On Bicircular Quartics with Collinear Foci.

By HENRY M. JEFFERY.

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1. Dr. Casey, in his classical treatise "On Bicircular Quartics" (Dublin, 1869), has omitted to treat of these degenerate forms, when the foci are collinear, with the single exception of Cartesians. Mr. Hart, to whom we are indebted for the complete analytical exposition of the theory by which bicirculars are generated (*Proceedings*, Vol. xi., p. 143), has also passed this group in silence.

Having had occasion to study this group of quartics in the dual form, I have proposed to obtain its equation (1) by Dr. Casey's method of generation, (2) in the vector form, and otherwise, in terms of the focal distances from a double focus.

2. It may be stated *in limine* that the group of bicirculars is degenerate, inasmuch as they may be generated by three, and not four, sets of orthotomic circles, and that all these three circles of reciprocation cannot be referred to the same self-conjugate triangles with the corresponding focal conics. See §§ 4, 9.

The several varieties of these quartics caused by the omission of terms in their equation will be considered. The variety, in which one of the single foci coincides with a double, and forms a triple focus, has been already discussed (*Proceedings*, Vol. xii., pp. 17—27). A separate section will be assigned to Cartesians, in which Dr. Casey's results will be obtained in a different way.

All singular forms will be exhibited, and a method given for determining the inflexions in these ovals.

3. Bicircular quartics with collinear foci may be expressed in either of these forms:

$$h\,(x^2+y^2)^2+(1+kx)\,(x^2+y^2)+u_1=0,$$

or

$$h\,(x^2+y^2)^2+x^2+y^2+u_2=0.$$

When the origin is arbitrary, these quartics have the fuller form

$$h\,(x^2+y^2)^2+(1+kx)(x^2+y^2)+mx^2+nx+p=0.$$

The double foci are found in the usual way by the intersection of two hyperbolæ

$$4h\,(x^2-y^2)+2kx+m=0,$$

$$4hxy+ky=0.$$

They lie on the (x) axis

$$4hx^2+2kx+m = 0.$$

If the origin be transposed to either double focus, the quartic takes the first form; if mid-way between them, the second form. The first form is used in this memoir, with the parameters κ, λ,

$$\lambda\,(x^2+y^2)^2+(1+fx)(1+fx)(x^2+y^2) = \kappa\,(1+ex).$$

If used to denote a group of class-quartics, this form would represent all those which have a triple and a single focus, and one of its satellite-foci coincident with the triple focus.

4. To generate a bicircular quartic with collinear foci by Dr. Casey's method of reciprocation.

There are in this case three focal conics, which are doubly confocal, since their common foci are the double foci of the quartic. (Casey, p. 469.)

$$\lambda\,(x^2+y^2)-(x-a)^2 = 0\;\ldots\ldots\ldots\;(F_1),$$
$$\mu\,(x^2+y^2)-(x-b)^2 = 0\;\ldots\ldots\ldots\;(F_2),$$
$$\nu\,(x^2+y^2)-(x-c)^2 = 0\;\ldots\ldots\ldots\;(F_3).$$

And there are three corresponding circles of reciprocation, which are mutually orthotomic (Casey, p. 469), and whose centres are collinear.

$$(x-f)^2+y^2 = \rho_1^2\;\ldots\ldots\ldots\;(J_1),$$
$$(x-g)^2+y^2 = \rho_2^2\;\ldots\ldots\ldots\;(J_2),$$
$$(x-h)^2+y^2 = \rho_3^2\;\ldots\ldots\ldots\;(J_3).$$

Transform F_1 and J_1 to the centre of J_1, and write r^2 for x^2+y^2,

$$\lambda\,(x+f)^2+\lambda y^2-(x+f-a)^2 = 0\;\ldots\ldots\ldots\;(F_1).$$

Its Boothian equivalent tangential equation is

$$\lambda\,(1+f\xi)^2-\{1+(f-a)\,\xi\}^2-\eta^2 = 0\;\ldots\ldots\ldots\;(F_1).$$

By reciprocating from the centre of J_1 by the relation

$$\frac{1}{p} = \frac{2r}{r^2+\rho_1^2}, \text{ where } \frac{1}{p^2} = \xi^2+\eta^2,$$

the bicircular is generated

$$\lambda\,(r^2+2fx+\rho_1^2)^2-\{r^2+2\,(f-a)\,x+\rho_1^2\}^2-4a^2y^2 = 0.$$

Again, transform to the common focus as origin:

$$\lambda\,(r^2+\rho_1^2-f^2)^2-(r^2-2ax+\rho_1^2-f^2+2af)^2-4a^2y^2 = 0.$$

Since the focal conics are confocal,

$$\frac{2a}{\lambda-1} = \frac{2b}{\mu-1} = \frac{2c}{\nu-1} = d,$$

if d denote the distance between their common foci.

The quartic may now be written

$$r^4+2dxr^2+2r^2(\rho_1^2-f^2-df-ad)+2dx(\rho_1^2-f^2+2af)$$
$$+(\rho_1^2-f^2-df)^2-f^2(d^2+2ad)=0.$$

Two other forms of the equation may be written, in which ρ_2, g, b; ρ_3, h, c; take the places of ρ_1, f, a. These forms are identified by equating their coefficients; so that

$$(2f+d)(2a+d)=(2g+d)(2b+d)=(2h+d)(2c+d)=S$$
$$=\sqrt{\{d(d+2a)(d+2b)(d+2c)\}}.$$

Since $4\rho_1^2-(2f+d)^2+d^2-4ad=-\dfrac{S^2}{(d+2a)(d+2b)}+d^2-2ad-2bd,$

by symmetry it also $=-\dfrac{S^2}{(d+2b)(d+2c)}+d^2-2bd-2cd;$

whence S is known.

For $\quad \rho_1^2=\dfrac{b-a}{4b+2d}\{(2f+d)^2-d(2b+d)\}=\dfrac{d(a-b)(a-c)}{2a+d},$

$$\rho_2^2=\frac{d(b-a)(b-c)}{2b+d};\quad \rho_3^2=\frac{d(c-a)(c-b)}{2c+d}.$$

Hence it appears that the conic F_2, which corresponds to the imaginary circle J_2, always lies between the two conics F_1 and F_3 (Mr. Hart, *Proceedings*, Vol. xi., p. 149). a, b, c are supposed to be in order of magnitude.

The circles of reciprocation are seen to be orthotomic.

For $\quad \rho_2^2+\rho_3^2=\dfrac{d(d+2a)(b-c)^2}{(d+2b)(d+2c)}=(g-h)^2;$

$$\rho_3^2+\rho_1^2=(h-f)^2;\quad \rho_1^2+\rho_2^2=(f-g)^2.$$

Finally, the constants in the bicircular equation may be expressed in terms of a, b, c, the given distances of the directrices and d, that between the foci

$$r^4+2dxr^2-(a+b+c)dr^2+dx(S-d^2-ad-bd-cd)$$
$$+\frac{d^2}{4}\{a^2+b^2+c^2-2bc-2ca-2ab-2(a+b+c+d)d+2S\}=0;$$

where $S=\sqrt{\{d(d+2a)(d+2b)(d+2c)\}}$, and $r^2=x^2+y^2$.

