Note on the Pellian Equation. *By* SAMUEL ROBERTS, F.R.S.

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1. It was naturally suggested by Euler and Wallis that the solution of the Pellian equation $q^2 - Dr^2 = 1$, where D is a non-quadrate number, might be often made less prolix by using both superior and inferior integer limits as the partial quotients belonging to the continued fraction representing $\sqrt{D}$. On the other hand, Lagrange (*Additions to Euler's Algebra*, Leonard's translation, Art. viii.) objects that, by indiscriminate use of both limits, we run the risk of never arriving at a form whose first coefficient is unity, as we require. This failure will happen, as he points out, whenever we use in the first transformation the inferior limit and then change to the superior limit throughout the succeeding steps of the operation. In fact, starting from a form whose leading coefficient is negative and which has a positive root, and proceeding by substitutions of the kind $\begin{vmatrix} \lambda, & -1 \\ 1, & 0 \end{vmatrix}$, where λ is the integer next $>$ the major positive root if both roots are positive, $>$ the positive root if the other is negative, all the leading coefficients of the resulting forms are negative. The form, therefore, which we set out with need not be of definite rank. Positive unity as a first coefficient is thus excluded, but negative unity may, and, in the cases we shall deal with, will occur, when it can be represented by the original form. Of course, there are many other cases in which a similar failure occurs. If, for example, we take for D a prime number, and proceed arbitrarily as to choice of integer limits, until, perchance, we arrive at a form which, by the use of one limit leads to a solution, and if then, instead of employing the successful limit, we pass to the alternative one, no form whose leading coefficient is unity will be arrived at.

Take as a type of the forms met with in such a process (a, b, c). The corresponding quadratic $aX^2 + 2bX + c = 0$ will have a major positive root >1. Let λ be the integer next greater than this root. If we transform successively the above form by the substitutions $\begin{vmatrix} \lambda, & -1 \\ 1, & 0 \end{vmatrix}$ and $\begin{vmatrix} \lambda-1, & 1 \\ & 1, & 0 \end{vmatrix}$, and equate the leading coefficients of the resulting forms, we get

$$a - 2a\lambda - 2b = 0, \text{ or } \lambda = -\frac{b}{a} + \frac{1}{2}.$$

Also, k being a positive quantity <1,

$$\lambda = \frac{-b \pm \sqrt{D}}{a} + k.$$

Therefore a is a factor of $2b$ and $2D$, and $\pm \frac{\sqrt{D}}{a}$ is $< \frac{1}{2}$. Consequently, if D is a prime, $\pm a = 2D$.

For the derivative (a', b', a), by the substitution $\begin{vmatrix} \lambda, & -1 \\ 1, & 0 \end{vmatrix}$, we have

$$b' = -a\lambda - b = -\frac{a}{2} = \mp D,$$

$$a' = \frac{b'^2 - D}{a} = \pm \frac{D-1}{2},$$

numerically >1 unless $D = 3$. But the conclusion holds also for 3, since 6 does not occur as a leading coefficient.

2. I shall now show that, if we always take the nearest integer limit, a form will be arrived at whose first coefficient is unity.*

The Pellian equation will, therefore, be solved, and the number of steps is in this case usually reduced.

We set out with the form $q^2 - Dr^2$. If the superior limit is the nearer one to $\sqrt{D}$, our first transformation is

$$(\lambda^2 - D, -\lambda, 1).$$

The substitution is $\begin{vmatrix} \lambda, & -1 \\ 1, & 0 \end{vmatrix}$, and ω, ω', the roots of the corresponding quadratic in order of magnitude, are both positive.

Since the substitution gives

$$\lambda - \sqrt{D} = \frac{1}{\omega}, \quad \lambda + \sqrt{D} = \frac{1}{\omega'},$$

ω is >2, and $\omega' < \frac{1}{2}$.

If the superior limit is the nearer one, the first transformation is

$$[-(D-\lambda^2), \lambda, 1].$$

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* I am sorry that I overlooked, until the time for sending this paper to the press was at hand, two memoirs, which anticipate me in showing that the Pellian equation is solved by using nearest limits exclusively or superior limits exclusively. The papers I refer to are—

I. "Ueber die Eigenschaften der periodischen negativen Kettenbruche, welche die Quadratwurzel aus einen ganzen positiven Zahl darstellen," von M. A. Stern. (*Abhandlungen der Kön. Gesells. d. Wiss. zu Göttingen*, Band 12, 1866, pp. 3—48.)

II. "Ueber einer neuen Methode die Pellische Gleichung aufzulösen," von B. Minnigerode. (*Nachr. der Kön. Gesells. d. Wiss. zu Göttingen*, 1873, pp. 619—642.)

It became a question whether the present paper should be withdrawn or retained. As the methods employed are somewhat different and some further results are given, it has been thought that the publication of the paper may still be acceptable to English readers. In an addition I give a brief account of the German memoirs, previous knowledge of which would, of course, have materially modified the form and arrangement of my paper.

The substitution is $\begin{vmatrix} \lambda, & 1 \\ 1, & 0 \end{vmatrix}$, and, if ω, ω' are the roots in order of magnitude of the corresponding quadratic, we have

$$\sqrt{D}-\lambda = \frac{1}{\omega} < \frac{1}{2}, \quad -\sqrt{D}-\lambda = \frac{1}{\omega'},$$

so that ω is >2, and ω' is negative and $<\frac{1}{2}$ numerically.

If, in like manner, any form which we reach, still following the major positive root, is (a, b, c), whose corresponding root ω is >2 and positive, either substitution $\begin{vmatrix} \lambda, & -1 \\ 1, & 0 \end{vmatrix}$ or $\begin{vmatrix} \mu, & 1 \\ 1, & 0 \end{vmatrix}$ gives a left associate form (a', b', a) which will have a corresponding root >2 and positive. For, if ω, ω' are the roots corresponding to the form (a, b, c), and Ω, Ω' those corresponding to the derivative form, we have either

$$\lambda-\omega = \frac{1}{\Omega} < \frac{1}{2} \text{ or } \omega-\mu = \frac{1}{\Omega} < \frac{1}{2}.$$

It appears then that, except in the first transformations for $D=2$ and $D=3$, viz., $(q+r)^2-2q^2$ and $(2q-r)^2-3q^2$, we shall never have occasion to make the substitutions $\begin{vmatrix} 2, & -1 \\ 1, & 0 \end{vmatrix}$ or $\begin{vmatrix} 1, & 1 \\ 1, & 0 \end{vmatrix}$, or, of course, $\begin{vmatrix} 1, & -1 \\ 1, & 0 \end{vmatrix}$.

That is to say, for inferior limits λ is >1, and for superior limits λ is >2.

Consequently, in the resulting continued fraction representing $\sqrt{D}$, no denominator will be <2 and no denominator immediately followed by a negative unit numerator will be <3. It will be understood that I attach signs to the numerators only.

3. We can now have recourse to the process for converting a continued fraction with mixed negative and positive numerators into one with positive numerators exclusively.

This depends on the identity

$$-\frac{b}{a+R} = -1 + \frac{1}{1+}\,\frac{b}{a-b+R},$$

where R may denote the residue of the continued fraction after the term $-\frac{b}{a}$. Therefore

$$a-\frac{1}{a_1-}\,\frac{1}{a_2+R} = a-1+\frac{1}{1+}\,\frac{1}{a_1-2+}\,\frac{1}{1+}\,\frac{1}{a_2-1+R},$$

$$a-\frac{1}{a_1+}\,\frac{1}{a_2-}\,\frac{1}{a_3+R} = a-1+\frac{1}{1+}\,\frac{1}{a_1-1+}\,\frac{1}{a_2-1+}\,\frac{1}{1+}\,\frac{1}{a_3-1+R},$$

and so forth.

