a = \rho_a - \rho_{a-1}$.

The symbol $(\rho_1, \rho_2, ..., \rho_a)$ represents a point-group whose successive ρ's are $\rho_1, \rho_2, ..., \rho_a$, every ρ before ρ_1 being zero, and ρ_{b+1} being equal to $\rho_b + 1$ when $b \geqslant a$. We suppose also that $\rho_1 > 0$, and $\rho_a > \rho_{a-1}+1$; for, if not, we can simply leave out the ρ's at the beginning and end until these inequalities are established.

The following properties of the point-group $N = (\rho_1, \rho_2, ..., \rho_a)$ will be easily seen by supposing it to be the same as $(\rho_l, \rho_{l+1}, ..., \rho_n)$, so that $\rho_1 = \rho_l$, $\rho_2 = \rho_{l+1}$, ..., $\rho_a = \rho_n = n+1$ (since $r_n = r_{n-1} = 0$), and $a = n-l+1$.

(α) ρ_a is the difference of the defects of N for a $(\rho_a - 1)$-ic and $(\rho_a - 2)$-ic.

(β) The highest curve for which N has excess is a $(\rho_a - 3)$-ic.

(γ) The lowest curve through N is a $(\rho_a - a)$-ic, *remembered as the last ρ diminished by the number of the ρ's.*

(δ) The number of the ρ's exceeds by 2 the number of the different orders of curves through N for which N has excess.

(ϵ) The number N is given by

$$N + q_{n-1} + 1 = \tfrac{1}{2} n (n+1),$$

i.e.,
$$N + \rho_l + ... + \rho_{n-1} = \tfrac{1}{2}\rho_n (\rho_n - 1),$$

i.e.,
$$N = \tfrac{1}{2}\rho_a (\rho_a - 1) - (\rho_1 + \rho_2 + ... + \rho_{a-1}).$$

(ζ) Thus the values of the ρ's in the point-group $N = (\rho_1, \rho_2, ..., \rho_a)$ express the *whole* effect of N in the determination of algebraic curves of all orders. The only restrictions on the values of the ρ's are $0 < \rho_1 < \rho_2 ... < \rho_a$, and $\rho_{b+1} = \rho_b + 1$ if $b \geqslant a$.

The point-groups (ρ_1, ρ_2) and (ρ_1), as indicated in (δ), are general, having zero excess for curves of any order which can be drawn through them. This is easily verified. The number N in (ρ_1, ρ_2) is $\tfrac{1}{2}\rho_2 (\rho_2 - 1) - \rho_1$, and in $(\rho_1) = (\rho_1, \rho_1 + 1)$ is $\tfrac{1}{2}\rho_1 (\rho_1 - 1)$.

It may also be verified that the point-group $(1, 2, ..., a, \rho_b, \rho_{b+1})$ consists of $\tfrac{1}{2}\rho_{b+1}(\rho_{b+1} - 1) - (1 + 2 + ... + a + \rho_b)$ general points on a $(\rho_{b+1} - a - 2)$-ic [*cf.* (γ) and (ϵ) above]. Similarly $(1, 2, ..., a, \rho_b)$, or $(1, 2, ..., a, \rho_b, \rho_b + 1)$, consists of $\tfrac{1}{2}\rho_b (\rho_b - 1) - \tfrac{1}{2} a (a+1)$ general points on a $(\rho_b - a - 1)$-ic. The point-group $(1, 2, ..., a)$ is a zero one.

Expressed in terms of the second differences of the defects $(\rho_1, \rho_2, ..., \rho_a, ..., \rho_b)$ becomes $(\delta_1, \delta_2, ..., \delta_a, ..., \delta_b)$; and it is easy to change from one expression to the other ($\delta_1 = \rho_1$, $\delta_a = \rho_a - \rho_{a-1}$). Al the δ's before δ_1 are zeros, and all the δ's after δ_b are units,* while $\delta_1 > 0$, $\delta_b > 1$. The only restriction on the values of the δ's is that they are all positive integers, not including zero.

The lowest curve through $(\delta_1, \delta_2, \ldots, \delta_b)$ is of order $l = \rho_b - b$, or

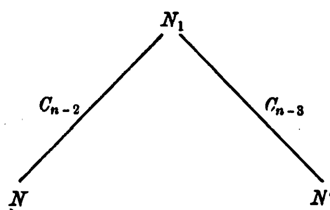
$$l = (\delta_1 - 1) + (\delta_2 - 1) + \ldots + (\delta_b - 1) = \Sigma(\delta - 1).$$

The point-group $(1, 2, \ldots, a, \rho_b, \rho_{b+1}, \ldots, \rho_c)$, when expressed in δ's, is written in the form $(1^a, \delta_b, \delta_{b+1}, \ldots, \delta_c)$; $\delta_b = \rho_b - a$.

IV. Reduction of Point-Groups.

Theorem I.—*A point-group* $(\rho_1, \rho_2, \ldots, \rho_{a-1}, \rho_a - 1)$ *can always be found coresidual to a given point-group* $N = (\rho_1, \rho_2, \ldots, \rho_a)$ *on a* $(\rho_a - 1)$*-ic.*

For $(\rho_1, \rho_2, \ldots, \rho_a)$ substitute $(\rho_l, \rho_{l+1}, \ldots, \rho_n)$; then $\rho_n = n+1$. The highest curve for which N has excess is a C_{n-2}; therefore N lies on a proper C_n [(β), p. 681]. Take this C_n as base-curve. We suppose that $a \geq 3$, otherwise the point-group N is a general one (p. 682); hence $n-2 \geq l$. Let a C_{n-2} be drawn through N. This will cut C_n again in a finite point-group N_1, since C_n is a proper curve. Also, since the excess of N for a C_{n-2} is not zero, N_1 must, by a known theorem, lie on a C_{n-3}. Let a C_{n-3} through N_1 cut C_n again in a finite point-group N'.†



Base-curve C_n.

