CRITICISMS AND DISCUSSIONS.

TIME AND SPACE.[1]

The conceptions of time and space which I wish to develop here have arisen on the basis of experimental physics. Therein lies their strength. Their tendency is radical. From now on space-in-itself and time-in-itself are destined to be reduced to shadows, and only a sort of union of the two will retain an independent existence.

I.

I wish first to show how from the mechanics now generally accepted we might arrive by purely mathematical considerations at a change in our ideas of space and time. The equations of Newton's mechanics show a double invariance. Their form is maintained, first, if we subject our system of original coordinates in space to any *change of position*; second, if we change its state of motion, that is to say, impart to it any uniform translation; neither does the zero-point of time play any part. We are accustomed to considering the axioms of geometry as settled before we approach the axioms of mechanics, and therefore these two invariances are seldom mentioned together. Each of them represents a certain group of transformations, which transform the differential equations of mechanics back into themselves. The existence of the first group is regarded as a fundamental property of space. It is usually preferred to treat the second group with contempt in order to

[1] Lecture delivered at the eightieth Congress of Naturalists at Cologne, September 21, 1908. Published in *Physikalische Zeitschrift*, X (1909, pp. 104-111, and *Jahresbericht der deutschen Mathematiker-Vereinigung*, Vol. XVIII, pp. 75-88; *Gesammelte Abhandlungen*, edited by D. Hilbert, pp. 431-444; also separately, Leipsic, B. G. Teubner, 1909. Translated from the German by Edward H. Carus who herewith expresses his gratitude to Prof. W. B. Smith of Tulane University for many suggestive criticisms.

pass lightly over the fact that we can never decide from physical phenomena whether the space we have assumed to be at rest is not after all in a state of uniform translation. Thus these two groups have an entirely separate existence, side by side. Their quite heterogeneous character may have discouraged their combination; but precisely this combination into one group gives us food for thought. We shall try to illustrate these relations graphically. Let x, y, z be rectangular coordinates of space and let t represent time. As they occur in our experience places and times are always combined. No one has ever observed a place except at a time, nor a time except in a place. But here I am still respecting the dogma that space and time have each an independent significance. I shall call a point in space at a definite time, that is, a system of values, x, y, z, t, a "world-point (*Weltpunkt*). The multiplicity of all possible systems of values x, y, z, t I shall call the world. I might boldly sketch four world-axes on the blackboard. Even *one* such axis consists merely of vibrating molecules and travels with the earth in space, thus alone furnishing us with sufficient food for abstract thought; the somewhat greater abstraction involved in the number four does not disturb the mathematician. In order not to have an empty void anywhere we shall assume that there is something perceptible everywhere and at all times. To avoid the terms matter or electricity we shall call this something substance. Let us direct our attention to the substance-point (*substantiellen Punkt*) at the world-point x, y, z, t, and imagine that we are able to recognize this substance-point at every other time. Let the changes dx, dy, dz, of the space coordinates of this substance-point correspond to an element of time dt. We thus obtain as a representation so to speak of the eternal course of the substance-point a curve in the world, a world-line whose points can be determined uniquely in terms of a parameter t from $-\infty$ to $+\infty$. The whole world stands resolved into such world-lines, and I wish at once to make the fundamental assertion that according to my opinion physical laws may find their most complete expression as mutual relations among these world-lines.

By the concepts space and time, the x, y, z-manifold $t=0$ and its two sides $t>0$ and $t<0$ become separated. If for simplicity we keep the zero point of time and space fixed, then the first mentioned group of mechanics means that we can give any rotation around the origin to the x, y, z-axes in $t=0$ corresponding to the homogeneous linear transformations of the expression $x^2+y^2+z^2$ into itself.

But the second group or invariance means that without changing the expressions of the laws of mechanics, we can replace x, y, z, t by $x-\alpha t$, $y-\beta t$, $z-\gamma t$, t, α, β, γ being any constants whatever. The time-axis can accordingly be given any direction whatever toward the upper half-world $t>0$. Now what connection has the condition of orthogonality in space with this complete upward freedom of the time-axis?