In this way we can provide for the case in which some numerators are positive and some negative. No partial quotient will become zero in the transformation unless an original partial quotient is unity or 2, immediately followed in the last case by a negative numerator.

4. Now, in the proposed process, we have seen that no partial quotient is unity or 2, immediately followed in the last case by a negative numerator. Moreover, it is accepted that a continued fraction can only be developed in one form with positive unit numerators and denominators >0 and integer, *i.e.* in the normal form.*

This being so, the continued fraction, which we obtain by taking the nearer integer limits, is converted into the normal form (1) by the introduction of fractional elements $1/1$, (2) by diminishing certain denominators by 1 or 2.

Hence we shall always have a partial quotient 2λ or $2\lambda+1$ or $2\lambda+2$, where λ is the greatest integer in $\sqrt{D}$; because we know that the partial quotient 2λ occurs in the normal development.

The denominator in such case is derived from a form having positive or negative unity for its first coefficient. For limits involving this can be established to the values of the middle coefficients of the forms. Thus, if a form has a negative root, the middle coefficient is numerically $<\sqrt{D}$, and $\omega < \dfrac{2\sqrt{D}}{a}$ numerically. If both roots are positive, one will be >2 and the other $<\frac{1}{2}$, and we may, without detriment to the argument, suppose the first coefficient to be positive and the form $(a, -b, c)$. To produce a denominator $=$ or $>2\lambda$, we must at least have $\omega > 2\lambda-\frac{1}{2}$, and because $\omega-\omega' = \dfrac{2\sqrt{D}}{a}$, $2\lambda-1$ is $< \dfrac{2\sqrt{D}}{a}$. If a be greater than unity, we must have $\lambda<2$. The cases in which $\lambda=1$, viz., $D=2$, $D=3$, may be separately verified.†

5. There may be, and in reality are, numbers for which the nearer limits are always inferior. There are also numbers for which the nearer limits are always superior. The former class is signalised by the absence of unit denominators in the normal development.

The economy of terms is not however measured directly by counting the unit partial quotients in the normal form. If there are $2n$ such units in consecutive order, we only remove n terms; if $2n+1$ such units, we remove $n+1$ terms.

* There is a generally trivial exception in the case of finite continued fractions. If the last fractional element is $1/m$, we may write for this $1/m-1+1/1$. This modification takes effect in the following theorem. If P, Q, R are positive integers in order of magnitude, and $Q^2-PR=\pm 1$, P/Q can be expressed by a normal symmetric continued fraction; *e.g.*, $7^2-8.6=1$ and $8/7$ is expressed by $1+1/7$, but also by $1+1/6+1/1$, which justifies the theorem in this case.

† It follows that the development under consideration affords the same series of solutions as the normal development and in the same order.

Unit denominators may have been derived from 2—1, or 3—2 occurring in the transformation to the normal form.

The prime 1153 affords a fair instance of abbreviation. In the usual period there are 43 terms and 26 units. Taking the line of nearest approximation, there are 24 terms in the period. The first period, of course, gives the least solution of $q^2 - 1153r^2 = -1$. The prime 97 is also a good instance.

In this process what we really effect is the removal of the elements 1/1 from the normal form. This we could do directly, if the normal form were given, by the identity

$$\frac{1}{1} + \frac{1}{a+R} = 1 - \frac{1}{a+1+R},$$

and it is a method given by Lagrange himself for abridging continued fractions in certain cases.*

6. I shall further show that, if we proceed by using superior limits only, the notation of the Pellian equation will be duly obtained by means of a leading coefficient equal to unity.

Our first transformation will now be

$$(\lambda^2 - D,\ -\lambda,\ 1),$$

and, if ω, ω' are the corresponding quadratic roots in order of magnitude, ω is > and $\omega' < 1$, both being positive. The next transformation gives a quadratic whose roots are determined by

$$\mu - \omega = \frac{1}{\Omega},\quad \mu - \omega' = \frac{1}{\Omega'},$$

$\Omega > 1$ and $\Omega' < 1$, both being positive, since μ cannot be < 2.

In like manner, every succeeding form will have two positive roots, one > 1 and the other < 1. Each form has only one left associate, and only one right associate by the law of formation. For, if we derive a form by $\quad \mu - \pi = \dfrac{1}{\Pi},\quad \mu - \pi' = \dfrac{1}{\Pi'},$

π, π' being the roots in order of magnitude corresponding to the given form, and Π, Π' in order of magnitude being those of the derived form, we go back from the derived form without ambiguity by

$$\mu - \frac{1}{\Pi} = \pi,\quad \mu - \frac{1}{\Pi'} = \pi',$$

and the substitution $\begin{vmatrix} 0, & 1 \\ 1, & \mu \end{vmatrix}$.

* *Traité de la Résolution des équations numériques*, Cap. vi., Art. 111, where it is applied to the ratio of the circumference of a circle to its diameter; or else *Additions to Euler's Algebra*, Chap. I.

If, then, we can show that the forms are finite in number, a period will be established. The right associate of the form $(\lambda^2-D, -\lambda, 1)$ is $(1, -\lambda, \lambda^2-D)$, so that the solution of the Pellian equation will follow.

Now, when μ is >2, the corresponding forms exist in finite number only, because then we have

$$\Omega-\Omega' = \frac{2\sqrt{D}}{a} > 1, \text{ or } a \text{ is } < 2\sqrt{D} \text{ and } b \text{ is } < a+\sqrt{D} < 3\sqrt{D}.$$

If however $\mu = 2$, we are unable to fix limits in this way, since $\frac{2\sqrt{D}}{a}$ may be <1.

But the substitutions $\begin{vmatrix} 2, & -1 \\ 1, & 0 \end{vmatrix}$ cannot follow in infinite succession,

since $$1-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\text{\&c. } ad\ inf.$$

is a development representing zero; it follows that there must be an infinite number of interruptions of the series by partial quotients exceeding 2. Since there is a limited number of forms giving such partial quotients, the series is periodic, and, each form having only one left and one right associate by the law of formation, the period includes the form $q^2-2\lambda q^2+(\lambda^2-D)\,r^2$.

The use of superior limits alone, however, may prolong the operation, and moreover does not directly furnish the solution of $q^2-Dr^2=-1$ when possible, since every leading coefficient is positive.

In one passage of his Algebra, Wallis expresses himself unguardedly, and lays himself open to the criticism of Lagrange. Yet his more formal statement appears to be correct. Referring to expedients for abridging the process, Dr. Wallis says:—"First, whereas in the former process, when a quantity fell between two limits, we always made use of the *lesser* limit (making addition to it), we might, if we had pleased, have always made use of the *greater* (making defalcation from it). But the most expedient way for shortening the process is to make use sometime of the one and sometime of the other, according as this or that comes nearest the truth."

It is nevertheless pretty clear that Wallis possessed no satisfactory proof to support his statement. I am inclined to think also that the advantage of any abridgment gained by these methods is slighter than was supposed, since we lose symmetry, or at least the symmetry is less familiar.