Then N, N' are coresidual on C_n, and have identically the same sets of residuals. Also we know that the multiplicity of a series of point-groups on C_n cut out by curves of order p through N is q_p when $p < n$, and $q_p - \frac{1}{2}(p-n+1)(p-n+2)$ when $p \geq n$. The same series is cut out by curves of order $p-1$ through N', and the multiplicity is q'_{p-1} or $q'_{p-1} - \frac{1}{2}(p-n)(p-n+1)$ according as $p < n$ or $p \geq n$. Equating these values,

$$q'_{p-1} = q_p \text{ when } p < n, \quad \text{and} \quad q'_{p-1} = q_p - (p-n+1) \text{ when } p \geq n.$$

---

* The full expression for $(\delta_1\, \delta_2, \ldots, \delta_b)$ is $(0, \delta_1, \delta_2, \ldots, \delta_b, 1^\infty)$, where 1^∞ stands for 1 repeated *ad inf.*

† *Cf. Proceedings*, Vol. xxvi., p. 525. The N points might necessarily comprise all the N' points. This would be the case if, and only if, N consisted of n points on a straight line, and $N-n$ remaining points having no excess for a C_{n-3}. The N' points would then be identical with the $N-n$ points, but the reasoning would not be affected.

The first of these equations holds not only when $p \geqslant l$, but also when $p < l$, for then $q'_{p-1} = q_p = -1$. Both equations hold when $p = n-1$.

Taking differences, we have

$$\rho'_{p-1} = \rho_p \text{ when } p < n, \text{ and } \rho'_{p-1} = \rho_p - 1 = p \text{ when } p \geqslant n.$$

The successive values of ρ' are therefore

$$\ldots,\ 0,\ \rho_l,\ \rho_{l+1},\ \ldots,\ \rho_{n-1},\ n,\ n+1,\ n+2,\ \ldots,$$

or

$$\ldots,\ 0,\ \rho_1,\ \rho_2,\ \ldots,\ \rho_{a-1},\ \rho_a - 1,\ \rho_a,\ \rho_a + 1,\ \rho_a + 2,\ \ldots;$$

hence

$$N' = (\rho_1, \rho_2, \ldots, \rho_{a-1}, \rho_a - 1),$$

which proves the theorem.

If $\rho_a - 1 = \rho_{a-1} + 1$, then $N' = (\rho_1, \rho_2, \ldots, \rho_{a-1})$; and, if $\rho_a - 1 > \rho_{a-1} + 1$, we can repeat the process on N', thus obtaining the point-group $(\rho_1, \rho_2, \ldots, \rho_{a-1}, \rho_a - 2)$. After $\rho_a - \rho_{a-1} - 1$ steps in the reduction we arrive at the point-group $(\rho_1, \rho_2, \ldots, \rho_{a-1})$. In this we leave out all ρ's at the end, if there are any, which exceed the preceding by 1, and continue the reduction. We ultimately arrive at the general point-group (ρ_1, ρ_2), which is a zero point-group if $\rho_1 = 1$ and $\rho_2 = 2$.

*This proves that the point-group $N = (\rho_1, \rho_2, \ldots, \rho_a)$ is a possible one, if $0 < \rho_1 < \rho_2 < \ldots < \rho_a$.**

It also follows that the point-group N is impossible if $\rho_{b+1} \leqslant \rho_b > 0$. For, if the point-group is possible, the process of reduction is valid. But if $\rho_{b+1} \leqslant \rho_b > 0$, then at some stage we arrive at a point-group for which ρ_b is the difference of the defects for curves of order $\rho_b - 2$ and $\rho_b - 1$, and ρ_{b+1} the difference of the defects for curves of order

* [*Multiple Points. III.*—In the reduction of a point-group which is *general of its kind* (that is, of the kind which has the assigned characterization), the distinguishing feature is that all the derived point-groups are general of *their* kind. It is this that leads to a determinate (not unique) construction for such point-groups. For point-bases the process of reduction is still valid, and so also is the reverse process of construction; but the derived point-bases may not be general of their kind (the kind which possesses a known characterization and consists of points of known order). In such a case the construction is not determined by the reduction alone. Thus if N, in the figure, is a point-base, then for any i-point belonging to N we have an $(i-1)$-point belonging to N_1. The orders of the points of N_1 are all known, and also the characterization (Theorem III.). But from N_1 we have to construct N by passing a C_{n-2} and C_n through it, having i-fold points at the $(i-1)$-points of N_1. Such curves could certainly be drawn through the general point-base of the kind of N_1 if the $(n-2)$-ic defect of N_1 is equal to or greater than Σi, the sum of the orders of all the points of N for which $i > 1$. There is also the condition, in order that the construction may be valid, that the C_n should be a proper curve, or at least should not have any fixed constituent. We return to this again in Note V., p. 688. The question of the generality or speciality of these derived point-bases is not correctly stated in Vol. XXVI., p. 541, § 30.—*October 13th.*]

$\rho_b - 1$ and ρ_b. The corresponding differences of the excesses are then 0 and $\rho_b + 1 - \rho_{b+1} > 0$, by (2), p. 675. Hence the excesses of the derived point-group for curves of order $\rho_b - 2$ and $\rho_b - 1$ are equal, but not zero, which is impossible, by (i.), p. 680.

General Theorems in Reduction.

Theorem II.—*If* $\rho_a - \rho_{a-1} = 1$, *then the point-group*

$$N = (\rho_1, \ldots, \rho_a, \ldots, \rho_b)$$

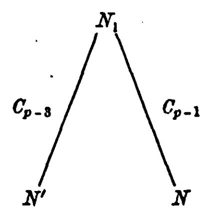
necessarily breaks up into the two independent point-groups

$$(\rho_1, \rho_2, \ldots, \rho_{a-1}) \quad \textit{and} \quad (1, 2, 3, \ldots, \rho_{a-1}, \rho_a, \rho_{a+1}, \ldots, \rho_b).*$$

If $\rho_1 = 1$, $\rho_2 = 2$, ..., $\rho_{a-1} = a-1$, then $(\rho_1, \rho_2, \ldots, \rho_{a-1})$ is a zero point-group, and the theorem is nugatory. We shall therefore assume the contrary.