To exhibit the connection we take a positive parameter c and consider the locus

$$c^2 t^2 - x^2 - y^2 - z^2 = 1.$$

It consists of two sheets separated by $t=0$ analogous to a hyperboloid of two sheets. Considering the sheet in the region $t>0$ we now conceive those homogeneous linear transformations of x, y, z, t into four new variables x', y', z', t', in which the expression for this sheet of the hyperboloid in the new variables corresponds to the original expression. Evidently the rotations of space about the origin belong to these transformations. We shall next obtain a full understanding of the remaining transformations by considering one in which y and z remain unchanged. Let us draw (Fig. 1) the intersection of this sheet with the plane of the x- and t-axes, the

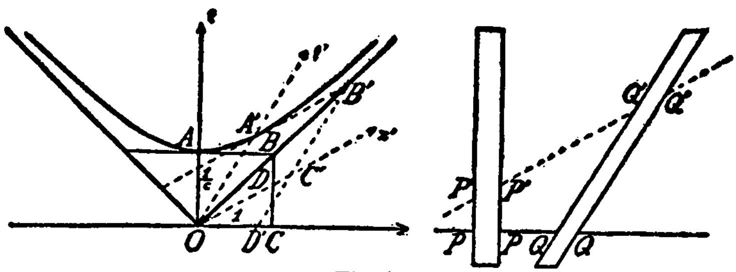

Fig. 1.

upper branch of the hyperbola $c^2 t^2 - x^2 = 1$ with its asymptotes. Then let any radius vector OA′ of this branch of the hyperbola be constructed from the origin O, let the tangent to the hyperbola at A′ be extended to the right until it intersects the asymptote at B′, let OA′B′ be completed to form the parallelogram OA′B′C′, and finally for later developments let B′C′ be continued to D′, its intersection with the *x-axis*. If we then take OC′ and OA′ as axes for parallel coordinates x' and t' with units $OC'=1$, $OA'=1/c$, then this branch of the hyperbola again has the equation $c^2 t'^2 - x'^2 = 1$, $t>0$, and the transition from x, y, z, t to x', y, z, t' is of the type under consideration. We now add to these transformations all arbitrary shiftings of the space and time origin, and in this way construct a group of transformations obviously still dependent on the parameter c, which I designate by G_c.

If we now let c increase to infinity, $1/c$ thus converging to zero, we see from the figure described that the branch of the hyperbola always approaches closer to the x-axis and the angle between the asymptotes widens into a straight angle. At the limit the special transformation changes into one in which the t'-axis can have any upward direction and x' steadily approaches nearer to x. In consequence of this it is clear that the group G_c, in the limit for $c=\infty$, thus as the group G_∞, becomes the complete group of Newton's mechanics. Under these circumstances and since G_c is mathematically more intelligible than G_∞, a mathematician in the free play of his imagination might well have had the idea that, after all, the phenomena of nature do not actually remain invariant for the group G_∞, but rather for a group G_c with a c that is definite and finite but *very large* if taken in the ordinary units. Such an idea would have been an extraordinary triumph of pure mathematics. Now, although mathematics has here been caught napping she still has the satisfaction that, owing to her happy antecedents, through senses made keen by their exercise in broad vistas, she is capable of grasping at once the far-reaching consequences of such a transformation of our conception of nature.

I shall now indicate what value of c will finally come into consideration. For c we shall substitute the *velocity of light in a vacuum.* In order to avoid the terms "space" and "void" we can define this magnitude as the ratio between the electromagnetic and the electrostatic units of electric quantity.

The existence of the invariance of natural laws for the group G_c under consideration would now be expressed as follows:

From the totality of natural phenomena we can derive with ever increasing exactitude by successively closer and closer approximations, a system of reference $x, y, z,$ and t, space and time, in terms of which these phenomena are then represented according to definite laws. But this system of reference is by no means uniquely determined thereby. *It is still possible to change this system of reference at will corresponding to the transformations of the above mentioned group* G_c, without changing thereby the expression of natural laws.