7. In the concrete example furnished by Lagrange, where q^2-6r^2 is the form, and the first transformation is by inferior limit, the rest

by superior limit, and in other instances of the same kind, the transformation to positive unit numerators introduces zero denominators. If we eliminate these by summation, we get the normal development.

In these cases, the term of the normal form $1/2\mu$, where μ is the greatest integer in $\sqrt{D}$, is disguised by decomposition. Its place is, however, indicated by a series of denominators 2 followed immediately by negative numerators. Thus $\sqrt{D}$ is represented by

$$2+\frac{1}{3}-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\frac{1}{4}-\frac{1}{2}-\&c.$$

Diminishing 3 by unity, $2+\frac{1}{2}$ gives a solution of $q^2-6r^2=1$. In like manner, though inconveniently, we could obtain a solution in other such cases. The form which commences the series representing $1/2\mu$ in a decomposed shape is characterised by the algebraical sum of the coefficients being equal to unity, and $q=1$, $r=1$ gives a representation of unity accordingly; so that a solution is furnished by this indication. Terms $-1/1$ cannot exist in successive series, being always immediately followed by a positive numerator, since the substitution $\begin{vmatrix} 1, & -1 \\ 1, & 0 \end{vmatrix}$ does not occur.

A process by integer limits, which does not furnish a series of terms $-1/2$ capable of accounting for $1/2\mu$, must include that term or $1/(2\mu+1)$ or $1/(2\mu+2)$. A variety of developments leading to a solution of the Pellian equation may therefore be determined. We can proceed, for instance, as far as we like by inferior limits, and then change to nearest limits, or, commencing arbitrarily as to choice of limits, we may change to inferior or nearest limits; or we may proceed alternately with inferior and superior limits; or, after reaching a form with a positive leading coefficient, we may proceed with superior limits. If, in fact, we go astray up to any point, we can regain the correct path, though possibly we may miss the least solution.

On the whole, then, it seems that the process by integer limits is theoretically arbitrary as to the choice of those limits. For in those cases in which a solution of the Pellian equation is not afforded by a form whose leading coefficient is unity, forms occur in which the algebraical sum of the coefficients is positive unity. And the conclusion is similar with respect to negative unity, when it can be represented.

8. If a is $<\sqrt{D}$, and the equation $q^2-Dr^2=\pm a$ is resolvable, q, r being relatively prime, $\pm a$ will be found as a leading coefficient in the forms belonging to the normal development of $\sqrt{D}$, or, as otherwise stated, a is the denominator of a complete quotient $\frac{\sqrt{D}+b}{a}$. We have

seen that, in the development of $\sqrt{D}$ by superior limits, the leading coefficients will be all positive. If, however, $q^2-Dr^2=+a$ is resolvable, and a is $<2\sqrt{D}$, a will be a leading coefficient of a derived form.

We can develop a fraction $\frac{p}{q}$, wher· p and q are integers, in the form

$$a-\frac{1}{b}-\frac{1}{c}-\frac{1}{d}-\ldots-\frac{1}{k},$$

a form which will be finite if we take the complete quotient when the denominator is unity. If we still continued to take the superior limit, the remainder of the continued fraction would have to be written *ad infinitum* in the form $-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}\ldots$ which is $=-1$

Taking then the finite form, suppose that $\frac{p_0}{q_0}$ is the reduced fraction next preceding $\frac{p}{q}$.

If now $\frac{p}{q}-\sqrt{D}=\frac{1}{q(q\mu-q_0)}$, where μ is >1, we have, since $pq_0-p_0q=-1$ in this case,

$$\mu=\frac{1+pq_0-\sqrt{D}qq_0}{q(p-q\sqrt{D})}=\frac{p_0-\sqrt{D}q_0}{p-\sqrt{D}q}$$
$$=\frac{pp_0-qq_0D+\sqrt{D}}{p^2-q^2D};$$

but this is the form of the complete quotient corresponding to $\frac{p}{q}$, $\frac{p_0}{q_0}$, on the supposition that the development in question is that of $\sqrt{D}$ by superior limits. Hence the only condition to be fulfilled is

$$p-q\sqrt{D}<\frac{1}{q-q_0},$$

q_0 being always $<q$.

We have also, suppose,

$$p-q\sqrt{D}=\frac{a}{p+q\sqrt{D}},$$

and the condition is

$$p+q\sqrt{D}-aq+aq_0>0.$$

But p is $>q\sqrt{D}$ and $2q\sqrt{D}-aq+aq_0>0$ if a is $<2\sqrt{D}$.

To provide similarly for the case of $q^2-Dr^2=-a$, where a is $<2\sqrt{D}$. We commence the development of $\sqrt{D}$ by taking the inferior limit and then proceed with superior limits. The leading coefficients of the forms will then be all negative. Following the same reasoning as before, we get for the condition that $\frac{p}{q}$, and $\frac{p_0}{q_0}$, the next preceding reduced fraction of the corresponding development of $\frac{p}{q}$, may belong to the development of $\sqrt{D}$,

$$q\sqrt{D}-p < \frac{1}{q-q_0},$$

and also we have $$q\sqrt{D}-p = \frac{a}{p+q\sqrt{D}}.$$

The condition to be fulfilled is again

$$p+q\sqrt{D}-aq+aq_0 > 0,$$

or $$2q\sqrt{D}-\frac{a}{p+q\sqrt{D}}-aq+aq_0 > 0,$$

or $$2q\sqrt{D}-a\left(q-q_0+\frac{1}{p+q\sqrt{D}}\right) > 0,$$

which is so if a is $< 2\sqrt{D}$ and $q_0 > 0$.

By the two developments, then, we solve, when possible, $q^2-Dr^2 = \pm a$, $a < 2\sqrt{D}$. The limit to which the normal development is subject is therefore doubled.

9. We will consider more particularly the case in which the first transformation is by inferior limit and the succeeding ones are by superior limits.

Then q^2-Dr^2 becomes in the first instance

$$-(D-\lambda^2)q^2+2\lambda qr+r^2,$$

where λ is the greatest integer in $\sqrt{D}$.

If κ be the integer next $> \frac{1}{\sqrt{D}-\lambda}$, we get, by the substitution

$$\begin{vmatrix} \kappa, & -1 \\ 1, & 0 \end{vmatrix};$$

$$-(D\kappa^2-\overline{\kappa\lambda+1}^2)q^2+2[(D-\lambda^2)\kappa-\lambda]qr-(D-\lambda^2)r^2 \ldots\ldots (\alpha).$$

Now put m for the value of this when $q=1$, $r=1$, so that

$$-(D-\lambda^2)(\kappa-1)^2+2\kappa\lambda+1-2\lambda = m,$$

m being positive. Then

$$-(D\kappa^2-\overline{\kappa\lambda+1}^2)m = [(D-\lambda^2)\kappa(\kappa-1)-2\kappa\lambda-1+\lambda]^2-D = \Pi^2-D,$$

$$-(D-\lambda^2)m = [(D-\lambda^2)(\kappa-1)-\lambda]^2-D = (\Pi+m)^2-D,$$

$$[(D-\lambda^2)\kappa-\lambda]m = D-\Pi(\Pi+m),$$

or the derived form is

$$-\frac{D-\Pi^2}{m}q^2+2\frac{D-\Pi(\Pi+m)}{m}qr-\frac{D-(\Pi+m)^2}{m}r^2.$$

If $\frac{m}{\sqrt{D}+\Pi}$ is < 1, we transform again by the substitution $\begin{vmatrix} 2, & -1 \\ 1, & 0 \end{vmatrix}$, obtaining $-\frac{D-(\Pi-m)^2}{m}q^2+2\frac{D-(\Pi-m)\Pi}{m}qr-\frac{D-\Pi^2}{m}r^2,$

and we may proceed in this way, as long as the leading coefficient remains negative, so that $\rho m-\Pi$ is $< \sqrt{D}$.