Let $(\rho_1, \ldots, \rho_a, \ldots, \rho_b)$ be the same as $(\rho_l, \ldots, \rho_p, \ldots, \rho_n)$, where $\rho_1 = \rho_l$, $\rho_a = \rho_p$, $\rho_p - \rho_{p-1} = 1$, $\rho_b = \rho_n = n+1$. We have $p \geqslant l+1$, since $a \geqslant 2$.

Then a C_p through N must have a fixed factor; for, if not, a proper C_p can be drawn through N [(α), p. 680], which is different from C_l, since $p > l$. Taking this C_p as base-curve, we can find on it a point-group $N' = N - 2p$ coresidual to N, as shown in the figure;† and, by the same reasoning as in the last theorem, we have



Base-curve C_p.

$$q'_{p-4} = q_{p-2}, \quad q'_{p-3} = q_{p-1}, \quad q'_{p-2} = q_p - 1;$$

therefore
$$\rho'_{p-3} = \rho_{p-1}, \quad \rho'_{p-2} = \rho_p - 1;$$

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* [*Multiple Points. IV.*—It is almost certain that this theorem, except as regards the independence of the constituents, holds in general for a point-base; but it is to be observed :—(i.) that the theorem does not determine in what manner the points of various orders are distributed among the two constituent point-bases, but only the totality of simple points in each; (ii.) that the assigning of orders to the points of N may cause N to break up further than would be required by the characterization alone; and (iii.) that there are limits to the possible values of the orders, combined with the numbers and the characterization, of the points, of which very little is at present known.

It would in many cases be feasible to effect such a distribution of the points of N among N' and N'' that no i-point should belong in part to one and in part to the other; but, if this were not possible, the theorem would not thereby be invalidated. —*October* 13*th.*]

† A C_{p-1} can be drawn through N, for $p-1 \geqslant l$; and N_1 lies on a C_{p-3}, since N has excess for a C_{p-1}. If N does not lie on a C_{p-2}, then N' does not lie on a C_{p-4}, and $q'_{p-4} = q_{p-2} = -1$.

therefore $\rho'_{p-3} = \rho'_{p-2} > 0,$

which is impossible (p. 684).

Then let $C_p = C_{p'} C_{p''}$, where $C_{p'}$ is fixed and $C_{p''}$ has no fixed constituent; and let N' be the number of the N points which lie on $C_{p'}$, and N'' the remainder which lie on $C_{p''}$.

All curves lower than a C_p through N must have the fixed factor $C_{p'}$. Consequently, $q_p, q_{p-1}, q_{p-2}, \ldots$, which are the defects of $N'+N''$ for curves of order $p, p-1, p-2, \ldots$, must be the defects of N'' for curves of order $p'', p''-1, p''-2, \ldots$. Hence the successive differences of the defects of N'' are

$$\rho_1, \rho_2, \ldots, \rho_{a-1}, \rho_a.$$

But $\rho_a - \rho_{a-1} = 1$; and a $C_{p''}$, to which ρ_a corresponds, has no fixed constituent, by hypothesis; therefore all the ρ's of N'' from ρ_{a-1} increase by units. Hence

$$N'' = (\rho_1, \rho_2, \ldots, \rho_{a-1}).$$

This determines the full characterization of N'', and the number N''.

Also, since ρ_{a-1} corresponds to a $C_{p''-1}$, we have

$$p''-1 = \rho_{a-1}-1 = \rho_{p-1}-1, \quad \text{and} \quad p' = p-p'' = p-\rho_{p-1}.$$

The fixed constituent of C_p is therefore of order $p-\rho_{p-1}$.

Again the excess of N'' for a $(\rho_{a-1}-2)$-ic, that is, a $C_{p''-2}$, is zero [(β), p. 682]. Therefore the excess of $N'+N''$ for a C_{p-2} is entirely contributed by N'. The same must be true for all curves higher than a C_{p-2}, whether degenerate, as in the case of a C_{p-2}, C_{p-1}, C_p, or not. But the defects of $N'+N''$ for curves of order $p-2, p-1, \ldots, n$ are

$$q_{p-2}, q_{p-1}, q_p, \ldots, q_n;$$

therefore the defects of N' for curves of the same orders are

$$q_{p-2}+N'', q_{p-1}+N'', q_p+N'', \ldots, q_n+N'',$$

and the differences of these are

$$\rho_{p-1}, \rho_p, \ldots, \rho_n, \quad \text{or} \quad \rho_{a-1}, \rho_a, \ldots, \rho_b.$$

And the differences of successive defects of N' before ρ_{a-1} are

$$1, 2, 3, \ldots, \rho_{a-1}-2, \rho_{a-1}-1,$$

since any curve lower than a C_p through N' has the fixed constituent $C_{p'}$. Hence

$$N' = (1, 2, 3, \ldots, \rho_{a-1}, \rho_a, \rho_{a+1}, \ldots, \rho_b).$$

This determines the full characterization of N'. And it can be seen that any two point-groups having the characterizations found for N' and N'', when placed in any positions relatively to one another, make up a composite point-group with the same characterization as that of

$$N = (\rho_1, \rho_2, \ldots, \rho_a, \ldots, \rho_b).$$

Expressed in terms of the second differences of the defects the theorem is, since $\rho_a = \delta_1 + \delta_2 + \ldots + \delta_a$:—

If $\delta_a = 1$, *then the point-group* $N = (\delta_1, \delta_2, \ldots, \delta_a, \ldots, \delta_b)$ *breaks up into*

$$(\delta_1, \delta_2, \ldots, \delta_{a-1}) + (1^{\delta_1+\delta_2+\ldots+\delta_a}, \delta_{a+1}, \delta_{a+2}, \ldots, \delta_b).$$

Here we may suppose that $\delta_{a+1}, \delta_{a+2}, \ldots, \delta_b$ are all greater than 1. The point-group $(\delta_1, \delta_2, \ldots, \delta_{a-1})$ may, of course, break up further.