For example, according to the described figure we can also call t the time, but then in connection with it we must necessarily define space by the manifold of the three parameters, x', y, z, in which case physical laws would be expressed in terms of x', y, z, t', exactly the same as in terms of x, y, z, t. According to this there would be in the world not that particular space but an infinite number of spaces, just as there is an infinite number of planes in three-dimensional space. Three-dimensional geometry becomes a chapter of four-dimensional physics. You now understand why I said at the outset that space and time are to fade away into mere shadows and that only a world-in-itself will exist.

II.

Now the question is, what circumstances force the changed conception of space and time on us? Does it never, as a matter of fact, contradict phenomena? And finally, has it advantages for the description of phenomena?

Before we enter into these questions, let us first make an important observation. When we individualize space and time in any manner, then a straight line parallel to the t-axis corresponds as world-line to a substance point at rest, a straight line inclined to the t-axis corresponds to a uniformly moving substance-point, and a world-line curved at will corresponds to a not-uniformly moving substance point. If we consider the world-line passing through any world-point x, y, z, t, and if we there find it parallel to any radius vector OA′ of the above-mentioned hyperboloid sheet, we may introduce OA′ as the new time-axis, and in the new conception of space and time thus obtained substance appears at rest at the world-point in question. Let us now introduce this fundamental axiom:

By a suitable determination of time and space, the substance present at any world-point whatever may always be conceived of as at rest.

This axiom means that in every world-point the expression

$$c^2\,dt^2 - dx^2 - dy^2 - dz^2$$

is always positive or, what amounts to the same thing, every velocity v is always less than c. According to this, c would exist as upper limit for all substance velocities and in this fact would lie the deeper significance of the magnitude c. In this other form the axiom has in it something which at first sight is unsatisfactory. But we must consider that now a modified mechanics will supersede the old—one into which will enter the square root of the above combination of differentials of the second degree, so that cases involving velocities exceeding that of light will only play some such part as figures with imaginary coordinates play in geometry.

The *impulse* and actual motive *for the assumption of the group* G_c originated through the fact[2] that the differential equation for the transmission of light-waves in empty space is actually characterized by the Group G_c. On the other hand the concept of rigid bodies has a meaning only in a mechanics with the group G_∞. If we have an optics with G_c and if on the other hand rigid bodies existed, it is easy to perceive that by the two hyperboloid sheets belonging to G and to G_∞ a definite t direction would be determined, and this would have the further result that we must be able to detect by means of suitable rigid optical instruments in the laboratory, a change in the phenomena at different orientations with reference to the direction of the earth's motion. All attempts, however, at this detection, especially a famous interference experiment of Michelson, had a negative result. To find an explanation for this, H. A. Lorentz constructed a hypothesis the value of which depends on the invariance of optics for the group G_c. According to Lorentz, every body in motion suffers a contraction in the direction of the motion, and for the velocity v this contraction is in the ratio

$$1 : \sqrt{1-(v^2/c^2)}.$$

This hypothesis sounds very fantastic, for the contraction is not to be regarded as a consequence of resistance in the ether but entirely as a gift from above, a phenomenon accompanying the state of motion.

I shall now show by our figure that the Lorentz hypothesis is entirely equivalent to the new conception of space and time through which it may much more readily be understood. If, for simplicity's sake we ignore y and z and consider a world of one space dimension, then parallel strips, an upright one like the t-axis, and one inclined to it (see Fig. 1) represent the path respectively of a stationary and a uniformly moving body which in both cases maintain a constant spatial extent. If OA′ is parallel to the second strip, we can introduce t' as time and x' as the space coordinate, and the

[2] What is practically an application of this fact is to be found as early as 1887 in a contribution by W. Voight in *Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen*, mathematisch-physikalische Klasse, page 41.

second body then appears at rest and the first in uniform motion. We now assume that the first body conceived as at rest has the length l, that is, the cross-section PP of the first strip on the x-axis $= l.\mathrm{OC}$ where OC denotes the unit on the x-axis; and on the other hand that the second body *conceived as at rest* has the same length l, that is, the cross-section of the secondstrip, measured parallel to the x'-axis gives the equation $\mathrm{Q'Q'} = l.\mathrm{OC'}$. We now have in these two bodies constructions of two *equal* Lorentz electrons, one at rest and one in uniform motion. If we keep the original coordinates x, t, fixed, then the section QQ of the respective strip *parallel to the x-axis*, must be regarded as an extension of the second electron. Now it is clear since $\mathrm{Q'Q'} = l.\mathrm{OC'}$ that $\mathrm{QQ} = l.\mathrm{OD'}$. A simple calculation shows that if $(dx/dt) = v$ for the second strip,