When the next leading coefficient would be positive under this process, let us write $\frac{D-\nu^2}{m}$ for $\frac{D-(\Pi-\rho m)^2}{m}$, and the form is, if

$$\nu = \rho m - \Pi,$$

$$-\frac{D-\nu^2}{m}q^2+2\frac{D-\nu(\nu-m)}{m}qr-\frac{D-(\nu-m)^2}{m}r^2;$$

and, if k is the integer next $> \frac{m}{\sqrt{D}-\nu}$, the derived form will be

$$-\left(\frac{Dk^2-(k\nu+m)^2}{m}\right)q^2+\frac{2[(D-\nu^2)k-m\nu]}{m}qr-\frac{D-\nu^2}{m}r^2 \;\ldots\ldots\; (a'),$$

which, as in the preceding case, becomes

$$-\frac{D-\Pi_1^2}{m_1}q^2+\frac{2[D-\Pi_1(\Pi_1+m_1)]}{m_1}qr-\frac{D-(\Pi_1+m)^2}{m_1}r^2,$$

m_1 being positive; and we see that (a') is the same in form as (a), when $m = 1$.

The structure of the series of forms is now clear, and, since

$$Q^2-DR^2=-D$$

is always possible, in integers, there will be a leading coefficient, $-D$, in the period, and all the others will be negative and $< D$ numerically, the last leading coefficient being $-(D-\lambda^2)$, where λ is the greatest integer in $\sqrt{D}$.

Moreover, the general form of the denominators of which m_1 is the type is $$-\frac{D-\nu^2}{m}(k-1)^2-2\nu+2\nu k+m.$$

If we substitute for $k-1$ its value $\frac{m}{\sqrt{D}-\nu}-\sigma$, where σ is positive and < 1, we get $$2\sigma\sqrt{D}-\sigma^2\frac{D-\nu^2}{m}=m_1 < 2\sqrt{D}.$$

If for $\frac{D-\nu^2}{m}$ we write a, ν being as above a maximum [*i.e.*, $D-(\nu+m)^2$ is negative], then $D^2=\nu^2+am$, where not only are 2ν, a and $m < 2\sqrt{D}$ respectively, but $\nu+m$ is by supposition $> \sqrt{D}$ and also $\nu+a > \sqrt{D}$.

For, $a=\frac{D-\nu^2}{m}=(\sqrt{D}-\nu)\left(\frac{\sqrt{D}+\nu}{m}\right)$ and $\frac{\sqrt{D}+}{m}$ is > 1, since the corresponding minor root is $1-\frac{m}{\sqrt{D}+\nu}$ and is positive and < 1.

These limiting coefficients therefore furnish values of a, which belong to a system $-a, \nu, m$ which are the numerical values of the coefficients of a reduced form. Such values of a would, therefore, be furnished by the normal development. But there may be cases in which $\frac{D-\nu^2}{m}$ is $< 2\sqrt{D}$, but ν is not a maximum relative to m, and these cannot be furnished by the normal development. The forms which arise when we use superior limits only, may be similarly investigated. The case is of course closely analogous to the one we have been discussing. The development when nearer limits are employed is also perhaps worth more detailed treatment, and it is not unlikely that analogous deviation from the normal procedure might be useful in the general theory of binary quadratic forms.

I append some numerical examples, reference to which may facilitate the reading of the previous articles.

By superior limits, we have, for q^2-13r^2,

$$\begin{aligned}
(4q-r)^2-13q^2 &= 3q^2-\ 8qr+r^2,\\
3\,(3q-r)^2-\ 8q\,(3q-r)+q^2 &= 4q^2-10qr+3r^2,\\
4\,(3q-r)^2-10q\,(3q-r)+3q^2 &= 9q^2-14qr+4r^2,\\
9\,(2q-r)^2-14q\,(2q-r)+4q^2 &= 12q^2-22qr+9r^2,\\
12\,(2q-r)^2-22q\,(2q-r)+9q^2 &= 13q^2-26qr+12r^2,\\
13\,(2q-r)^2-26q\,(2q-r)+12q^2 &= 12q^2-26qr+13r^2,\\
12\,(2q-r)^2-26q\,(2q-r)+13q^2 &= 9q^2-22qr+12r^2,\\
9\,(2q-r)^2-22q\,(2q-r)+12q^2 &= 4q^2-14qr+9r^2,\\
4\,(3q-r)^2-14q\,(3q-r)+9q^2 &= 3q^2-10qr+4r^2,\\
3\,(3q-r)^2-10q\,(3q-r)+4q^2 &= q^2-\ 8qr+3r^2,\\
\&c. \qquad & \qquad \&c.
\end{aligned}$$

All the leading coefficients are positive,* but, observing that the sum of the coefficients of $9q^2-14qr+4r^2$ is negative unity, we have

$$4-\cfrac{1}{3-\cfrac{1}{2}} = 18/5,$$

giving the least solution of $q^2-13r^2=-1$.

The theorem of Professor Smith (*Proceedings*, Vol. vii., p. 197) will apply to these processes as well as to the normal form. Thus, proceeding in the case of q^2-13r^2 with the nearest limits, the product

* In general, for superior limits, $\frac{Q}{R}-\sqrt{N}$ is positive, if $\frac{Q}{R}$ is a reduced fraction.

of the three complete quotients $\frac{4+\sqrt{13}}{3}$, $\frac{5+\sqrt{13}}{4}$, $\frac{3+\sqrt{13}}{1}$ is equal to $18+5\sqrt{13}$, giving the above solution of $q^2-13r^2=-1$.

If we take q^2-44r^2, the superior limits are also the nearer ones, and we have the forms

$$q^2-44r^2,$$
$$(5,\ -7,\ 1)^*,$$
$$(4,\ -8,\ 5),$$
$$(5,\ -8,\ 4),$$
$$(1,\ -7,\ 5)^*.$$

The product $\frac{7+\sqrt{44}}{5}\cdot\frac{8+\sqrt{44}}{4}\cdot\frac{8+\sqrt{44}}{5}\cdot\frac{7+\sqrt{44}}{1}$

gives $199+30\sqrt{44}$ and $q=199$, $r=30$ constitute the lowest solution of $q^2-44r^2=1$.

In the following numerical example, I write down the substitution opposite the form on which it operates, so that each form is derived from the preceding one.