The lowest curve through $N = (\delta_1, \ldots, \delta_a, \ldots, \delta_b)$ is a C_l, where $l = \Sigma(\delta-1)$; and this is a proper curve if $\delta_1, \delta_2, \ldots, \delta_a$ are all equal to 1 and $\delta_{a+1}, \delta_{a+2}, \ldots, \delta_b$ all greater than 1. It follows from what is proved in the next theorem that the lowest curve through N without any fixed constituent is a C_m, where $m = a + \Sigma(\delta-1)$ if δ_a is equal to 1 and $\delta_{a+1}, \ldots, \delta_b$ are all greater than 1, and $m = \Sigma(\delta-1) = l$ if $\delta_1, \ldots, \delta_a, \ldots, \delta_b$ are all greater than 1. If $\delta_a = 1$, the fixed curve common to $C_l, C_{l+1}, \ldots, C_{m-1}$ is the lowest curve through

$$N' = (1^{\delta_1+\delta_2+\ldots+\delta_a}, \delta_{a+1}, \ldots, \delta_b),$$

and is therefore of order

$$(\delta_{a+1}-1) + (\delta_{a+2}-1) + \ldots + (\delta_b - 1).$$

THEOREM III.—*If* $\delta_b, \delta_{b+1}, \ldots, \delta_c$ *are all positive integers, not including zero, then*

$$(1^a, \delta_b+1, \delta_{b+1}+1, \ldots, \delta_c+1) + (\delta_c, \delta_{c-1}, \ldots, \delta_b) = I(\Sigma\delta, a+\Sigma\delta),$$

$I(l, m)$ *denoting the complete intersection of a* C_l *and* C_m.

This can be deduced from the following theorem (*Proc. Lond. Math. Soc.*, Vol. XXVI., p. 526):—

If a C_l *and* C_m *can be drawn through a point-group* N, *cutting again in a finite point-group* N' $(N+N' = lm)$, *then*

$$q'_{l+m-p-3} = r_p - 1 + [\tfrac{1}{2}(l-p-1)(l-p-2)] + [\tfrac{1}{2}(m-p-1)(m-p-2)],$$

square brackets indicating, as before, that the product enclosed is to be retained only if its individual factors are positive. This theorem is true for all values of p from 0 to $l+m-3$, taking $q_p + 1 = 0$ if N does not lie on a C_p.

In applying the theorem we take

$$N = (1^a, \delta_b+1, \delta_{b+1}+1, \ldots, \delta_c+1) = (\delta_l, \delta_{l+1}, \ldots, \delta_m, \ldots, \delta_n),$$

and assume for the moment that a proper C_l and C_m can be drawn through N, where $l = \Sigma\delta$, $m = a+\Sigma\delta$; the assumption being justified at the end. We then have

$$\delta_{l-1} = 0,\ \delta_l = \delta_{l+1} = \dots = \delta_{m-1} = 1,$$

$$\delta_m = \delta_b+1,\ \dots,\ \delta_n = \delta_c+1,\ \ \delta_{n+1} = \delta_{n+2} = \dots = 1.$$

Taking differences in the above theorem, we have

$$q'_{l+m-p-2} - q'_{l+m-p-3} = r_{p-1} - r_p + [l-p-1] + [m-p-1],$$

or $$\rho'_{l+m-p-2} = p+1-\rho_p + [l-p-1] + [m-p-1].$$

Let p diminish from $l+m-3$ to 1, so that $l+m-p-2$ increases from 1 to $l+m-3$. Then

from $p = l+m-3$	to n,	$\rho'_{l+m-p-2} = p+1-\rho_p$;
,, $p = n$	,, $m-1$,	$\rho'_{l+m-p-2} = p+1-\rho_p$;
,, $p = m-1$	,, $l-1$,	$\rho'_{l+m-p-2} = m-\rho_p$;
,, $p = l-1$	,, 1,	$\rho'_{l+m-p-2} = l+m-p-1-\rho_p$.

Again taking differences, subtracting each equation from the next succeeding, we have

from $p = l+m-3$	to $n+1$,	$\delta'_{l+m-p-1} = \delta_p - 1 = 0, 0, 0, \dots, 0$;
,, $p = n$	,, m,	$\delta'_{l+m-p-1} = \delta_p - 1 = \delta_c, \delta_{c-1}, \dots, \delta_b$;
,, $p = m-1$	,, l,	$\delta'_{l+m-p-1} = \delta_p = 1, 1, \dots, 1$;
,, $p = l-1$	,, 2,	$\delta'_{l+m-p-1} = \delta_p + 1 = 1, 1, \dots, 1$.