Cor.— Conversely, by this form of the equation, a fundamental theorem of Dr. Casey's is illustrated; the foci of the confocal focal conics are double foci of the bicircular quartic.

5. The bicircular quartic is also generated in three ways as the envelope of a variable circle, whose centre lies on (F), a focal conic, and which cuts (J), the corresponding circle of reciprocation, orthogonally. (Casey, p. 469.)

Let the generating circle be

$$(x-\alpha)^2+(y-\beta)^2=\gamma^2,$$

where the parameters are subject to two conditions:

$$\lambda(\alpha^2+\beta^2)=(\alpha-a)^2,$$

and $$2\alpha f-\alpha^2-\beta^2+\gamma^2-\rho_1^2-f^2=0;$$

because it cuts orthogonally the circle

$$(x-f)^2+y^2=\rho_1^2.$$

The resulting quartic is found to be identical with the preceding

$$\lambda(r^2+\rho_1^2-f^2)^2-(r^2-2ax+\rho_1^2-f^2+2af)^2-4a^2y^2=0.$$

Hence it appears that the form of reciprocation

$$\frac{1}{p}=\frac{2r}{r^2+\rho}$$

expresses both the properties stated in the enunciation.

The dual theorem may be obtained from this genesis. Class-quartics which have a triple and single focus, and whose satellite conic has one of its collinear foci coincident with the triple focus, as denoted by the Boothian equation

$$\kappa(1+e\xi)=(1+f\xi)(\xi^2+\eta^2)+\lambda(\xi^2+\eta^2)^2,$$

are generated in three ways. They are envelopes of variable singly confocal conics, whose directrix touches F a dirigent circle, and whose common tangent with J subtends a right angle at the common focus.

6. In the special case, when there is a triple and a double focus, the coefficient of x vanishes,

and $$S=d(d+a+b+c);$$

whence $$d(a^2+b^2+c^2-2bc-2ca-2ab)=8abc,$$

and d is thus a function of a, b, c.

The equation to the bicircular becomes

$$\frac{1}{d}(x^2+y^2)^2+(2x-a-b-c)(x^2+y^2)+2abc=0.$$

Let this form be compared with the equation (*Proceedings*, Vol. xii., p. 18) in which the constants are functions of the focal distances,

$$\Sigma(a'^2-2b'c')(x^2+y^2)^2-2a'b'c'(a'+b'+c'+4x)(x^2+y^2)+a'^2b'^2c'^2=0.$$

It follows that $$a:b:c:1::a':b':c':2.$$

The distances of the single foci from the triple focus are therefore double the distances of the directrices of the focal conics from the same triple focus.

7. In the case, when the bicircular passes through a double focus,

the constant term disappears, d is a function of a, b, c, and the quartic becomes

$$r^4+2dxr^2-(a+b+c)\,dr^2-\tfrac{1}{2}dx\,(a^2+b^2+c^2-2bc-2ca-2ab)=0,$$

where $$4d\Sigma\,[a\,(a^2-b^2-c^2)]=\{\Sigma\,(a^2-2bc)\}^2.$$

8. If the focal conics be parabolæ, since d is infinite,

$$a+f=b+g=c+h=\tfrac{1}{2}(a+b+c)=s \text{ (suppose)},$$

$$\rho_1^2=(a-b)(a-c),\ \rho_2^2=(b-a)(b-c),\ \rho_3^2=(c-a)(c-b).$$

The bicircular quartic degenerates into the circular cubic with four single foci collinear with its double focus:

$$(x-s)(x^2+y^2)-\frac{x}{4}(a^2+b^2+c^2-2bc-2ca-2ab)-(s-a)(s-b)(s-c)=0;$$

where a, b, c denote the distances of the directrices of the focal conics from the double focus.

9. Let the case be considered where the focal conics are denoted, as before:

$$\lambda\,(x^2+y^2)-(x-a)^2=0 \quad \ldots\ldots\ldots\ldots (F_1),$$

$$\mu\,(x^2+y^2)-(x-b)^2=0 \quad \ldots\ldots\ldots\ldots (F_2);$$

but the orthotomic circles of reciprocation J_1, J_2 are referred to the same self-conjugate triangles

$$x^2+y^2-2bx+ab=0 \quad \ldots\ldots\ldots\ldots (J_1),$$

$$x^2+y^2-2ax+ab=0 \quad \ldots\ldots\ldots\ldots (J_2).$$

The bicircular quartic, generated from either pair, is

$$(x^2+y^2)^2+2\,(dx-ab-ad-bd)(x^2+y^2)+2abdx+a^2b^2=0.$$

The third pair F_3 and J_3, from which the same quartic may be generated, are not, as in the general theorem, referred to the same self-conjugate triangle.

10. To determine the vector equation; *i.e.*, to obtain the linear relation between the distances from three of its single foci of a point in a bicircular quartic whose foci are collinear.

Let the two forms of the quartic equation be compared, in the first of which the origin is a double focus.

$$\lambda\,(x^2+y^2)^2+(1+fx)(x^2+y^2)=\kappa\,(1+ex),$$

$$lr_1+mr_2+nr_3=0,$$

where $r_1^2=(x+a)^2+y^2$, and r_2, r_3 have similar values; a, b, c are now the distances of three foci from the double focus. Then, on expanding, it will be found that

$$l\sqrt{a}+m\sqrt{b}+n\sqrt{c}=0,$$

since there is no coefficient of x^2, other than that of x^2+y^2. Let d

denote the fourth focal distance; the fourth focus may be expressed as a point-circle in either of the forms

$$(x+d)^2+y^2=0,$$
$$\lambda r_1^2+\mu r_2^2+\nu r_3^2=0.$$

Since this point-circle touches the quartic $(lr_1+mr_2+nr_3=0)$,

$$\frac{l^2}{\lambda}+\frac{m^2}{\mu}+\frac{n^2}{\nu}=0.$$

The discriminant of this circle denotes that it is the intersection of two imaginary straight lines. This condition of linear factors is

$$\frac{(b-c)^2}{\lambda}+\frac{(c-a)^2}{\mu}+\frac{(a-b)^2}{\nu}=0.$$

Equate coefficients in the two forms of the point-circle

$$\lambda a^2+\mu b^2+\nu c^2=(\lambda+\mu+\nu)\,d^2,$$
$$\lambda a+\mu b+\nu c=(\lambda+\mu+\nu)\,d;$$

whence $\lambda(a-d):\mu(b-d):\nu(c-d)::b-c:c-a:a-b,$

i.e. $\frac{1}{\lambda}:\frac{1}{\mu}:\frac{1}{\nu}::AB\,.\,AC\,.\,AD:BA\,.\,BC\,.\,BD:CA\,.\,CB\,.\,CD$