Taking the form q^2-151r^2, we have for the normal development

	$(1,\ 0,\ -151)$ ...... $12q+r,\ q$			$(-15,\ 11,\ 2)$ ...... $q+r,\ q$
*	$(-7,\ 12,\ 1)$ ...... * $3q+r,\ q$			$(9,\ -4,\ 15)$ ...... $q+r,\ q$
	$(10,\ -9,\ -7)$ ...... $2q+r,\ q$			$(-14,\ 6,\ 9)$ ...... $q+r,\ q$
	$(-3,\ 11,\ 10)$ ...... $7q+r,\ q$			$(5,\ -9,\ -14)$ ...... $4q+r,\ q$
	$(17,\ -10,\ -3)$ ...... $q+r,\ q$			$(-6,\ 11,\ 5)$ ...... $3q+r,\ q$
	$(-6,\ 7,\ 17)$ ...... $3q+r,\ q$			$(17,\ -7,\ -6)$ ...... $q+r,\ q$
	$(5,\ -11,\ -6)$ ...... $4q+r,\ q$			$(-3,\ 10,\ 17)$ ...... $7q+r,\ q$
	$(-14,\ 9,\ 5)$ ...... $q+r,\ q$			$(10,\ -11,\ -3)$ ...... $2q+r,\ q$
	$(9,\ -5,\ -14)$ ...... $q+r,\ q$			$(-7,\ 9,\ 10)$ ...... $3q+r,\ q$
	$(-15,\ 4,\ 9)$ ...... $q+r,\ q$		*	$(1,\ -12,\ -7)$ ...... *$24q+r,\ q$
	$(2,\ -11,\ -15)$ ...... $11q+r,\ q$			

The asterisks indicate the periods.

Employing nearest limits for the same form, we have

	$(1,\ 0,\ -151)$ ...... $12q+r,\ q$			$(9,\ -13,\ 2)$ ...... $3q-r,\ q$
*	$(-7,\ 12,\ 1)$ ...... * $3q+r,\ q$			$(5,\ -14,\ 9)$ ...... $5q+r,\ q$
	$(10,\ -9,\ -7)$ ...... $2q+r,\ q$			$(-6,\ 11,\ 5)$ ...... $4q-r,\ q$
	$(-3,\ 11,\ 10)$ ...... $8q-r,\ q$			$(-3,\ 13,\ -6)$ ...... $8q+r,\ q$
	$(-6,\ 13,\ -3)$ ...... $4q+r,\ q$			$(10,\ -11,\ -3)$ ...... $2q+r,\ q$
	$(5,\ -11,\ -6)$ ...... $5q-r,\ q$			$(-7,\ 9,\ 10)$ ...... $3q+r,\ q$
	$(9,\ -14,\ 5)$ ...... $3q-r,\ q$		*	$(1,\ -12,\ -7)$ ...... *$24q+r,\ q$
	$(2,\ -13,\ 9)$ ...... $13q-r,\ q$			

Using superior limits, we get

(1, 0, −151)	$13q-r,\ q$	(9, −13, 2)	$3q-r,\ q$
*(18, −13, 1)	* $2q-r,\ q$	(5, −14, 9)	$6q-r,\ q$
(21, −23, 18)	$2q-r,\ q$	(21, −16, 5)	$2q-r,\ q$
(10, −19, 21)	$4q-r,\ q$	(25, −26, 21)	$2q-r,\ q$
(29, −21, 10)	$2q-r,\ q$	(17, −24, 25)	$3q-r,\ q$
(42, −37, 29)	$2q-r,\ q$	(34, −27, 17)	$2q-r,\ q$
(49, −47, 42)	$2q-r,\ q$	(45, −41, 34)	$2q-r,\ q$
(50, −51, 49)	$2q-r,\ q$	(50, −49, 45)	$2q-r,\ q$
(45, −49, 50)	$2q-r,\ q$	(49, 51, 50)	$2q-r,\ q$
(34, −41, 45)	$2q-r,\ q$	(42, 47, 49)	$2q-r,\ q$
(17, −27, 34)	$3q-r,\ q$	(29, −37, 42)	$2q-r,\ q$
(25, −24, 17)	$2q-r,\ q$	(10, −21, 29)	$4q-r,\ q$
(21, −26, 25)	$2q-r,\ q$	(21, −19, 10)	$2q-r,\ q$
(5, −16, 21)	$6q-r,\ q$	(18, −23, 21)	$2q-r,\ q$
(9, −14, 5)	$3q-r,\ q$	*(1, −13, 18)	*$26q-r,\ q$
(2, −13, 9)	$13q-r,\ q$		

From the above we gather that the positive numbers $<2\sqrt{151}$ represented by q^2-151r^2 are

1, 2, 5, 9, 10, 17, 18, 21.

Transforming first of all by inferior limits, and afterwards by superior limits, we get

	For q.	For r.		For q.	For r.
(1, 0, −151)	$12q+r$,	q	(−15, 26, −35)	$3q-r$,	q
(−7, 12, 1)	$4q-r$,	q	(−14, 19, −15)	$3q-r$,	q
*(−15, 16, −7)	* $2q-r$,	q	(−27, 23, −14)	$2q-r$,	q
(−3, 14, −15)	$9q-r$,	q	(−30, 31, −27)	$2q-r$,	q
(−6, 13, −3)	$5q-r$,	q	(−23, 29, −30)	$2q-r$,	q
(−23, 17, −6)	$2q-r$,	q	(−6, 17, −23)	$5q-r$,	q
(−30, 29, −23)	$2q-r$,	q	(−3, 13, −6)	$9q-r$,	q
(−27, 31, −30)	$2q-r$,	q	(−15, 14, −3)	$2q-r$,	q
(−14, 23, −27)	$3q-r$,	q	(−7, 16, −15)	$5q-r$,	q
(−15, 19, −14)	$2q-r$,	q	(−30, 19, −7)	$2q-r$,	q
(−35, 26, −15)	$2q-r$,	q	(−51, 41, −30)	$2q-r$,	q
(−51, 44, −35)	$2q-r$,	q	(−70, 61, −51)	$2q-r$,	q
(−63, 58, −51)	$2q-r$,	q	(−87, 79, −70)	$2q-r$,	q
(−71, 68, −63)	$2q-r$,	q	(−102, 95, −87)	$2q-r$,	q
(−75, 74, −71)	$2q-r$,	q	(−115, 109, −102)	$2q-r$,	q
(−75, 76, −75)	$2q-r$,	q	(−126, 121, −115)	$2q-r$,	q
(−71, 74, −75)	$2q-r$,	q	(−135, 131, −126)	$2q-r$,	q
(−63, 68, −71)	$2q-r$,	q	(−142, 139, −135)	$2q-r$,	q
(−51, 58, −63)	$2q-r$,	q	(−147, 145, −142)	$2q-r$,	q
(−35, 44, −51)	$2q-r$,	q	(−150, 149, −147)	$2q-r$,	q

	For q.	For r.		For q.	For r.
$(-151, 151, -150)$	$2q-r$,	q	$(-102, 109, -115)$	$2q-r$,	q
$(-150, 151, -151)$	$2q-r$,	q	$(-87, 95, -102)$	$2q-r$,	q
$(-147, 149, -150)$	$2q-r$,	q	$(-70, 79, -87)$	$2q-r$,	q
$(-142, 145, -147)$	$2q-r$,	q	$(-51, 61, -70)$	$2q-r$,	q
$(-135, 139, -142)$	$2q-r$,	q	$(-30, 41, -51)$	$2q-r$,	q
$(-126, 131, -135)$	$2q-r$,	q	*$(-7, 19, -30)$	* $5q-r$,	q
$(-115, 121, -126)$	$2q-r$,	q			

From which it appears that the numbers $< 2\sqrt{151}$ represented by q^2-151r^2 negatively are

$$3, \quad 6, \quad 7, \quad 14, \quad 15, \quad 23.$$

I have written out the example fully, and in several forms for the purpose of comparison.