Hence the successive values of δ', for ascending orders of curves, are

$$\dots, 0, 0, \delta_c, \delta_{c-1}, \dots, \delta_b, 1, 1, 1, \dots;$$

therefore $$N' = (\delta_c, \delta_{c-1}, \dots, \delta_b).\text{*}$$

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* [*Multiple Points. V.*—We have given only that form of the theorem which is adapted for the quickest reduction of a given point-group. The general theorem is

$$(\dots \delta_a, \dots, \delta_b, \dots, \delta_c) + (\delta_c - 1, \dots, \delta_b, \dots, \delta_a + 1, \dots)$$
$$= I(f + \Sigma\overline{\delta-1}, g + \Sigma\overline{\delta-1}). \quad \text{.........(A)}$$

Here, in order to be perfectly general, we do not suppose the suffixes ... a ... b ... c to be consecutive positive integers. The δ's of $N' = (\delta_c - 1, \dots, \delta_b, \dots, \delta_a + 1, \dots)$, taken in reversed order, are formed from the δ's of $N = (\dots \delta_a, \dots, \delta_b, \dots, \delta_c)$, in direct order, those up to δ_a being increased by 1, thence up to δ_b being unchanged, and thence up to the last δ_c being diminished by 1; f is the number of the δ's of N which are increased, *i.e.*, the number of the δ's in ... δ_a, and g is the number of the δ's of N which are not diminished, *i.e.*, the number of the δ's in ... $\delta_a, \dots, \delta_b$. The δ's may have any values which make both N and N' possible; *i.e.*, in the case of point-groups, any number of the δ's at the beginning of both N and N' may be

Hence also we see that it is possible to draw proper curves of order $\Sigma\delta$ and $a+\Sigma\delta$ through $N = (1^a, \delta_b+1, \ldots, \delta_c+1)$; for two proper curves of such order can certainly be drawn through $N' = (\delta_c, \delta_{c-1}, \ldots, \delta_b)$,

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zeros, but after once ceasing to be zeros they must be positive integers, excluding zero.

The specially important case in which $f = 0$ gives

$$(\ldots \delta_b, \ldots, \delta_c) + (\delta_c - 1, \ldots, \delta_b, \ldots) = I(\Sigma\,\overline{\delta-1},\ g + \Sigma\,\overline{\delta-1}), \quad \ldots\ldots\ldots\ldots\text{(B)}$$

where g is the number of the δ's in N which remain unchanged in N'.

Taking g equal to the whole number of the δ's in N, we have

$$(\delta_1, \ldots, \delta_c) + (\delta_c, \ldots, \delta_1) = I(\Sigma\,\overline{\delta-1},\ \Sigma\delta). \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(C)}$$

If the two curves of order $\Sigma(\delta-1)$, $\Sigma\delta$ touch at all the points where they meet, and if N is the point-group formed by all the points of contact (each counted once only), then N' coincides with N; hence the series $\delta_1, \ldots, \delta_c$ is unaltered when reversed. The general case, as may be seen from (A), is almost as simple. In order that a point-group N may be such that two curves can touch at all the N points, without further intersection, then either $N = (\delta_1, \delta_2, \ldots, \delta_2, \delta_1)$, as above, or

$$(\delta_1, \ldots, \delta_{a-1}, \delta_a, \ldots, \delta_a, \delta_{a-1}+1, \ldots, \delta_1+1), \quad \text{or} \quad (\delta_1, \ldots, \delta_a, \delta_a+1, \ldots, \delta_1+1),$$

or one of the two last reversed. It is very remarkable that we can apparently assume any characterization for N, provided it comes under one of these forms and gives the correct value of the number N, without increasing the total number of independent interconnexions of the N points, that is, without increasing the specialization of N.

{For, let C_n, $C_{n'}$ $(n \leqslant n')$ be two curves which cut altogether in two point-groups N, N' having any the same characterization, and let k be the number of independent interconnexions of either point-group due to this characterization. The number of points which can be chosen at will on a given C_n which form part of a point-group on C_n with the same characterization as N is $N+r_n-k$ (*Proc. Lond. Math. Soc.*, Vol. xxvi., p. 529). If therefore a $C_{n'}$ can be drawn through N which touches any C_n through N at $N+r_n-k$ of the N points, it will touch it at the remainder. But a $C_{n'}$ can be drawn through N touching C_n at $N-\frac{1}{2}(n-1)(n-2)$ of the N points. Hence the number of conditions that a $C_{n'}$ can be drawn touching C_n at all the N points is

$$(N+r_n-k)-(N-\tfrac{1}{2}\,\overline{n-1}\,.\,\overline{n-2}) = r_n-k+\tfrac{1}{2}(n-1)(n-2) = N-k+q_n+1-3n.$$

Hence, since the degree of freedom of C_n accounts for q_n of the conditions, it follows that $N-k-3n+1$ is the number of conditions to be satisfied by the N points; and to this we can now add the k conditions due to the characterization. Thus the total number is $N-3n+1$, which remains unaltered, whatever the assumed characterization may be.}

The properties expressed by (A), (B), (C) hold equally for point-groups and point-bases; and (B), (C) have applications especially to the latter. What is required to complete the theory in regard to point-bases is, first, to determine the limits of possibility of the δ's as depending on the assigned orders of the points, and, second, to show how to deal with specialized derived point-bases when they cannot be excluded, as in the case of any nine 2-points which lie on a proper sextic.

Whether we are given the characterization of a point-group, or of a point-base, we know the order $\Sigma(\delta-1)$ of the lowest curve which passes through it. Also in the case of a point-group we know the order of the lowest curve without fixed constituents which passes through it; but we do not know it at present with any certainty for a given point-base. Hence for a point-base we have to use (B) in the place of the theorem in the text; and we can only apply (B) by way of trial, for we do not know the lowest value of g which will make the derived point-base N' possible.

We give now an example of a point-base which does not contain any points of since N' has no excess for a curve of order $\Sigma\delta-2$ [*cf.* (β), p. 682, and (β), p. 681]; and these determine N, and pass through it.