$$\mathrm{OD'} = \mathrm{OC}\sqrt{1-(v^2/c^2)},$$

and therefore also $\mathrm{PP}:\mathrm{QQ} = 1 : \sqrt{1-(v^2/c^2)}$. But this is the meaning of the hypothesis of Lorentz on the contraction of electrons in motion. If, on the other hand, adopting the system of reference $x'\,t'$, we regard the second electron as at rest, then the length of the first will be denoted by the cross section P′P′ of its strip parallel to OC′, and we would find the first electron shortened in exactly the same proportion with reference to the second. For it is according to the figure:

$$\mathrm{P'P'}:\mathrm{Q'Q'} = \mathrm{OD}:\mathrm{OC'} = \mathrm{OD'}:\mathrm{OC} = \mathrm{QQ}:\mathrm{PP}.$$

Lorentz called the combination t' of x and t the *place-time* of the uniformly moving electron and used a physical construction of this conception for the better understanding of the contraction hypothesis. But it remained for A. Einstein[8] to recognize clearly that the time of one electron was just as good as that of the other, that is, that t and t' are to be treated alike. Thus time was the first to be discarded as a concept determined uniquely by phenomena.

Neither Einstein nor Lorentz disturbed the conception of space, perhaps for the reason that in the special transformation where the x', t' plane coincides with the x, t plane it is possible to interpret the x-axis of space as remaining fixed in its position. To loftily ignore the conception of space in similar wise is doubtless due to the boldness of mathematical discipline. After this further step which however is indispensable for a true understanding of the group G_c, the expression *postulate of relativity* for the demand for an invariance

[8] A. Einstein, *Annalen der Physik*, XVII, 1905, p. 891; *Jahrbuch der Radioaktivität und Electronik*, IV, 1907, p. 411.

in the group G_c, seems to me very weak. Since the postulate comes to mean that phenomena occur only in the four-dimensional world of space and time but the projection into space and into time can still be assumed with a certain degree of freedom, I would rather call this proposition the *postulate of the absolute world* (or for short, *world-postulate*).

III.

Through the world-postulate a similar kind of treatment of the four determining elements x, y, z, t, becomes possible. Through it, as I shall now show, we gain an insight into the forms under which physical laws operate. Above all, the conception of *acceleration* becomes sharply defined.

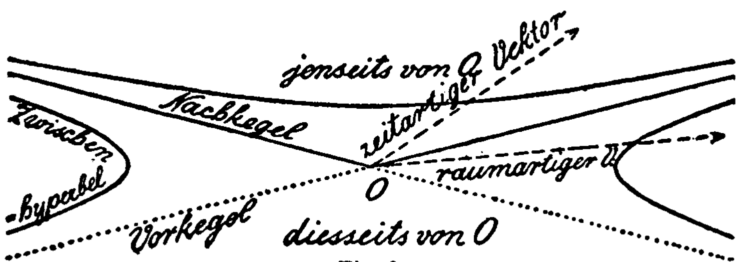

Fig. 2.

I shall use a geometrical mode of expression which at once suggests itself, at the same time tacitly ignoring z in the triplet x, y, z. I take any world-point, O, as the space-time origin. The *cone* $c^2 t^2 - x^2 - y^2 - z^2 = 0$ with O as vertex (Fig. 2) consists of two parts, one with the values of $f < 0$, another with the values of $t > 0$. The first, the *"past" cone* (*Nachkegel*) *of* O consists, let us say, of all world-points which "send light to O"; the second, the *"future" cone* (*Vorkegel*) *of* O, consists of all points which "receive light from O." The region bounded only by the future cone of O may be designated *this side of* O (*diesseits von* O), and that bounded only by the past cone, *the other side of* O (*jenseits von* O). The hyperboloid sheet considered above,

$$F = c^2 t^2 - x^2 - y^2 - z^2 = 1, \; t > 0, \text{ falls to the other side of O.}$$

The region between the cones is filled with the hyperboloidic forms of one sheet

$$-F = x^2 + y^2 + z^2 - c^2 t^2 = k^2$$

for all constant positive values of k^2. Of importance for us are the hyperbolas with O as center which lie on the latter loci. The single branches may be called briefly *interhyperbolas* (*Zwischen-hyperbeln*) *with center* O. Such a branch of a hyperbola, considered as the world-line of a substance-point, would represent a motion which, for $t=-\infty$ and $t=+\infty$ aproaches asymptotically the velocity of light, c.