As an example involving an arbitrary number, we may take

$$D = \overline{4n+1}^2 + \overline{3n+1}^2 = 25n^2+14n+2.$$

The normal development of $\sqrt{D}$ is

$$5n+1+\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{10n+2+}\ldots.$$

Proceeding by superior limits, we have

$$
\begin{array}{ll}
[1,\ 0,\ -(25n^2+14n+2)], & (5n+2)\,q-r,\ \ q, \\
[6n+2,\ -(5n+2),\ 1]^*, & {}^*2q-r,\ \ q, \\
[4n+1,\ -(7n+2),\ 6n+2], & 4q-r,\ \ q, \\
[14n+2,\ -(9n+2),\ 4n+1], & 2q-r,\ \ q, \\
\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots & \ldots\ \ \ldots \\
\big[14n+10kn-(k^2-2), & \\
-14n+5\,(2k-1)\,n-2k^2+2k+4, & 2q-r,\ \ q, \\
\quad 14n+10\,(k-1)\,n-\overline{k-1}^2+2\big], &
\end{array}
$$

until $k = 10n+1$ gives

$$
\begin{array}{ll}
[4n+1,\ -(9n+2),\ 14n+2], & 4q-r,\ \ q, \\
[6n+2,\ -(7n+2),\ 4n+1], & 2q-r,\ \ q, \\
[\ 1,\ -(5n+2),\ 6n+2]^*, & {}^*(10n+4)\,q-r,\ \ q.
\end{array}
$$

Taking the inferior limit at the first step, and afterwards superior limits, we get

$$
\begin{array}{ll}
[1,\ 0,\ -(25n^2+14n+2)], & (5n+1)\,q+r,\ \ q, \\
[-(4n+1),\ 5n+1,\ 1], & 3q-r,\ \ q, \\
[-(6n+2),\ 7n+2,\ -(4n+1)], & 2q-r,\ \ q, \\
[\ -1,\ 5n+2,\ -(6n+2)], & (10n+4)\,q-r,\ \ q, \\
\&c., & \&c.
\end{array}
$$

The sequel is indicated by the result for superior limits exclusively.

Addition I.

1. Dr. Stern, in his memoir before mentioned, compares the continued fraction $a+\frac{1}{a_1+}\frac{1}{a_2+}\dots+\frac{1}{a_m}$

with one of equal value of the form

$$b-\frac{1}{b_1}-\frac{1}{b_2}-\dots-\frac{1}{b_n}.$$

The first form is called positive and denoted by $(a, a_1, a_2, \dots a_m)$, the second is called negative and denoted by $[b, b_1, b_2, \dots b_n]$.

By means of the identity

$$a+\frac{1}{a_1+}\frac{1}{R}=a+1-\frac{1}{2}-\frac{1}{1}+\frac{1}{a_1-2}+\frac{1}{R},$$

[It seems simpler to take the identity

$$a+\frac{1}{a_1+}\frac{1}{R}=a+1-\frac{1}{1}+\frac{1}{a_1-1}+\frac{1}{R}\Big]$$

the author obtains the following rule:—Let $\overline{a_k-1}$ denote a succession of a_k-1 denominators each $=2$, then we have generally

$$(a, a_1 \dots a_{2k}) \;= [a+1, \overline{a_1-1}, a_2+2, \overline{a_3-1} \dots a_{2k}+1],$$

$$(a, a_1 \dots a_{2k+1}) = [a+1, \overline{a_1-1}, a_2+2 \dots \overline{a_{2k+1}-1}].$$

When k is infinite, the forms coalesce.

The periodicity of the positive fraction involves that of the negative fraction, and the author compares in detail the denominators and numerators of the complete quotients belonging to a positive and a negative fraction, representing in each case the square root of the same non-quadrate integer A.

The method therefore is the same in principle as that employed in §§ 3, 4, in reference to the continued fraction obtained by the nearest limits.

The negative period is evidently analogous to that of the continued fraction obtained by first of all using the inferior limit, and afterwards the superior limits—a case briefly treated in § 9. Thus no leading coefficient of a form is $>A$, and no middle coefficient $>2A$; and, if a leading coefficient is $=A$, then $x^2-Ay^2=-1$ is resoluble, as, of course, is obvious.

The author notes that, if $x^2-Ay^2=d_n$, when d_n is a positive integer $<\sqrt{A}$, then d_n will appear as the denominator of a complete quotient of the negative continued fraction. I do not find it remarked, that the same is true as long as d_n is $< 2\sqrt{A}$.*

The cases in which the period has one mid-term or two mid-terms are discussed, and it is shown that, when there are two mid-terms, A is the difference of two squares determined.

Analogues to Goepel's properties of the normal fraction and other interesting results are given, and the memoir concludes with a table for non-quadrate numbers up to 100, for the negative development of $\sqrt{A}$, similar to Degen's "Canon Pellianus."

2. Dr. Minnigerode notices that Herr Stern had shown how the continued fraction with negative unit numerators and positive integer denominators, representing the square root of a non-quadrate number, may be used to solve the corresponding Pellian equation.

The author then proceeds to prove that the equivalent continued fraction with negative denominators may also be used when the integer absolutely nearest to the irrational is taken as the limit.

He treats the subject from the point of view afforded by the theory of Binary Quadratic Forms.

Each substitution $\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix}$, where $\alpha\delta-\beta\gamma = 1$, which transforms (a, b, c), determinant D, of the species σ, into itself, yields a solution of

$$t^2 - Du^2 = \sigma^2, \qquad \sigma = 1 \text{ or } 2,$$

by means of

$$\alpha = \frac{t-bu}{\sigma}, \quad \beta = -\frac{cu}{\sigma},$$

$$\gamma = \frac{au}{\sigma}, \qquad \delta = \frac{t+bu}{\sigma}.$$

The form need not belong to the class of $(1, 0, -D)$ (Dirichlet's *Vorlesungen*, p. 148).

The author puts

$$\omega = a_0 - \frac{1}{\omega_1}, \quad \omega_1 = a_1 - \frac{1}{\omega_2}, \; \ldots\ldots \; \omega_{r-1} = a_{r-1} - \frac{1}{\omega_r}, \text{ \&c.}$$

If $\omega, \omega_1, \omega_2 \ldots$ are >2, not having reference to sign, the correspond-

---

* The author finds the condition $d_n\,(y-y_0) < x+y\sqrt{A}$ in order that $\frac{x}{y}$ may be an approximate fraction to $\sqrt{A}$, and argues (p. 28),—"Da nun $y-y_0$ positiv und kleiner als y ist, so ist, wenn $d_n < \sqrt{A}$, $d_n\,(y-y_0) < y\sqrt{A}$, und um so mehr $d_n\,(y-y_0) < x+y\sqrt{A}$, also ist auch $\frac{x}{y}$ ein Näherungswerth von $\sqrt{A}$." But, since $x-y\sqrt{A}$ is positive, x is $> y\sqrt{A}$, and $d_n\,(y-y_0)$ is to be less than $2y\sqrt{A}+$.

ing continued fraction

$$a_0 - \frac{1}{a_1} - \frac{1}{a_2} - \dots - \frac{1}{a_{r-1}} - \frac{1}{a_r},$$

is called "regular" (*regelmässige*).

Now to (a, b, a_1) corresponds a quadratic $a+bx+a_1x^2=0$, and the roots $\frac{-b-\sqrt{D}}{c}$, $\frac{-b+\sqrt{D}}{c}$ are termed respectively the first and second roots of the form.