We add an example of the reduction of a point-group. Take the

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order 1. Very few such examples of a constructional kind are known, the best known being that of the nine 2-points on a proper sextic. Constructional point-bases which include 1-points can be obtained in any number by the methods we have described; not so those which do not include 1-points. (*Cf.* Cayley, *Proc. Lond. Math. Soc.*, Vol. III., p. 197; *Collected Works*, Vol. VII., p. 254.) Consider the point-base N formed by lm 2-points, situated at the lm points in which a C_l and C_m intersect, these lm points being all finitely separated. We know the form of the equation of the general algebraic curve through the point-base, viz.,

$$C_l^2 S + C_l C_m S' + C_m^2 S'' = 0;$$

and we can thence deduce the characterization. The result is

(i.) $N = (3, 3, \ldots \text{ repeated } l \text{ times}) = (3^l)$, if $l = m$;

(ii.) $N = (1^{m-l}, 2^{m-l}, 3^{2l-m}, 2^{m-l})$, if $l < m$, $2l > m$;

(iii.) $N = (1^{m-l}, 2^l, 1^{m-2l}, 2^l)$, if $2l \leqslant m$.

Suppose now that all that is given with respect to N is its characterization, viz. that in (i.), (ii.), or (iii.), and the fact that N is made up of lm separate 2-points. (The degree $N = 3lm$.) We shall consider first the application of (B) to case (ii.). The lowest curve through N is of order $\Sigma(\delta-1) = 2l$; call it C_{2l}. (If N is constructed as originally supposed, $C_{2l} = C_l^2$.) If N were a point-group, the lowest curve without fixed constituents through it would be of order $m-l+2l = l+m$. But, as regards the point-base, this is too low a limit; for, since the two curves must intersect in $4lm$ points at the least, the curve without fixed constituents must be at least of order $2m$. Assume then, by way of trial, that a C_{2m} without fixed constituents can be drawn through N, *i.e.*, that the value of g in (B) is $2(m-l)$. Then the first $2(m-l)$ of the δ's in N are to remain unchanged in N', and the rest are to be diminished each by 1. Thus (B) gives

$$(1^{m-l}, 2^{m-l}, 3^{2l-m}, 2^{m-l}) + (1^{m-l}, 2^{2l-m}, 2^{m-l}) = I(2l, 2m).$$

Hence $$N' = (1^{m-l}, 2^{2l-m}, 2^{m-l}) = (1^{m-l}, 2^l) = I(l, m).$$

Now it is possible to draw through $N' = I(l, m)$ two curves C_{2l}, C_{2m} which have double points at all the points of N', and which have no common constituent. The point-base N can therefore be reduced in a single step to an unspecialized point-group $N' = I(l, m)$; and the construction thus found for N is the one originally supposed.

This $N' = I(l, m)$ is the smallest derived of N. The next smallest N'' is obtained by drawing a C_{2l} and C_{2m+1} through N. For this, (B) gives

$$(1^{m-l}, 2^{m-l}, 3^{2l-m}, 2^{m-l}) + (1^{m-l}, 2^{2l-m-1}, 3, 2^{m-l}) = I(2l, 2m+1).$$

It can be proved that this $N'' = (1^{m-l}, 2^{2l-m-1}, 3, 2^{m-l})$ must be specialized. For N'' is a point-group containing $lm+2l$ points, and, if general of its kind, lies on a proper C_{l+1} and a proper C_{m+1}. But through N'' a curve C_{2l} can be drawn, having double points at lm of the N'' points. This C_{2l} and the proper C_{l+1} therefore cut in

$$2lm+2l = 2l(m+1) > 2l(l+1) \text{ points, since } l < m.$$

Hence C_{2l} must have the proper C_{l+1} for a constituent. Thus

$$C_{2l} \equiv C_{l+1}\, C_{l-1}.$$

But again, since C_{2l} has double points at each of the lm points, C_{l-1} must pass through the lm points; and C_{l-1} has only l^2-1 points in all in common with the proper C_{l+1}. Thus $l^2-1 \geqslant lm$, and $l^2 > lm$, which is not true. Thus N'' must be point-group, expressed in terms of the second differences of the defects, $N = (4, 4, 6, 1, 1, 5, 1, 3, 7, 2)$. Working upwards to the defects, we find that this is a group of 404 points whose defects fo curves of order 23 to 33 are $-1, 3, 11, 25, 40, 56, 77, 99, 124, 156, 190$.

$$N = (4, 4, 6) + (1^{16}, 5) + (1^{22}, 3, 7, 2) \quad \text{(Theorem II.)}.$$

(α) (i.) $(4, 4, 6) + (5, 3, 3) = I\,(11, 11)$ (Theorem III.);

(ii.) $(5, 3, 3) + (2, 2, 4) = I\,(8, 8)$ „

(iii.) $(2, 2, 4) + (3, 1, 1) = I\,(5, 5)$ „

(iv.) $(3, 1, 1) = (3, 4)$ in first differences

$= 3$ general points (p. 682).

This gives us the construction of the point-group $(4, 4, 6)$. The number of its points is $11^2 - 8^2 + 5^2 - 3 = 79$.

(β) $(1^{16}, 5) = (1, 2, \dots 16, 21)$ in first differences

$= 74$ general points on a C_4 (p. 682).

(γ) (i.) $(1^{22}, 3, 7, 2) + (1, 6, 2) = I\,(9, 31)$ (Theorem III.);

(ii.) $(1, 6, 2) = (1, 7, 9)$ in first differences

$= 28$ general points on a C_6 (p. 682).

Thus $(1^{22}, 3, 7, 2)$ is constructed by drawing a C_9 and C_{31} through 28 general points on a C_6 to cut again in $9 \times 31 - 28 = 251$ points.

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specialized; in fact, C_{l+1} cannot be a proper curve. If the construction found above for N is the only solution, N'' consists of the lm points in which C_l, C_m cut, and l point-pairs on a straight line, viz., at the points where any straight line cuts C_l. This straight line is an l-fold tangent to C_{2m+1}.

This reasoning suggests the inference that the *smallest* derived point-base N' of a given point-base N is the one which *is the most likely to be general of its kind.* Hence the importance of discovering the order of the lowest curve without any fixed constituent which passes through a given point-base.