If now in analogy to the concept of a vector in space, we call a directed tract (*gerichtete Strecke*) in the manifold x, y, z, t, a *vector*, then we must differentiate between time vectors (*zeit-artigen Vektoren*) with a direction from O to the sheet $+F=1$, $t>0$, and the space-vectors (*raumartigen Vektoren*) with a direction from O to $-F=1$. The time-axis can be parallel to any vector of the first kind. Every world-point between the past cone and future cone of O can be arranged by the system of reference to be *simultaneous* with O, but equally well as *previous to* O or *later than* O. Every world-point on this side of O is necessarily always previous to O, every world-point on the other side of O necessarily always later than O. Passing the limit for $c=\infty$ would correspond to the complete closing up of the wedge-shaped section between the cones into the plane manifold $t=0$. In our figures this section has purposely been made of different widths.

Let us resolve any vector whatever as from O to x, y, z, t, into the four *components*, x, y, z, t. If the directions of two vectors are respectively those of a radius vector OR from O to one of the surfaces $\mp F=1$ and of a tangent RS at the point R of the surface concerned, then the vectors shall be called *normal* to each other. Accordingly

$$c^2\, tt_1 - xx_1 - yy_1 - zz_1 = 0$$

is the condition that the vectors with the components x, y, z, t, and x_1, y_1, z_1, t_1 are normal to each other.

The *unit measures* for the scalars of vectors of different directions are to be so determined that the scalar 1 shall always be given to a space-vector from O to $-F=1$, and $1/c$ to a time-vector from

$$\text{O to } F=1,\ t>0.$$

If we now consider the world-line of a substance point passing through a world-point P (x, y, z, t), the scalar

$$d\tau = (1/c)\sqrt{c^2\, dt^2 - dx^2 - dy^2 - dz^2}$$

accordingly then corresponds to the differential time-vector dx, dy, dz, dt in passing along the line.

The integral $\int d\tau = \tau$ of this quantity on the world-line measured from any fixed initial point P_0 to a variable terminal point P, we call the characteristic time (*Eigenzeit*) of the substance-point at P. On the world-line we consider x, y, z, t (the components of the vector OP) as functions of the characteristic time τ and designate their first derivatives with respect to t by $\dot{x}, \dot{y}, \dot{z}, \dot{t}$, the second derivatives with respect to t by $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{t}$, and call the vectors formed from these, the derivative of the vector OP with respect to τ the *velocity--vector at* P and the derivative of this velocity-vector with respect to τ the *acceleration-vector at* P. Then the relations hold:

$$c^2 \dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2 = c^2$$
$$c^2 \dot{t}\ddot{t} - \dot{x}\ddot{x} - \dot{y}\ddot{y} - \dot{z}\ddot{z} - 0,$$

that is, the velocity-vector is the time-vector in the direction of world-line at P of unit length and the acceleration-vector at P is normal to the velocity-vector of P, therefore certainly a space-vector.

Now there exists, as is easily seen, a definite hyperbola branch which has three consecutive points in common with the world-line (*Weltlinie*) at P and whose asymptotes are generators of a past and future cone (see Figure 3 below). Let this hyperbola branch be called the *hyperbola of curvature* (*Krümmungshperbel*) *at* P. If M is the center of this hyperbola we are here concerned with an interhyperbola with its center at M. Let ρ be the length of the vector MP, *then we find the acceleration-vector at* P *to be the vector in the direction* MP *of length* c^2/ρ.

If $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{t}$, are all zero, then the hyperbola of curvature reduces to the straight line touching the world-line at P, and ρ is to be put equal to ∞.