(Salmon's *Higher Plane Curves*, p. 237). The relative values of l, m, n are found from the relations

$$l\sqrt{a}+m\sqrt{b}+n\sqrt{c}=0,$$
$$\frac{l^2(a-d)}{b-c}+\frac{m^2(b-d)}{c-a}+\frac{n^2(c-d)}{a-b}=0.$$

The linear relation required is found to be

$$r_1(b-c)\{\sqrt{(ad)}-\sqrt{(bc)}\}$$
$$+r_2(c-a)\{\sqrt{(bd)}-\sqrt{(ca)}\}+r_3(a-b)\{\sqrt{(cd)}-\sqrt{(ab)}\}=0.$$

By forming three other linear relations by symmetry, and adding all the four, we obtain the most general vector equation

$$Ar_1+Br_2+Cr_3+Dr_4=0.$$

Cor. 1.—If $d=\infty$, the vector equation to Cartesians is, as in § 28,

$$(b-c)\sqrt{a}\,r_1+(c-a)\sqrt{b}\,r_2+(a-b)\sqrt{c}\,r_3=0.$$

Cor. 2.—If $a^{-\frac{1}{2}}+b^{-\frac{1}{2}}+c^{-\frac{1}{2}}+d^{-\frac{1}{2}}=0$, and the constant term vanishes; the bicircular passes through a double focus (§ 11), and its vector equation is $\frac{\sqrt{b}-\sqrt{c}}{\sqrt{a}}r_1+\frac{\sqrt{c}-\sqrt{a}}{\sqrt{b}}r_2+\frac{\sqrt{a}-\sqrt{b}}{\sqrt{c}}r_3=0.$

The result may be simply verified by comparing the forms

$$\lambda(x^2+y^2)^2+(1+fx)(x^2+y^2)=\kappa ex,$$
$$lr_1+mr_2+nr_3=0.$$

The conditions are $la+mb+nc=0,$

$$l\sqrt{a}+m\sqrt{b}+n\sqrt{c}=0.$$

Cor. 3.—If $d=0$, or the bicircular have a triple and a double focus, the vector equation is

$$\frac{b-c}{\sqrt{a}}r_1+\frac{c-a}{\sqrt{b}}r_2+\frac{a-b}{\sqrt{c}}r_3=0.$$

This form is otherwise obtained (Vol. xii., p. 18).

Cor. 4.—If the term involving x^2+y^2 be removed, no vector equation is possible.

The values of l, m, n cannot be real, since the conditions

$$l\sqrt{a}+m\sqrt{b}+n\sqrt{c}=0,$$
$$\Sigma l^4a^2-\Sigma m^2n^2(b^2+c^2)=0,$$

when combined, lead to the sum of squares

$$\frac{1}{l^2}(b-c)^2+\frac{1}{m^2}(c-a)^2+\frac{1}{n^2}(a-b)^2=0.$$

In class-quartics, this group has a satellite-focus at infinity.

Cor. 5.—If an apse be at infinity, $\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}=0$, the bicircular degenerates into the circular cubic with collinear foci,

$$r_1(\sqrt{b}-\sqrt{c})+r_2(\sqrt{c}-\sqrt{a})+r_3(\sqrt{a}-\sqrt{b})=0.$$

11. To determine the focal distances of the apses in terms of the distances of the four single foci from a double focus.

At an apse, $y=0$, r_1, r_2, r_3 become $x+a$, $x+b$, $x+c$ severally. The vector equation of § 10 becomes

$$\Sigma(x+a)(b-c)\{\sqrt{(ad)}-\sqrt{(bc)}\}=0.$$

After rejecting the factor

$$(\sqrt{b}-\sqrt{c})(\sqrt{c}-\sqrt{a})(\sqrt{a}-\sqrt{b}),$$

$$x(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d})+\sqrt{(bcd)}+\sqrt{(acd)}+\sqrt{(abd)}+\sqrt{(abc)}=0.$$

The other three apses are determined, by altering the signs of the radicals by pairs. Thus

$$x(\sqrt{a}+\sqrt{b}-\sqrt{c}-\sqrt{d})+\sqrt{(abcd)}\left\{\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{c}}-\frac{1}{\sqrt{d}}\right\}=0.$$

Cor. 1.—For Cartesians, when $d=\infty$, the distances of the apses are thus obtained:

$$x=\sqrt{(bc)}+\sqrt{(ca)}+\sqrt{(ab)}.$$

Cor. 2.—If the origin be an apse, so that

$$a^{-\frac{1}{2}}+b^{-\frac{1}{2}}+c^{-\frac{1}{2}}+d^{-\frac{1}{2}}=0,$$

the distances of the other three apses have the type

$$x\{(\sqrt{a}+\sqrt{b})(\sqrt{a}+\sqrt{c})-2a\}=2a\sqrt{(bc)}.$$

Cor. 3.—If $d = 0$, for the apses

$$x(\sqrt{a}+\sqrt{b}+\sqrt{c}) = \sqrt{(abc)}.$$

Cor. 4.—If an apse be at infinity, the distances of the three other apses from the double focus of the circular cubic have the type

$$\tfrac{1}{2}\sqrt{a}(\sqrt{a}+\sqrt{b}+\sqrt{c})-\tfrac{1}{2}\sqrt{(bc)}.$$

12. If the bicircular cubic be nodal, as for instance the pedal curve of a central conic, two foci, as well as two apses, unite at the node and disappear.* (Casey, p. 474.)

Its vector equation is reduced, when $c = d$, by rejecting the factor $\sqrt{a}-\sqrt{b}$,

$$(r_1 b+r_2 a)\sqrt{c}-(r_1+r_2)c\sqrt{c}+r_3(\sqrt{a}+\sqrt{b})\{c-\sqrt{(ab)}\} = 0.$$

13. To determine the collinear foci in these bicircular quartics.

The equation to the group being

$$\lambda(x^2+y^2)^2+(1+fx)(x^2+y^2) = \kappa(1+ex),$$

it may be proved that there are two double foci, at the intersection of the quadrics

$$2\lambda(x^2-y^2)+fx = 0,$$

$$4\lambda xy+fy = 0.$$

These are the origin (0, 0), and the point $\left(-\frac{f}{2\lambda}, 0\right)$.

These two double foci and the four single foci may be obtained from the equivalent class-equation

$$S^3-27T^2 = 0,$$

where S, T denote the quartic and sextic invariants.

$$3S = Q^2-\tfrac{3}{4}RV-3\kappa\lambda R^2,$$

$$27T = -Q^3+\tfrac{9}{8}QRV+9\kappa WR^2,$$

and

$$Q = 2\lambda+\xi^2+\eta^2+f\xi,$$

$$R = \xi^2+\eta^2,$$

$$V = (\xi+f)^2+\eta^2+4\kappa\lambda e\xi-\kappa ef\eta^2,$$

$$W = \lambda^2+\tfrac{1}{2}\lambda(\xi^2+\eta^2+f\xi)-\tfrac{3}{16}\eta^2(f^2-ef-e^2\kappa\lambda).$$

Neglecting for the time the third terms in both S and T, which involve κ, we find from the expansion of $(3S)^3-(27T)^2$,

$$\tfrac{27}{64}R^2V^2(Q^2-RV).$$

Hence R^2 or $(\xi^2+\eta^2)^2$ measures the expansion, and the class of te

* It is incorrectly stated (*Proceedings*, Vol. xiii., p. 71) that this node is a double focus. The equivalent class-curve is a sextic, with two double and two single foci.

equivalent equation is the eighth, as was known, since there are two double points at infinity.