Taking case (i.), $N = (3^l)$, the lowest curve through N is a C_{2l}, as before. If we assume, by way of trial, that C_{2l} has no fixed constituent, (B) gives (taking $g = 0$)

$$(3^l) + (2^l) = I\,(2l, 2l).$$

Here $N' = (2^l) = I(l, l)$ satisfies the premised conditions, and is general of its kind. The C_{2l} through N is not a proper curve, but has the requisite property that it does not possess any fixed constituent.

Taking case (iii.), $N = (1^{m-l}, 2^l, 1^{m-2l}, 2^l)$, we see that N breaks up, by Theorem II., into

$$(1^{m-l}, 2^l) + (1^{2m-l}, 2^l) = I\,(l, m) + I\,(l, 2m).$$

The two constituents of N, $I\,(l, m)$ and $I\,(l, 2m)$, are not independent of one another, nor is the second general of its kind. If the first is general of its kind, the second must consist of point-pairs having the same situation as the single points of the first. This gives a correct analysis of N.—*October* 13*th*.]

V. Number of Interconnexions of a Point-Group.

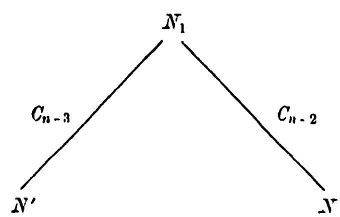
The number of the independent interconnexions of the points of the point-group $(\rho_1, \rho_2, \ldots, \rho_a)$, *due to the characterization, is*

$$\rho_1(\rho_3-\rho_2-1)+\rho_2(\rho_4-\rho_3-1)+\ldots+\rho_{a-2}(\rho_a-\rho_{a-1}-1).$$

To find the number of independent interconnexions of the points of

$$N=(\rho_1, \rho_2, \ldots, \rho_a)=(\rho_l, \rho_{l+1}, \ldots, \rho_n)$$

we go back to Theorem I. Suppose that k is the required number of interconnexions of N, and k' that of N'. We find the value of $k-k'$ by obtaining and equating two different expressions for the least number of parameters in terms of which the base-curve C_n, and the positions of all the points of N and N' upon C_n, can be expressed.



Base-curve C_n.

Taking any two coordinate axes, the least number of parameters in terms of which the positions of the N points in the plane can be expressed is $2N-k$, since k is the number of independent interconnexions of the N points. These $2N-k$ parameters determine the N points; q_n more parameters, and not less, will determine the C_n, since q_n is the defect of N for a C_n; and r'_{n-3} more parameters, and not less, will determine N'. This last result follows from the fact that r'_{n-3} is the multiplicity of N' on C_n.* Thus one of the required expressions is $2N-k+q_n+r'_{n-3}$; and the other is $2N'-k'+q'_n+r_{n-3}$, obtained by starting with N'. Equating these, and noticing that $N+q_n=N'+q'_n$, we have

$$k-k'=(N-r_{n-3})-(N'-r'_{n-3})$$

$$=q'_{n-3}-q_{n-3}, \text{ from (1), p. 675,}$$

$$=q_{n-2}-q_{n-3}=\rho_{n-2}=\rho_{a-2}.$$

Hence, in changing from the point-group $(\rho_1, \ldots, \rho_a)$ to $(\rho_1, \ldots, \rho_{a-1}, \rho_a-1)$, k is diminished by ρ_{a-2}. After $\rho_a-\rho_{a-1}-1$ such steps $(\rho_1, \ldots, \rho_a)$ is reduced to $(\rho_1, \ldots, \rho_{a-1})$, and k is diminished by $\rho_{a-2}(\rho_a-\rho_{a-1}-1)$. But, when $(\rho_1, \ldots, \rho_a)$ has been reduced to the

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* A concise statement of the Riemann-Roch theorem is that *the multiplicity of any point-group on a curve* C_n *is equal to the* $(n-3)$*-ic excess of the point-group.* (*Proceedings*, Vol. xxvi., p. 523.)

general point-group (ρ_1, ρ_2), k is diminished to zero. Hence

$$k = \rho_1(\rho_3-\rho_2-1)+\rho_2(\rho_4-\rho_3-1)+\ \dots\ +\rho_{a-2}(\rho_a-\rho_{a-1}-1).*$$

Also, since $\quad k-k' = \rho_{a-2}$, and $N-N' = \rho_a-1$,

therefore $\quad (N-k)-(N'-k') = \rho_a-\rho_{a-2}-1 \geqslant 1;$

therefore $\quad N-k > N'-k' > \dots \geqslant 0.$

It can be easily proved that

$$N-k = \tfrac{1}{2}(\delta_1-1)\delta_1+\tfrac{1}{2}(\delta_2-1)(\delta_2+2\delta_1)+\ \dots\ +\tfrac{1}{2}(\delta_a-1)(\delta_a+2\delta_{a-1}).$$

VI. Rational Transformation of Point-Bases.†

In rational transformation the whole system of curves of order n which satisfy given conditions transforms into the whole system of curves of another order n' satisfying another set of conditions. If the original conditions are simply those of passage through a given point-base, the transformed conditions are also, so long as n remains fixed, *simply* those of passage through another point-base. But, as the orders n of the original curves increase by units, the orders n' of the transformed curves increase in arithmetical progression, while also the orders of their multiple points may some increase in arithmetical progression, and others remain constant. As n varies, the orders of the points of the transformed point-base also vary.

Rational transformation thus leads to a generalized view of the questions treated above. Instead of investigating the properties in respect to excess and defect of a *simple* point-base, whose points are all of fixed orders, we have to consider these same properties for a

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* [*Multiple Points. VI.*—In the application of this result to point-bases each i-point is to be regarded as a single point. But the reasoning by which the result is obtained fails in the majority of cases, since the proof depends on the use of a slow process of reduction, which would generally cause the derived point-bases to be specialized. The proof can, however, be extended to any reduction of a point-group. Thus it appears that one condition (and probably not the only one) for the correctness of the result, when applied to a given point-base N, is that it should be possible to reduce N by means of a series of unspecialized point-bases; and, for this purpose, as we have seen (Note V., p. 691), the most rapid reduction seems the most likely of any to prove effective.