IV.

To show that the assumption of the group G_c as holding in the laws of physics does not lead to a contradiction, it is indispensable to undertake a revision of the whole of physics on the basis of this assumption. This revision has already been successfully carried out within a certain region for questions of thermo-dynamics and radiation of heat,[4] for electromagnetic processes and finally for mechanics with retention of the concept of mass.[5]

[4] M. Planck, "Zur Dynamik bewegter Systeme," *Sitzungsberichte der k. preussischen Akademie der Wissenschaften zu Berlin,* 1907, p. 542; also *Annalen der Physik,* Vol. XXVI, 1908, p. 1.

[5] H. Minkowski, "Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern," *Nachrichten der k. Gesellschaft der Wissenschaften*

In the last-named field the first question that arises is: If a force with components X, Y, Z, along the space-axes is applied at a world-point P (x, y, z, t) where the velocity-vector is $\dot{x}, \dot{y}, \dot{z}, \dot{t}$, as what force is this to be conceived under any possible change of the system of reference? Now there exist tested lemmas about ponderomotive force in the electromagnetic field in the cases where the group G_c is certainly to be allowed. These lemmas lead to the simple rule: *On changing the system of reference the said force is to be applied in the new space coordinates, so that the vector pertaining thereto with the components.*

$$\dot{t}X,\ \dot{t}Y,\ \dot{t}Z,\ \dot{t}T$$

where $$T = 1/c^2\ (\dot{x}/\dot{t}X + \dot{y}/\dot{t}Y + \dot{z}/\dot{t}Z)$$

is the work that the force divided by c^2 performs at the world-point, all remain unchanged. This vector is always normal to the velocity-vector at P. Such a vector belonging to a force at P shall be called a *moving force-vector at* P.

Now let the world-line running through P be described by a substance-point with a constant *mechanical mass m.* Let *m* times the velocity-vector at P be called the *impulse-vector at* P, and the *m* times the acceleration-vector at P be called the *force-vector of the motion at* P. According to these definitions the law describing the motion of a mass point with a given moving force-vector reads:[6] *The force-vector of the motion is equal to the moving force-vector.*

This statement summarizes four equations for the components along the four axes, of which the fourth (because both of the described vectors were *a priori* normal to the velocity-vector) can be regarded as a consequence of the first three. According to the above meaning of T the fourth equation undoubtedly expresses the law of energy. The *kinetic energy* of point-mass is therefore to be defined as c^2 *times the component of the impulse-vector along the t-axis.* The expression for this is

$$mc^2(dt/d\tau) = mc^2/\sqrt{1-(v^2/c^2}\,,$$

which, after subtracting the additive constant mc^2 and neglecting quantities of the order $1/c^2$ is the expression of kinetic energy in

zu Göttingen (mathematisch-physikalische Klasse) 1908, p. 53, and *Mathematische Annalen,* Vol. LXVIII, 1910, p. 527; H. Minkowski, *Gesammelte Abhandlungen,* Vol. II, p. 352.

[6] H. Minkowski, *Gesammelte Abhandlungen,* Vol. II, p. 400. Compare also M. Planck, *Verhandlungen der Physikalischen Gesellschaft,* Vol. IV, 1906, p. 136.

Newtonian mechanics $\frac{1}{2}mv^2$. In this the dependence of energy on the system of reference appears obvious. But since the t-axis can now be taken in the direction of any time-vector, the law of energy, on the other hand, formulated for every possible system of reference, contains the entire system of equations of motion. This fact retains its significance in the above-mentioned limiting case for $c=\infty$, also for the deductive development of the Newtonian mechanics, and in this sense it has already been noted by J. R. Schütz.[7]

We can from the start so determine the relation of unit length to unit time, that the natural limit of velocity becomes $c=1$. If we then introduce $\sqrt{-1}.t$ s in place of t the quadratic differential expression becomes

$$d\tau^2 = -dx^2 - dy^2 - dz^2 - ds^2$$

thus completely symmetrical in x, y, z, s, and this symmetry now enters into every law which does not contradict the world-postulate. Accordingly we can express the essence of this postulate very significantly in the mystical formula:

300,000 kilometers $\sqrt{-1}$ second.