Reject R^2, and also suppress R or $\xi^2+\eta^2$, where it occurs, to obtain the foci. There results

$$\tfrac{27}{64}\,(f^2+2f\xi+4\kappa\lambda e\xi-\kappa ef\eta^2)^2\,(4\lambda^2+4\lambda f\xi-f^2\eta^2)$$
$$-\tfrac{27}{16}\kappa\eta^2\,(2\lambda+f\xi)^3\,(f^2-fe-e^2\kappa\lambda).$$

Write $(\xi^2+\eta^2)-\xi^2$ for η^2, and again reject $\xi^2+\eta^2$. Then $(2\lambda+f\xi)^2$ is a common factor, and denotes a double focus.

The four single foci are thus expressed:

$$(f^2+2f\xi+4\kappa\lambda e\xi+\kappa ef\xi^2)^2+8\kappa\xi^2\,(f^2-ef-e^2\kappa\lambda)\,(2\lambda+f\xi),$$

or $f^2+4\,(f+2\kappa\lambda e)\,\xi+(4+2\kappa ef+16\kappa\lambda)\,\xi^2+4\kappa\,(2f-e)\,\xi^3+\kappa^2e^2\xi^4=0.$

The origin is another double focus, since the class is the eighth.

14. To express the equation to the group of bicircular quartics in terms of the distances of the single foci from a double focus.

Let the binary quartic of § 13 be compared with the form which denotes the product of these focal distances,

$$(1+a\xi)\,(1+b\xi)\,(1+c\xi)\,(1+d\xi)=0,$$

or

$$1+p\xi+q\xi^2+r\xi^3+s\xi^4=0,$$

where p, q, r, s mean the usual combinations of the roots. The following relations may be established:

$$\lambda : f : \kappa : \kappa e : 1$$
$$:: \tfrac{1}{2}p^2-2q+4s^{\frac{1}{2}} : 4\,(ps^{\frac{1}{2}}-r) : 2qs-4s^{\frac{3}{2}}-\tfrac{1}{2}r^2 : 4\,(ps-rs^{\frac{1}{2}}) : 4qs^{\frac{1}{2}}-8s-rp.$$

The given bicircular may take these values of its constants:

$$\lambda\,(x^2+y^2)^2+(1+fx)\,(x^2+y^2)=\kappa\,(1+ex).$$

The distances of the apses from the double focus at the origin may be expressed implicitly in terms of the distances of the four single foci:

$$(p^2-4q+8s^{\frac{1}{2}})\,x^4+8\,(ps^{\frac{1}{2}}-r)\,x^3+2\,(4qs^{\frac{1}{2}}-8s-rp)\,x^2$$
$$=8\,(ps-rs^{\frac{1}{2}})\,x+4qs-8s^{\frac{3}{2}}-r^2.$$

This quartic has the same discriminant as $1+p\xi+q\xi^2+r\xi^3+s\xi^4=0$, since two foci unite at a node.

The roots of this quartic are given in § 11,

$$x\,(\sqrt{a}\pm\sqrt{b}\pm\sqrt{c}\pm\sqrt{d})+\sqrt{(abcd)}\left(\frac{1}{\sqrt{a}}\pm\frac{1}{\sqrt{b}}\pm\frac{1}{\sqrt{c}}\pm\frac{1}{\sqrt{d}}\right)=0.$$

15. For the circular cubic, $p^2-4q+8s^{\frac{1}{2}}=0$. Its equation is found after rejecting $ps^{\frac{1}{2}}-r$,

$$(8x+2p)\,(x^2+y^2)=8s^{\frac{1}{2}}x+ps^{\frac{1}{2}}+r,$$

where the focal distances have the relation

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}=0.$$

If $d = 0$, or the quartic have a triple focus at the origin, the quartic of § 14 becomes $(p^2-4q)\,x^4-8rx^3-2rpx^2+r^2=0$,

whose roots are $x\,(\pm\sqrt{a}\pm\sqrt{b}\pm\sqrt{c}) = \sqrt{abc}$.

Other modifications occur when other terms in the quartic are removed.

I next proceed to obtain relations between two parameters in the quartic, which express general properties in the form of curves.

16. To obtain the relation between the parameters, when bicircular quartics with collinear foci are resolved into pairs of circles.

Resume the ordinary form of these quartics (§ 13),

$$\lambda r^4 + r^2\,(1+bx) = \kappa\,(1+ax).$$

This will be a rational function of r^2 or x^2+y^2, if

$$(1+bx)^2 + 4\kappa\lambda\,(1+ax)$$

be a complete square.

The required relation is expressed by the hyperbolæ,

$$\kappa\lambda\,(\kappa\lambda a^2 - b^2 + ab) = 0.$$

17. To determine the first discriminating curve, or the mutual relation between the parameters, when the bicircular quartics in this group are singular. (Fig. 1.)

At a singular point $\frac{d\phi}{dx} = 0,\ \frac{d\phi}{dy} = 0.$

For the quartic $\lambda r^4 + r^2\,(1+bx) - \kappa\,(1+ax) = 0 \equiv \phi$,

$$\kappa a = 2x\,(1+bx) + 4\lambda x r^2 + br^2,$$

$$0 = 2y\,(1+bx) + 4\lambda y r^2.$$

Hence also $\kappa\,(4+3ax) = (2+bx)\,r^2.$

[The second condition is resolved into two parts

$$1+bx+2\lambda r^2 = 0 \ldots\ldots (1); \text{ and } y = 0 \ldots\ldots (2).$$

The first yields an hyperbola as a portion of curve (D_1),

$$\kappa\lambda a^2 + b\,(a-b) = 0,$$

and all nodes lie in the line $abx+2b+a=0$.

The second part gives a sextic as a second portion.

$$4\lambda x^3 + 3bx^2 + 2x - \kappa a = 0,$$

$$bx^3 + 2x^2 - 3\kappa ax - 4\kappa = 0.]$$

The parameters κ, λ are connected, as the coordinates of a unicursal sextic, or quintic, if $a = 0$. The equation may be obtained, without the intermediary (x), as the discriminant of the binary quartic (ϕ), when $y = 0$.

Besides the axes of coordinates, there is a linear asymptote

$$27a^4\kappa + 256\lambda = 48a\,(4b-3a),$$

which corresponds to the value $\left(-\frac{4}{3a}\right)$ of x; but not when $a = \infty$.

At singular points on the sextic

$$\frac{d\kappa}{dx} = 0, \quad \frac{d\lambda}{dy} = 0.$$

These conditions determine two values of x,

$$3abx^2 + 6bx + 3ax + 8 = 0,$$

to each of which a cuspidal value of (κ, λ), and thereby a corresponding cusped quartic, corresponds, if the values of (x) are real. Various modifications occur, when a and b have the values zero or infinity.