If $\delta_0, \delta_1 \dots$ are the nearest integers to $\omega, \omega_1 \dots$, the respective first roots of the associated forms

$$(a, b, a_1)\ (a_1, b_1, a_2) \dots (a_r, b_r, a_{r+1}),$$

or, say, $f, f_1 \dots f_r$, then

$$\begin{array}{ll} b+b_1 = -\delta\ a_1, & b_1^2 - a_1 a_2 = 0, \\ b_1+b_2 = -\delta_1 a_2, & b_2^2 - a_2 a_3 = 0, \\ \dots\ \dots\ \dots\ \dots & \dots\ \dots\ \dots \\ b_{r-1}+b_r = -\delta_r a_{r+1}, & b_r^2 - a_r a_{r+1} = D, \end{array}$$

and $\omega_1, \omega_2 \dots$ are >2, irrespectively of sign.

If the development is continued, we shall arrive at a form f_r, in which, irrespectively of sign, $a_{r+1} \overline{\gtrless} a_r$. For, otherwise, we should have $a_1 > a_2 > a_3 > \dots$ ad inf., which cannot be.

But forms so specialised are finite in number, being subject to the double inequality $\frac{3}{5}\sqrt{D} < b_r > \frac{5}{3}\sqrt{D}$.

From this follows at once the periodicity of the continued fraction, representing the first root of (a, b, a_1). All this is closely analogous to the usual process.

The forms from f_r to its next recurrence are called "reduced forms" by analogy. They are fully characterised by the fact that they give rise to purely periodic regular continued fractions.

The author shows further that, if we start with a so-called "reduced form," all the solutions of $t^2 - Du^2 = \sigma^2$ are obtainable, or, what is the same thing, we get the least solution by means of the first period.

The proof (pp. 627—652) commences by taking a specialised "reduced form" (a, b, c), in which c is negative and absolutely $\overline{\gtrless}\ a$. The cases have to be separately considered in which b is $<\sqrt{D}$ and b is $>\sqrt{D}$; so also the cases in which $\alpha = 0$ or $\frac{\gamma}{\alpha}$ is integer in the substitution $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$, which transforms (a, b, c) into itself.

Finally, it is shown that any "reduced form" may be taken with similar result.

The general proposition would, I think, be more readily established by comparing the "regular" continued fraction with its equivalent, having positive numerators and positive denominators, which I have called the normal continued fraction.

Addition II.

The statements in § 7 require amplification.

Any form belonging to the development of $\sqrt{D}$ in a continued fraction by means of integer limits is properly or improperly equivalent to the original form $q^r - Dr^r$. Let (a, b, c) be such a form; then (1) if we express the major positive root by means of inferior limits as a continued fraction, we shall, according to the usual theory, obtain the solution of the Pellian equation. The forms of the period will alternately have negative middle terms. If the signs of these are changed, the forms become reduced. (2) If we proceed by nearest limits, the development will, in accordance with the results of § 4, be definite in relation to the normal positive development, and in such a way that we must arrive at a leading coefficient = unity. (3) Similarly, if we proceed by inferior and superior limits alternately, we have

$$a - \frac{1}{b+}\,\frac{1}{c-}\,\frac{1}{d+}\,\frac{1}{R} = a-1+\frac{1}{1+}\,\frac{1}{b-1+}\,\frac{1}{c-1+}\,\frac{1}{1+}\,\frac{1}{d-1+}\,\frac{1}{R}.$$

Two successive letters will not represent unity. In fact, the letters with positive numerators, in the left-hand expression cannot be less than 2.

If $b = 1$, the result is

$$a-1+\frac{1}{c+}\,\frac{1}{1+}\,\frac{1}{d-1+}\,\frac{1}{R}.$$

If also $d = 1$, then it is

$$a-1+\frac{1}{c+}\,\frac{1}{1+R}.$$

When therefore a partial quotient a occurs in the normal development, there will be one $= a$ or $a+1$ in the development by alternate limits, and the form to which this partial quotient belongs will have unity for a leading coefficient when a is twice the greatest integer in $\sqrt{D}$. (4) Suppose, again, we proceed arbitrarily as to limits, and obtain the form (a, b, c) where a is positive. If now we go on with superior limits, the normal fraction being

$$a+\frac{1}{a_1+}\,\frac{1}{a_2}+\dots,$$

our result will be of the form

$$[a+1,\ \overline{a_1-1},\ a_2+2,\ \overline{a_3-1}\ \dots],$$

and we can apply the same reasoning as in Addition I. (1).

(5) We may proceed arbitrarily as to limits, except the condition that, when the irrational is < 2, the inferior limit must be taken. For in this case no partial quotient $= 2$ is immediately followed by a negative numerator. The transformation of such a continued fraction into the normal fraction equivalent to it is similar to that of the continued fraction obtained by taking the nearest limits, and no zero denominator is introduced except in the case of three successive terms of the form

$$\frac{1}{a-}\,\frac{1}{1+}\,\frac{1}{R}=\frac{1}{a-1+}\,\frac{1}{1+}\,\frac{1}{0+}\,\frac{1}{R}=\frac{1}{a-1+}\,\frac{1}{1+R}.$$

(6) There is another case worth noting, and with more detail. We may proceed by taking as limits the integers next to, but furthest from, the true values of the irrationals.

Every partial quotient is then 1 or 2. When the partial quotient is unity, it is always immediately followed by a positive unit numerator; when it is 2, by a negative unit numerator.

Suppose the negative continued fraction (to use Herr Stern's expression) $[a+1,\ \overline{a_1-1},\ a_2+2,\ \overline{a_3-1},\ \dots]$ ....(*a*)

is given. Applying repeatedly the identity

$$\frac{1}{K+}\,\frac{1}{R}=1-\frac{1}{1+}\,\frac{1}{K-1+}\,\frac{1}{R}$$

to the terms $$-\frac{1}{a_{2m}+2-}\,\frac{1}{R},$$

we get $$-\left(1-\frac{1}{2-}\,\frac{1}{2-}\dots-\frac{1}{2-}\,\frac{1}{1+}\,\frac{1}{2-}\,\frac{1}{R}\right),$$

the denominator 2 being repeated $a_{2m}-2$ times.

Hence the fraction (*a*) is equivalent to

$$[a+1,\ \overline{a_1-2},\ 1,\ 2^{+},\ \overline{a_2-2},\ 1,\ 2^{+},\ \overline{a_3-2},\ 1,\ 2^{+},\ \dots]\ \dots\dots(b),$$

where 2^+ is written to indicate that the corresponding numerator is positive.

We will suppose at present that no a is <2. The above fraction fulfils the conditions required.

Now any form (A, B, C) is identical with the form

$$\left(\frac{(A+B)^2-D}{m},\ \frac{(A+B)(B+C)-D}{m},\ \frac{(B+C)^2-D}{m}\right),$$

where m is written for $A+2B+C$, and D is the determinant.

Consider the form into which (A, B, C) is changed by the substitution $\begin{pmatrix}\lambda, & -1\\ 1, & 0\end{pmatrix}$.

The resulting form will have for its leading coefficient $A\lambda^2+2B\lambda+C$, and the sum of the coefficients is $A(\lambda-1)^2+2B(\lambda-1)+C$, which would have been the leading coefficient, if we had transformed by the substitution $\begin{pmatrix}\lambda, & +1\\ 1, & 0\end{pmatrix}$.

These leading coefficients will be of opposite signs when λ is the superior limit and consequently $\lambda-1$ the inferior limit. But, having regard to the genesis of the fraction (a), one sees that the fraction

$$[a+1,\ \overline{a_1-1},\ a_2+2\ \dots\ a_{2m}+1]$$

is equal to the fraction

$$(a,\ a_1,\ a_2\ \dots\ a_{2m}).$$

Also, if $\lambda=2$, the sum of the coefficients of the derived form is the same as that of the coefficients of the form from which it is derived.