V.

Perhaps the advantages secured by the world-postulate are nowhere show more impressively than in stating the effect according to the Maxwell-Lorentz theory of a point-charge moving at will. Let us consider the world-line of such a paint-electron with the charge e and introduce the characteristic time τ from any initial point. To obtain the field determined by the electron at any world-point P_1 we construct the past cone P_1 (Fig. 4). This meets the infinite world-line of the electron at a single point P because its directions are everywhere those of a time vector. We construct the tangent at P to the world-line and through P_1 the normal P_1Q to this tangent. Let the scalar of P_1Q be r. Then, according to the definition of a past cone we must take the scalar value of PQ as r/c.

Now the vector in the direction PQ of length e/r represents in its components along the x-, y-, z-axes the vector potential multiplied by c, and in the component along the t-axis the scalar potential

[7] J. R. Schütz, "Das Prinzip der absoluten Erhaltung der Energie" in *Nachrichten der k. Gesellschaft der Wissenschaften zu Göttingen* (mathematisch-physikalische Klasse), 1897, p. 110.

of the field produced by e for the world-point P_1. This is the basis of the fundamental laws established by A. Liénard and E. Wiechert.[8]

In the description of the field itself produced by the electron it is clearly seen that the separation of the field into electric and magnetic forces is a relative one depending on the time axis of reference. Both forces can be described together most luminously after the analogy, however imperfect, of a force screw in mechanics.

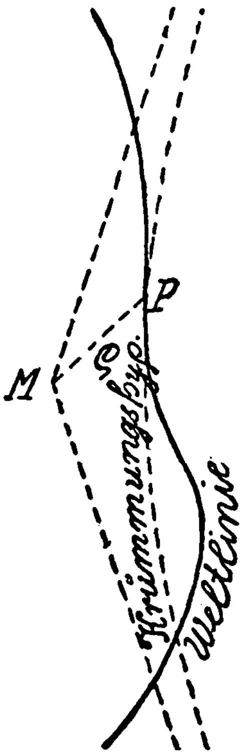

Fig. 3.

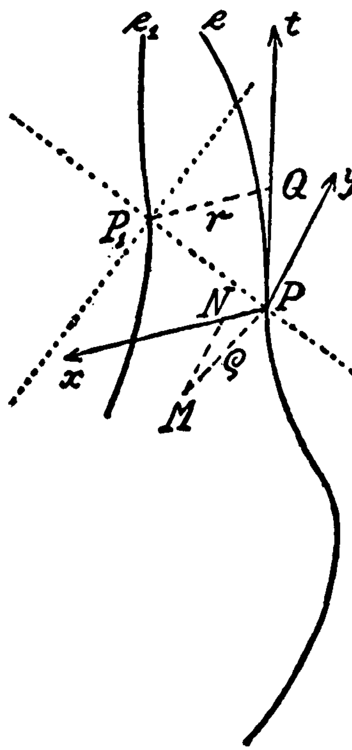

Fig. 4.

I shall now describe the *ponderomotive effect of one point-charge moving at will on another point-charge moving at will.* Let us take the world-line of the second point-electron of charge e_1, passing through the world-point P_1. Let us determine P, Q, r as before, then (Fig. 4) construct the center M of the hyperbola of

[8] A. Liénard, "Champ électrique et magnétique produit par une charge concentrée en un point et animée d'un mouvement quelconque," in *L'éclairage électrique,* Vol. XVI, 1898, pp. 5, 53, 106; E. Wiechert, "Elektrodynamische Elementargesetze" in *Archives néerlandaises des sciences exactes et naturelles* (2), Vol. V, 1900, p. 549.