18. The cases should be noticed when these values of x unite. When this occurs, (1) $a = 6b$, and (2) $3a = 2b$.

For (1) the cusps unite in a point of undulation, and for (2) in a stationary point.

The corresponding quartics have, the one a cusp, and the other a folium-point, but they are reduced to points.

For the value (2), the sextic curve (D_1) degenerates into the quartic

$$(12\kappa\lambda + 3)^2 = 32b^2\kappa \ldots\ldots\ldots\ldots \text{(A)}$$

and the line

$$48\lambda + b^4\kappa = 12b^2 \ldots\ldots\ldots\ldots \text{(B).*}$$

19. To determine the second discriminating curve, or the mutual relation between the parameters, when bicircular quartics with collinear foci have points of undulation.

This curve (D_2) consists of two portions, obtained by different processes. The ordinary tests for undulation will be first taken.

I. At a folium-point, $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} = 0$.

Let these tests be applied to the quartic,

$$\lambda r^4 + (1 + bx)\, r^2 = \kappa\,(1 + ax),$$

or, when solved,

$$2\lambda r^2 + (1 + bx) = \sqrt{\{(1 + bx)^2 + 4\kappa\lambda\,(1 + ax)\}} \equiv 2\sqrt{U}.$$

By successive differentiation,

$$2\lambda \frac{d}{dx}(r^2) + b = U^{-\frac{1}{2}} \frac{dU}{dx},$$

$$2\lambda \frac{d^2}{dx^2}(r^2) = U^{-\frac{1}{2}} \frac{d^2U}{dx^2} - \tfrac{1}{2} U^{-\frac{3}{2}} \left(\frac{dU}{dx}\right)^2,$$

$$0 = -\tfrac{3}{2} U^{-\frac{3}{2}} \frac{dU}{dx} \frac{d^2U}{dx^2} + \tfrac{3}{4} U^{-\frac{5}{2}} \left(\frac{dU}{dx}\right)^3.$$

This condition is resolved into two factors,

$$2U \frac{d^2U}{dx^2} - \left(\frac{dU}{dx}\right)^2 = 0, \qquad \frac{dU}{dx} = 0.$$

---

* The quartics, whose parameters are thus related (B), are singular, but not cuspidal.

The first factor gives the relation of § 16, when the quartic is resolved into two circles. We are concerned with the second factor,

$$b(1+bx)+2\kappa\lambda a = 0.$$

In this case,

$$r^2 = \frac{\kappa a}{b} \pm \frac{1}{\lambda}\, U^{\frac12} = \frac{\kappa a}{b} \pm \sqrt{\left\{\frac{\kappa}{\lambda}\left(1-\frac{a}{b}\right) - \frac{\kappa^2 a^2}{b^2}\right\}},$$

$$2\lambda \frac{d}{dx}(r^2) + b = 0, \quad \text{or} \quad x + y\frac{dy}{dx} + \frac{b}{4\lambda} = 0,$$

$$2\lambda \frac{d^2}{dx^2}(r^2) = \frac{b^2}{2}\, U^{-\frac12}, \quad \text{or} \quad 1 + \left(\frac{dy}{dx}\right)^2 = \frac{b^2}{8\lambda}\, U^{-\frac12}.$$

The last may take the form

$$y^2 + \left(x + \frac{b}{4\lambda}\right)^2 = \frac{b^2 y^2}{8\lambda}\, U^{-\frac12},$$

or

$$r^2 + \frac{bx}{2\lambda} + \frac{b^2}{16\lambda^2} = \frac{b^2}{8\lambda}(r^2 - x^2)\, U^{-\frac12}.$$

That is,

$$\frac{1}{\lambda}\, U^{\frac12} - \frac{1}{2\lambda} + \frac{b^2}{16\lambda^2} = \left\{\frac{\kappa a}{b} + \frac{1}{\lambda}\, U^{\frac12} - \frac{1}{b^4}(2\kappa\lambda a + b)^2\right\}\frac{b^2}{8\lambda}\, U^{-\frac12},$$

or

$$8U - 4U^{\frac12} - \frac{b^2}{2\lambda}\, U^{\frac12} = \kappa ab - \frac{1}{b^2}(2\kappa\lambda a + b)^2.$$

After further reduction, this part of curve (D_2) is thus defined:

$$\left\{1 - \kappa ab + 4\left(2 - \frac{a}{b}\right)\kappa\lambda - \frac{4a^2}{b^2}\kappa^2\lambda^2\right\}^2$$

$$= \frac{1}{b^2}\kappa\lambda\,(b^2 - ab - a^2\kappa\lambda)\left(4 + \frac{b^2}{2\lambda}\right)^2. \quad \text{(Fig. 2.)}$$

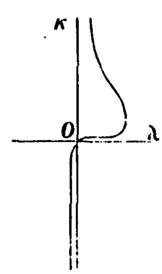

Fig. 2.

The curve has only one branch, with three inflexions, and (since it depends on the product κ, λ) lies in the first and third, or in the second and fourth quadrants, according as $a < > b$. The curve has two acnodes, whose abscissa is $-\frac{b^2}{8\lambda}$.

If $a = 0$ or b, this portion of curve (D_2) does not exist.

II. The preceding process is applicable to folium-points in companion-curves to lemniscatoids (see § 20); but for the neighbourhood of limaçonoids the conditions may be thus found.

The bitangents touch the apses in such folium-points.

The conditions are

$$4U = (1+bx)^2 + 4\kappa\lambda\,(1+ax) = 0,$$

and $\quad 2\lambda x^2 + 1 + bx = 0, \qquad$ since $y = 0$.

The eliminant is a quintic. (Fig. 3.)

$$\lambda\,(1-4\kappa\lambda)^2$$
$$= -2\kappa\lambda b\,(3a-4b) + \kappa b^3 (a-b) - 8\kappa^2\lambda^2 a\,(a-b).$$

If $a = 0$, the figure has no asymptote but the axes.*

$a = b$, curve (D_3) becomes a bicusped quartic, as in Fig. 4.

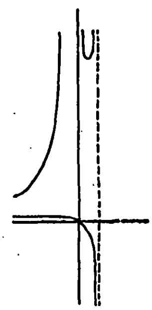

Fig. 3.

20. To determine all bicircular quartics with collinear foci by the aid of the two discriminating curves.

The singular quartics are crunodal and acnodal. The former are lemniscatoid for values of (κ, λ) in the first quadrant, and limaçonoid in the second quadrant. The acnodal quartics shrink to acnodes for points (κ, λ) on the bounding lines NQ, PL; for ulterior values of (κ, λ) no corresponding quartics are possible. (Fig. 1.)

The quartics may be bi-, uni-, and non-folium.

If (κ, λ) lie between the curve (D_3) of Fig. 2, the quartic is bifolium; if (κ, λ) lie between the branches of curve (D_3) of Fig. 3, the quartic is unifolium. For values of (κ, λ) exterior to the branches of Figs. 2 and 3, the corresponding quartics are non-folium.