It seems probable that the result holds for a point-base N so long as it does not exceed twice the number of the points of order 1 contained in N. Further it appears that the correct result, if different from, is less than that found above. Thus in case (i.) of the example in Note V. the value of k, given by the formula $\Sigma\rho_{a-2}(\rho_a-\rho_{a-1}-1)$, is just three times the correct value; and in cases (ii.) and (iii.) the formula gives a value which is *more* than three times too great.—*October 13th.*]

† See footnotes on "Multiple Points" for the meaning of *point-base*, and the applicability to point-bases of the results proved for point-groups.

generalized point-base, including some points of fixed, and others of variable, order. The orders of the curves drawn through the point-base increase with constant difference ν, while the orders of the points of the base increase correspondingly, but each with its own constant difference ι, which varies for different points, and may, in particular, be zero.

In this generalized view the *virtual* number of the conditions supplied for a C_n is $N_n = \frac{1}{2}\Sigma i\,(i+1)$, while the *actual* number is $N_n - r_n$, r_n being the n-ic excess. The n-ic defect, q_n, is still the degree of freedom of a C_n which satisfies the N_n conditions. Instead of formula (1), p. 675, we have

$$N_n - r_n + q_n + 1 = \tfrac{1}{2}(n+1)(n+2).$$

It is evident that all the defects are invariants in rational transformation, since the number of general points through which a C_n, satisfying the N_n conditions, can be drawn is equal to the number of general points through which the transformed $C_{n'}$, satisfying the transformed $N'_{n'}$ conditions, can be drawn, and *vice versâ*. The invariance of the excesses is not so evident; but this can be easily shown by proving it to hold for any quadric transformation. Thus q_n, r_n, ρ_n, δ_n, $N_n - \frac{1}{2}(n+1)(n+2)$ are all invariants, while N_n and n are not. Here $\rho_n = q_n - q_{n-\nu}$, $\delta_n = \rho_n - \rho_{n-\nu}$; and we may further put $\sigma_n = r_{n-\nu} - r_n$, $e_n = \sigma_{n-\nu} - \sigma_n$.

The invariant $N_n - \frac{1}{2}(n+1)(n+2)$ involves three others, viz., $\nu^2 - \Sigma\iota^2$, $3\nu - \Sigma\iota$, and $n\nu - \Sigma i\iota$. In a simple point-base $\nu = 1$, and all the ι's vanish; hence $\nu^2 - \Sigma\iota^2 = 1$, and $3\nu - \Sigma\iota = 3$; and a generalized point-base derived by rational transformation from a simple one must satisfy these equations. Conversely, if the equations hold for a generalized point-base, it can be rationally transformed into a simple one; for the equations $\nu^2 - \Sigma\iota^2 = 1$, $3\nu - \Sigma\iota = 3$ show that a net of curves C_ν can be described with multiple points of order $\iota, \ldots$ at the points of the point-base. This net rationally transforms the point-base into a simple one; for C_ν transforms to $C_{\nu'}$, consequently $\nu' = 1$, and all the ι''s vanish.

I do not know whether generalized point-bases with similar properties of excess and defect, but having other values than 1 and 3 for $\nu^2 - \Sigma\iota^2$ and $3\nu - \Sigma\iota$, are possible, or not. Assuming them to be possible, the curves C_ν, if they exist, still transform into the curves $C_{\nu'}$, but cannot themselves be used for rational transformation. The numbers 1, 3 are perhaps the lowest possible values of $\nu^2 - \Sigma\iota^2$, $3\nu - \Sigma\iota$ respectively; and, this being so, the curves C_ν certainly exist. If $\nu^2 - \Sigma\iota^2$ were negative, it would appear that there must be a *superior* limit to the order n of a curve which could satisfy the N_n conditions.

If we write π for the deficiency of C_ν, we have $\delta_n - \epsilon_n = \nu^2 - \Sigma\iota^2 =$ *constant* number of ordinary points in which two curves C_ν intersect, and $\rho_n + \sigma_n + \pi - 1 = n\nu - \Sigma i\iota =$ number of ordinary points in which C_n and C_ν intersect. Reasoning from analogy we should expect 0 to be the least possible value of ϵ_n, corresponding to the breaking-point (p. 679), and $\nu^2 - \Sigma\iota^2$ to be the least possible value of δ_n.

The Conformal Representation of a Pentagon on a Half Plane. *By* Miss M. E. Barwell. Read June 9th, 1898. Received, in revised form, September 15th, 1898.

1. The conformal representation of a rectilinear polygon on a half plane was first attempted by Schwarz and Christoffel, who arrived independently at the same result. They have shown that the area of the w-plane included by a polygon, whose sides do not cross, can be conformally represented by the northern half of the z-plane, the boundary of the polygon corresponding to the axis of real quantities on the z-plane.

The necessary transformation is

$$w = M\int (z-a)^{a-1}(z-b)^{\beta-1}\dots(z-b)^{\lambda-1}\,dz + M',$$

where $a, b, \dots, l$ are the points on the real axis of z corresponding to the angular points of the polygon taken in order, and all lying in the finite part of the z-plane.

$a\pi, \beta\pi, \dots, \lambda\pi$ are the internal angles of the polygon at the respective points. The constant M' is determined by fixing the origin in the w-plane. Any three of the real quantities $a, \dots, l$ may be chosen arbitrarily, and the remainder must be determined in terms of these three, and the constants of the polygon $a\,.\,\beta\dots\lambda$. The case of the quadrilateral is given in Forsyth's *Theory of Functions*, p. 546. There is one unknown quantity besides M to be determined, and the solution involves Gauss' hypergeometric functions.