curvature at P, and finally the normal MN from M to a straight line through P parallel to QP_1. Let us next determine with P as origin a system of reference with the t-axis in the direction of PQ, the x-axis in the direction of QP_1, the y-axis in the direction of MN, so that finally the direction of the z-axis is determined as normal to the t-, x-, y-axes. Let the acceleration vector at P be $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, $\ddot{t}$, and the velocity-vector at P_1 be $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$, $\dot{t}_1$. Now the action of the moving force-vector of the first electron e moving at will on the second electron e_1 moving at will at P_1 is formulated thus:

$$-ee_1(\dot{t}_1 - \dot{x}_1/c)\mathfrak{K},$$

in which the three relations between the components $\mathfrak{K}x, \mathfrak{K}y, \mathfrak{K}z, \mathfrak{K}t$, of the vector $\mathfrak{K}$ are: $c\mathfrak{K}t - \mathfrak{K}x = 1/r^2$, $\mathfrak{K}y = \ddot{y}/c^2r$, $\mathfrak{K}z = 0$ and lastly, this vector $\mathfrak{K}$ is normal to the velocity-vector at P_1 and through this circumstance alone is dependent on the latter velocity-vector.

If we compare this statement with the previous formulation[9] of the same fundamental law of the ponderomotive effect of moving point-charges on each other, we cannot but grant that the relations here coming under observation do not manifest their intrinsic character of utter simplicity except in four dimensions, but throw a very complicated projection upon a tri-dimensional space preimposed upon them.

In mechanics reformed according to the world-postulate the disagreements which have caused friction between the Newtonian mechanics and modern electrodynamics disappear of their own accord. I shall touch upon the relation of the *Newtonian law of attraction* to this postulate. I shall assume that when two point masses m and m_1 describe their world-lines a moving force-vector acts from m on m_1 just as in the above expression in the case of electrons, except that now mm_1 is to be substituted for $-ee_1$.

We shall now consider especially the particular case where the acceleration-vector of m is constantly zero, in which case we can so introduce t that m is conceived of as at rest, and the motion of m_1 depends only on the moving force-vector proceeding from m. If we modify this vector first by the factor

$$\dot{t}^{-1} = \sqrt{1 - v^2/c^2},$$

[9] K. Schwarzschild, *Nachrichten der k. Gesellschaft der Wissenschaften zu Göttingen* (mathematisch-physikalische Klasse), 1903. p. 132. H. A. Lorentz, *Enzyklopädie der mathematischen Wissenschaften*, Vol. V, Art. 14, p. 199.

which, up to quantities of the order $1/c^2$ is equal to 1, then it follows[10] that for positions x_1, y_1, z_1 of m_1 and their corresponding time-positions, Kepler's laws would again obtain, except that in place of the times t_1 the characteristic time τ_1 of m would be substituted.

On the basis of this simple observation we can see that the proposed law of attraction in conjunction with the new mechanics would be no less suitable for explaining astronomical observations than Newton's law of attraction in conjunction with the Newtonian mechanics.

The fundamental equations for electromagnetic processes in ponderable bodies are likewise in complete harmony with the world-postulate. Even the derivation of these equations, as taught by Lorentz, on the basis of conceptions of the electron theory need not for this end by any means be abandoned, as I shall show elsewhere.[11]

The universal validity of the world-postulate is, I should believe, the true core of an electromagnetic world-picture; first discovered by Lorentz, then further developed by Einstein, it is now clearly discernible. In the future development of its mathematical consequences enough indications will be found for experimental verification of the postulate to reconcile by the idea of a pre-established harmony between pure mathematics and physics even those to whom a surrender of old accustomed view-points is uncongenial or painful.

HERMANN MINKOWSKI.

SUGGESTIONS FOR A NEW LOGIC.

The world of logic is in a state of disturbance. A new logic is wanted and anxiously sought after. The logisticians are active and non-Aristotelian thinkers are presenting solutions. Among those dissatisfied with both the traditional and modern logic there is one man of particular originality and distinction. It is Dr. Charles Mercier of Charing Cross Hospital, London, and we take pleasure in presenting a review of his work.

DR. MERCIER'S LOGICAL WORK.

Dr. Charles A. Mercier is a physician whose specialty is mental

[10] H. Minkowski, *Ges. Abhandlungen*, II, p. 403.

[11] This idea is developed in the paper: "Eine Ableitung der Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern vom Standpunkte der Elektronentheorie. Aus dem Nachlass von Hermann Minkowski bearbeitet von Max Born in Göttingen. *Mathematische Annalen*, Vol. LXVIII, 1910, p. 526; *Ges. Abhandlungen*, Vol. II, p. 405.