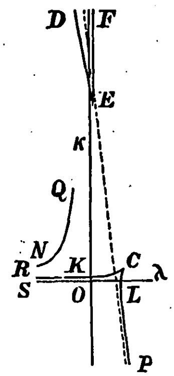

Fig. 1.

The sequence has been minutely exhibited in *Proceedings*, Vol. xii., pp. 21, 22, with illustrations, for the several quadrants, with the omission of the limitations created by the second branch (Fig. 3) of curve (D_3).

The cuspidal portion DEF of Fig. 1 protrudes through the apsidal portion of Fig. 3; the corresponding quartics have the same character as for points in and near the cuspidal part KCL.

It is easy to anticipate the number of folia in the neighbourhood of critical quartics, by considering the shapes of these last. Near lemniscatoids, the companion-curves are bifolium; near limaçonoids, unifolium.

21. For two relative values of the constants, the first discriminating curve is modified, as was pointed out in § 18.

* In the description of the curve (D_2) (*Proceedings*, Vol. xii., p. 25), this portion was omitted. Without its aid, the limiting forms of Fig. 4 (p. 22) cannot be found.

If $3a = 2b$, $2\kappa = x^2$, $6\lambda x^2 + 4bx + 3 = 0$.

The eliminant defines curve (D_1) as the bicusped quartic $(12\kappa\lambda + 3)^2 = 32\kappa b^2$. (Fig. 4.)

In the second quadrant, the sequence of values of the corresponding quartics is unaltered; the two cusps of Fig. 1 have united in a point of inflexion. In the first quadrant, critical quartics are acnodal, and the companion-curves are bifolium, according to the position of (κ, λ) on curve (D_2). (Fig. 2.)

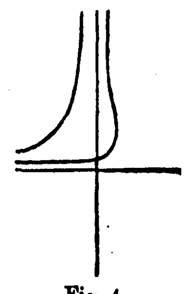

Fig. 4.

If $a = 6b$, the sextic is also modified: the two cusps of Fig. 1 have united in a point of undulation. (Fig. 5.)

The quartics, which correspond to the point of inflexion in Fig. 4, and the folium-point in Fig. 5, are cuspidal and undulational, but both mere point-quartics, as these points (κ, λ) are on the limiting branches of curve (D_1),

$$5b^4y^4 + 2b^2y^2(5b^2x^2 + 12bx + 12) + (bx+2)^3(5bx - 6) = 0,$$

$$81b^4y^4 + 54b^2y^2(3b^2x^2 + 4bx + 4) + (3bx + 2)^4 = 0.$$

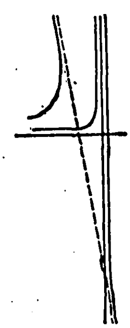

Fig. 5.

22. In the case, when the bicirculars pass through the origin,

$$\lambda(x^2+y^2)^2 + (1+bx)(x^2+y^2) = \kappa x.$$

The curve (D_1) is the tricuspidal quartic (see § 17)

$$(b + 9\kappa\lambda)^2 = 4(b^2 - 3\lambda)(1 + 3b\kappa).$$

[To values of (κ, λ) on curve (D_1) in the first quadrant there correspond acnodal quartics: in the fourth quadrant, as (κ, λ) on curve (D_1) approaches the cusp, an acnodal quartic and a lemniscatoid gradually unite in the cusped quartic

$$b^3y^2(y^2 + 2x^2) + 3by^2(1+bx) + x(bx+1)^3 = 0.]$$

To critical values in the second and third quadrants, limaçonoids correspond.

23. In the case where both $\frac{1}{a}$ and $\frac{1}{b} = 0$,

or $$\lambda(x^2+y^2)^2 + bx(x^2+y^2) = \kappa x,$$

the curve (D_1) is the cubic hyperbola $27\kappa\lambda^2 = 4b^3$,

the curve (D_2) is the homothetic cubic $8\kappa\lambda^2 = b^3$,

since the other portion (§ 19, II.) does not exist.

If the coordinates be considered tangential, this group comprises those class-quartics which have a triple focus and a single focus at infinity, and whose satellite-conics have the same as single foci. The critical quartics are acnodal; the companion-curves, whether single, or pairs of ovals, non-folium or uni-folium as (κ, λ), are above or below curve (D_2). The origin is in all cases a folium-point.

[If $\frac{1}{b} = 0$ only, there is a cusp on curve (D_1), and the corresponding quartic is resolved into two circles

$$a^2(x^2+y^2) = 4, \quad (ax+2)^2+y^2 = 0.$$

The case when $a = 0$ has been studied (Vol. xii., pp. 17–27); Cartesians, in which $b = 0$, will be examined in § 29.]

24. Bicircular quartics, which have the same single foci, but distinct double foci, are mutually orthotomic.

This is an extension of Prof. Crofton's theorem for Cartesians (*Proceedings*, Vol. i., 6).

Let the vector equation to one of the group be

$$lr_1+mr_2+nr_3 = 0,$$

where (by § 10) $l : m : n :: (b-c)\{\sqrt{(ad)}-\sqrt{(bc)}\}$
$: (c-a)\{\sqrt{(bd)}-\sqrt{(ca)}\}$
$: (a-b)\{\sqrt{(cd)}-\sqrt{(ab)}\}.$

The equation to a tangent circle at the point (R_1, R_2, R_3) is

$$\frac{lr_1^2}{R_1}+\frac{mr_2^2}{R_2}+\frac{nr_3^2}{R_3} = 0.$$

Then another quartic of the group may be written

$$l'r_1+m'r_2+n'r_3 = 0,$$

where $l' \propto (b-c)\{\sqrt{[(a+h)(d+h)]}-\sqrt{[(b+h)(c+h)]}\}$,

the triple focus being moved, but r_1, r_2, r_3 being unaltered.

Its tangent circle is $\frac{l'r_1^2}{R_1}+\frac{m'r_2^2}{R_2}+\frac{n'r_3^2}{R_3} = 0.$

The condition that the tangent circles, or, as a consequence, that the two quartics intersect orthogonally, is

$$2\left(\frac{la}{r_1}+\frac{mb}{r_2}+\frac{nc}{r_3}\right)\left(\frac{l'a}{r_1}+\frac{m'b}{r_2}+\frac{n'c}{r_3}\right)$$
$$= \left(\frac{l}{r_1}+\frac{m}{r_2}+\frac{n}{r_3}\right)\left(\frac{l'a^2}{r_1}+\frac{m'b^2}{r_2}+\frac{n'c^2}{r_3}\right)$$
$$+\left(\frac{l'}{r_1}+\frac{m'}{r_2}+\frac{n'}{r_3}\right)\left(\frac{la^2}{r_1}+\frac{mb^2}{r_2}+\frac{nc^2}{r_3}\right).$$

This condition may be simplified:

$$\frac{1}{r_2 r_3}(mn'+m'n)(b-c)^2+\frac{1}{r_3 r_1}(nl'+n'l)(c-a)^2+\frac{1}{r_1 r_2}(lm'+l'm)(a-b)^2=0.$$

But, from the two quartics, r_1, r_2, r_3 are the minors of the determinant

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \end{vmatrix}.$$

The condition of orthotomy becomes

$$\Sigma\,(m^2n'^2-m'^2n^2)\,(b-c)^2=0.$$

It may be proved that this is an identity.

Substitute for m, n, m', n', their values, and reject the common factor $(b-c)^2\,(c-a)^2\,(a-b)^2$,

$$\{\sqrt{(bd)}-\sqrt{(ca)}\}^2\{\sqrt{[(c+h)(d+h)]}-\sqrt{[(a+h)\,(b+h)]}\}^2,$$
$$-\{\sqrt{(cd)}-\sqrt{(ab)}\}^2\{\sqrt{[(b+h)(d+h)]}-\sqrt{[(c+h)\,(a+h)]}\}^2,$$

together with two other sets of values, may be shown to be zero.

The collected values of the sets may be reduced to the factors

$$\{(bd+ac-cd-ab)+(cd+ab-ad-bc)+(ad+bc-bd-ac)\},$$
$$\{h\,(a+b+c+d)+2h^2+2\sqrt{(abcd)}-2\sqrt{[(a+h)(b+h)(c+h)(d+h)]}\};$$

of which one is identically zero.

The general problem is considered in *Higher Plane Curves*, p. 238.

25. To generate a Cartesian by Dr. Casey's method of reciprocation. It will be noticed that the method of § 4 fails for generating Cartesians, since $d=0$, when a double becomes a triple focus.

In this case, three concentric circles take the place of confocal conics:

$$x^2+y^2=A^2 \quad \ldots\ldots\ldots\ldots (F_1),$$
$$x^2+y^2=B^2 \quad \ldots\ldots\ldots\ldots (F_2),$$
$$x^2+y^2=C^2 \quad \ldots\ldots\ldots\ldots (F_3);$$

and there are three corresponding circles of reciprocation, which are mutually orthotomic:

$$(x-f)^2+y^2=\rho_1^2 \quad \ldots\ldots\ldots\ldots (J_1),$$
$$(x-g)^2+y^2=\rho_2^2 \quad \ldots\ldots\ldots\ldots (J_2),$$
$$(x-h)^2+y^2=\rho_3^2 \quad \ldots\ldots\ldots\ldots (J_3).$$

The Cartesian is generated in three ways as the envelope of the variable circle $\quad (x-\alpha)^2+(y-\beta)^2=\gamma^2,$

which is subject to the conditions that its centre is in F,

$$\alpha^2+\beta^2=A^2,$$

and that it cuts J orthogonally,

$$2\alpha f-\alpha^2-\beta^2+\gamma^2+\rho_1^2-f^2=0.$$

The Cartesian envelope is found to be

$$(x^2+y^2)^2+2(\rho_1^2-f^2-2A^2)(x^2+y^2)+8A^2fx+(\rho_1^2-f^2)^2-4A^2f^2=0.$$

Dr. Casey has been anticipated in this method of generating Cartesians by Quetelet (Chasles, *Aperçu Historique*, p. 350), if in his theorem we measure the distance from the centre of the variable circle to the circumference of the orthotomic circle along a tangent.

In like manner, its equation is found in terms of the constants

$$\rho_2,\ g,\ B;\quad \rho_3,\ h,\ C.$$

By equating the coefficients of like terms in these three forms of the same Cartesian,

$$\rho_1^2=f^2-fg-B^2+A^2=f^2-fh-C^2+A^2,$$
$$A^2f=B^2g=C^2h=fgh.$$

The (F) circles are

$$x^2+y^2=gh,\quad x^2+y^2=fh,\quad x^2+y^2=fg.$$

The (J) circles are $(x-f)^2+y^2=(f-g)(f-h)$,

$$(x-g)^2+y^2=(g-f)(g-h),$$
$$(x-h)^2+y^2=(h-f)(h-g).$$

The Cartesian thus generated is defined by the form

$$(x^2+y^2)^2-2(gh+hf+fg)(x^2+y^2)$$
$$+8fghx+h^2g^2+f^2h^2+f^2g^2-4fgh(f+g+h)=0.$$

It will appear (§ 28) that f, g, h are the distances of the single foci from the triple focus at the origin.

26. The dual theorem may be noticed.

If the variables be interpreted as Boothian tangential coordinates, the (F) equations still denote circles, and the (J) equations singly confocal conics, with parallel directrices.

Thus the class-quartic

$$\kappa(1+e\xi)=\xi^2+\eta^2+\lambda(\xi^2+\eta^2)^2,$$

which has a triple and a single focus, and for which the centre of its satellite-circle coincides with the triple focus, may be generated in three ways.

It is the envelope of a variable conic, with a fixed focus, whose directrix touches a fixed dirigent circle (F), and which has a common tangent with a fixed confocal conic (J), such that the points of contact subtend a right angle at the common focus.

27. The distances of the apses from the triple focus are found by writing $y=0$, in the Cartesian of § 25.

The reducing cubic of this binary quartic by Euler's method is

$$z^3-qz^2+prz-r^2=0.$$

Its roots are gh, fh, fg, the squares of the radii of the (F) circles. (Casey, p. 491.)

Both quartic and cubic have the same discriminant

$$(g-h)^2(h-f)^2(f-g)^2.$$

28. The equation to the bicircular (§ 25) may assume various forms

$$(x^2+y^2+gh-hf-fg)^2=4gh\{(x-f)^2+y^2\}.$$

Hence, if ρ_1, ρ_2, ρ_3 denote the focal distances of a current point,

$$2\rho_1\surd(gh)=x^2+y^2+gh-hf-fg;$$

so also

$$2\rho_2\surd(hf)=x^2+y^2+hf-fg-gh,$$

$$2\rho_3\surd(fg)=x^2+y^2+fg-gh-hf.$$

Again, by subtracting these equations in pairs, we obtain the usual vector forms

$$h\surd g\rho_2-g\surd h\rho_3=\surd(fgh)(h-g),$$

$$f\surd h\rho_3-h\surd f\rho_1=\surd(fgh)(f-h),$$

$$g\surd f\rho_1-f\surd g\rho_2=\surd(fgh)(g-f).$$

Lastly, by addition, the equation of § 10, Cor. I., is derived:

$$(g-h)\surd f\rho_1+(h-f)\surd g\rho_2+(f-g)\surd h\rho_3=0.$$

(Casey, p. 486.)

29. To determine the foci in Cartesians.

The equation to the group is taken to be

$$\lambda(x^2+y^2)^2+(x^2+y^2)=\kappa(1+ex).$$

Since the circular points at infinity are cusps, Cartesians are class-sextics, and not class-octavics,* as ordinary bicircular quartics are. The origin is a triple focus, and there are three single foci.

Proceed to find the equivalent class-sextic as in § 13, when $f=0$,

$$S^3-27T'^2=0.$$

If the terms be omitted at first from S and T', which contain κ, the expanded form is measured by $(\xi^2+\eta^2)^3$, or the class is diminished from 12 to 6 dimensions.

The residue, after rejecting this factor R^3, is the equivalent class-

* In the second edition of the *Aperçu Historique*, p. 353 (Paris, 1875), the error is repeated, which was long ago corrected by Prof. Cayley, *Liouville*, Vol. xv., p. 354.

sextic
$$16\kappa^2\lambda^3e^2+8\kappa\lambda^2e\,(1+4\kappa\lambda)\,\xi+16\kappa^2\lambda^2e^2\xi^2\,(1-\kappa e\xi)$$
$$+(\xi^2+\eta^2)\,\{\lambda\,(1+4\kappa\lambda)^2+8\kappa\lambda\,(1+4\kappa\lambda)\,e\xi\,(1-\kappa e\xi)-4\kappa^3\lambda^2e^2\xi^3\}$$
$$+(\xi^2+\eta^2)^2\,\{(1+4\kappa\lambda)^2\,(1-\kappa e\xi)-8\kappa^2\lambda\,(1+4\kappa\lambda)\,e\xi\}$$
$$-(\xi^2+\eta^2)^3\,\kappa\,(1+4\kappa\lambda)^2$$
$$=-6\kappa^2\lambda^2e^2\eta^2\,(4-3\kappa e\xi)$$
$$+(\xi^2+\eta^2)\,\{\tfrac{27}{16}\kappa^4\lambda e^2\eta^4-\kappa^2\lambda e^2\eta^2\,(8-9\kappa e\xi)+\tfrac{1}{2}\kappa^2\lambda e^2\eta^2\,(1+36\kappa\lambda)\}$$
$$+(\xi^2+\eta^2)^2\,(1+36\kappa\lambda)\,\tfrac{1}{4}\kappa^2e^2\eta^2.$$

The focal lengths are obtained by substituting $(\xi^2+\eta^2)-\xi^2$ for η^2, and rejecting all the terms which involve $\xi^2+\eta^2$,
$$8\kappa\lambda e+4\,(1+4\kappa\lambda)\,\xi-4\kappa e\xi^2+\kappa^2e^2\xi^3=0.$$
Its discriminant, $27\kappa^3\lambda e^4+4\kappa e^2\,(1+36\kappa\lambda)+16\,(1+4\kappa\lambda)^2$, is the same as that of the expression for the apsidal distances
$$\lambda x^4+x^2-\kappa ex-\kappa=0.$$

30. To determine the first discriminating curve, when Cartesians have singular points. (Fig. 6.)

This curve (D_1) is defined by the discriminant of the binary quartic of § 29 $(y=0)$ as a bicusped quartic
$$16\,(1+4\kappa\lambda)^2+4\kappa e^2\,(1+36\kappa\lambda)+27\kappa^3\lambda e^4=0.$$
It is drawn more simply as a unicursal curve by the conditions of § 17, by equating b to zero,
$$\kappa=\frac{2x^2}{4+3ex},\quad -\lambda=\frac{ex+2}{x^2\,(4+3ex)}.$$
The linear asymptote has the form
$$27e^4\kappa+256\lambda+144e^2=0.$$
This quartic in (κ,λ) has two cusps, one at infinity, and the other $\left(-\frac{32}{9e^2},\ -\frac{3e^2}{128}\right)$, when $3ex+8=0$.

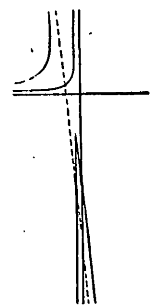

Fig. 6.

To these related values of κ, λ, limaçon Cartesians correspond; and to the finite cusp the cardioid,
$$27e^4y^4+18e^2y^2\,(3e^2x^2-64)+(ex-8)\,(3ex+8)^3=0.$$
There are no lemniscatoid Cartesians, since no part of curve (D_1) is in the first quadrant.

31. To determine the second discriminating curve, when Cartesians have points of undulation. (Fig. 7.)

The second process of § 19 is alone applicable.

The conditions, that the bitangents touch the apses in folium-points, are
$$1+4\kappa\lambda\,(1+ex)=0,\quad 2\lambda x^2+1=0.$$
The eliminant defines curve (D_2) as the quartic $(1+4\kappa\lambda)^2+8\kappa^2\lambda e^2=0$, and the (κ) axis.

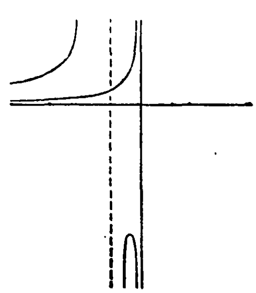

Fig. 7.

The abscissa is essentially negative, and the ordinate has a minimum value. For values of (κ, λ) within the branches of Fig. 7 the corresponding quartics have veri-bitangents; for exterior values of (κ, λ), acu-bitangents.

32. By the aid of these two discriminating curves, all Cartesians may be exhibited.

In the first quadrant of Figs. 6, 7, there is no singular Cartesian, nor has any curve a folium-point. Each is an oval without a folium, and has an acu-bitangent.

In the second quadrant, for values of (κ, λ) on or above the upper branch of Fig. 6, the corresponding Cartesians are points or do not exist; for values below and above the lower branch of Fig. 6, the Cartesians are single ovals. If (κ, λ) is on this branch of Fig. 6, the curves are limaçons, and for lower values, two ovals.

The single ovals have imaginary or real bitangents, as (κ, λ) is above or below the branch of Fig. 7; the pairs of ovals have veri- or acu-bitangents, as (κ, λ) is above or below the lower branch, terminating in two circles, when $\kappa = 0$.

In the third quadrant, the Cartesians consist of an oval with an acu-bitangent, unless (κ, λ) falls within curve (D_2).

If (κ, λ) is within the loop, and beyond the cusp of Fig. 6, the Cartesian (which protrudes in the former loop) is a single veri-bitangential oval.

If (κ, λ) is within the cuspidal part of Fig. 6, the Cartesian has a pair of ovals, terminating for nodal values in a limaçon and an acnodal quartic. If (κ, λ) be the cusp, the nodal values unite in the cusped cardioid.

In the fourth quadrant, no Cartesian is possible for values of (κ, λ) above curve (D_1). For values of (κ, λ) within that curve and the (κ) axis, the curves have a single acu-bitangential oval, commencing with acnodes and ending in pairs of circles.

33. If the origin is on the Cartesian, its equation becomes

$$\lambda\,(x^2+y^2)^2+x^2+y^2 = \kappa x.$$

The curves (D_1), (D_2) are two homothetic cubics:

$$27\kappa^2\lambda+4 = 0, \quad 8\kappa^2\lambda+1 = 0.$$

Cartesians occur for all values of the parameters. For points (κ, λ) beyond both branches of curve D_1, they are single ovals with veri-bitangents; for intermediate values of (κ, λ), the Cartesians consist of a pair of ovals, with real or imaginary bitangents, as (κ, λ) is beyond or within curve (D_2).

34. If $e = 0$, the Cartesian is resolved into two circles.

35. The classification of this memoir is applicable to those cyclides, which are generated by the revolution about their axes of bicircular quartics with collinear foci.