It follows, then, that the sum of the coefficients of any form belonging to (a) is the same as the leading term of the corresponding form of the positive normal fraction, and that, as often as λ is successively $=2$, the sum of the coefficients is constant.

Moreover, since the leading terms of the forms belonging to (a) are positive, the sums of the coefficients of the forms are negative, and equal respectively to the negative leading coefficients of the forms of the equivalent normal continued fraction representing $\sqrt{D}$.

We have seen that, whenever $q^2-Dr^2=k$ is possible, k being a positive integer $<2\sqrt{D}$, then k will appear as a leading term of a form belonging to the continued fraction (a); *à fortiori*, all the positive leading terms of forms belonging to the continued fraction with positive unit numerators, and representing $\sqrt{D}$, will appear among the leading terms of forms belonging to the negative continued fraction representing $\sqrt{D}$. These leading terms occur when we stop at the last partial quotient in a series of the value 2; for

$$[a+1,\ \overline{a_1-1},\ a_2+2,\ \dots\ \overline{a_{2m+1}-1}]$$
$$=(a,\ \ a_1,\ \ a_2,\ \ \dots\ a_{2m+1}).$$

The same reasoning precisely may be applied to the continued fraction obtained by taking first of all the inferior limit, and then proceeding exclusively by superior limits.

We have a fraction

$$[a,\ a_1^+ +1,\ \overline{a_2-1},\ \overline{a_3+2},\ \overline{a_4-1},\ \ldots] \ldots\ldots\ldots\ldots\ldots\ldots (c),$$

and it will be seen that the sums of the coefficients of all the forms belonging to it are equal to the corresponding positive leading terms of the normal forms, while all the negative leading terms of the normal forms are found among the leading terms of the forms belonging to (*c*). These are given by the fractions

$$[a,\ a_1^+ +1,\ \overline{a_2-1},\ \overline{a_3+2}\ \ldots\ \overline{a_{2m}-1}].$$

In the same way as (*a*), we can transform (*c*) into (*b*).

Still, supposing that no a is <2, we can now see without much difficulty that every fraction

$$[a+1,\ \overline{a_1-2},\ 1,\ 2^+\ \ldots\ \overline{a_{2m+1}-2},\ 1,\ 2^+]$$
$$= [a+1,\ \overline{a_1-1},\ a_2+2\ \ldots\ \overline{a_{2m+1}-1}],$$

and every fraction

$$[a+1,\ \overline{a_1-2},\ 1,\ 2^+\ \ldots\ \overline{a_{2m}-2},\ 1,\ 2^+]$$
$$= [a,\ a_1^+ +1,\ \overline{a_2-1},\ a_3+2\ \ldots\ \overline{a_{2m}-1}].$$

Also as to fractions terminating with an element $-\frac{1}{2}$, each one, derived from (*b*), has one equivalent fraction terminating with an element $-\frac{1}{2}$ derived from (*a*).

But there remains a species of fraction derived from (*a*) (and having the last of a series of elements $-\frac{1}{2}$ for its final term), which is equivalent to a fraction derived from (*b*) of the form

$$[a+1,\ \overline{a_1-2},\ 1,\ 2^+ \ldots 1,\ 2^+,\ \overline{a_k-2},\ 1].$$

For $$-\frac{1}{1}+\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\ldots-\frac{1}{2}-\frac{1}{1}=-\frac{1}{2}.$$

The result is similar when we compare (*b*) with (*c*), and we conclude generally that all the leading terms of forms belonging to (*a*) and (*c*) reappear as leading terms of forms belonging to (*b*), and no others.

Moreover, the sums of the coefficients of the forms, excluding the first belonging to (*b*) are equal to the leading terms of the forms belonging to the normal positive continued fraction representing $\sqrt{D}$.

But the case of $a_k = 1$ remains to be considered. If $a_k = 2$, $\overline{a_k-2}$ simply means that there are no elements $-\frac{1}{2}$ corresponding to $\overline{a_k-2}$, and we pass at once to $-\frac{1}{1}$. When $a_k=1$, the matter is of more consequence. Yet we only have to leave these terms $\frac{1}{1}$ as they are. Thus $(a, a_1, a_2, 1, 1, 1 \dots 1, a_3, a_4 \dots)$, when 1 is k times repeated in succession, becomes

$$(a+1,\ \overline{a_1-2},\ 1,\ 2^+,\ \overline{a_2-2},\ 1,\ 1^+,\ 1^+,\ 1^+ \dots$$
$$\dots 2^+,\ 2,\ \overline{a_3-2},\ 1,\ 2^+,\ a_4-2,\ 1,\ 2^+ \dots),$$

where 1^+ is also k times repeated.

If, however, a series of units begins directly after a, the transformation is $(a,\ 1^+,\ 1^+ \dots 1^+,\ 2^+,\ \overline{a_1-2},\ 1,\ 2^+ \dots)$,

or $(a,\ 1^+,\ 1^+ \dots 1^+,\ 2^+,\ \overline{a_1-2},\ 1,\ 1^+ \dots)$,

if $\frac{1}{a_1}$ is followed by terms $\frac{1}{1}$ immediately.

Now these changes do not affect the fact that all the leading terms of the forms belonging to the two negative continued fractions reappear as leading terms in the forms belonging to the continued fraction obtained by taking as limits the integers next to, but furthest from, the values of the irrationals.

This being so, the conclusion is reached, that such a development gives, when possible, the solution of $q^2-Dr^2=\pm k$, when k is a positive integer $<2\sqrt{D}$.

As an example take $D=19$; the normal development is

$$(4,\ 2,\ 1,\ 3,\ 1,\ 2,\ 8,\ 2 \dots),$$

so that we have partial quotients $=1$ and 2. Paying attention to this, we have for the three negative developments

$$[5,\ 2,\ 3,\ 2,\ 2,\ 3,\ 2,\ 10,\ 2 \dots],$$
$$[4,\ 3^+,\ 5,\ 4,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 4,\ 5 \dots],$$
$$[5,\ 1^+,\ 1^+,\ 2^+,\ 2,\ 1,\ 1^+,\ 2^+,\ 1,\ 2^+,\ \overline{8-2},\ 1,\ 2^+],$$

and the approximations derived from the third development are those of the first two together. In obtaining such developments, care must be taken to adhere to the positive root when the two first coefficients of the form operated on are of the same sign. *E.g.*, in the case of $-2q^2-2qr+9r^2$ the transformation is obtained from

$$\frac{1-\sqrt{19}}{-2}.$$

Thursday, June 12th, 1884.

Prof. HENRICI, F.R.S., President, in the Chair.

Mr. G. S. Ely, Ph.D., Professor of Mathematics in Buchtel College, Akron, Ohio, was elected a Member of the Society.

The President announced that the Council had awarded the De Morgan Medal to Prof. Cayley, F.R.S.

Mr. Tucker communicated an abstract of a paper by Prof. H. Lamb, entitled "Note on the Induction of Electric Currents in a Cylinder placed across the Lines of Magnetic Force."

Mr. J. Hammond gave some results of a paper by him in the *American Journal of Mathematics.*

The following presents were received :—

"Proceedings of the Royal Society," Vol. xxxvi., No. 230.

"Proceedings of the Physical Society of London," Vol. v., Pt. 5, Oct. 1883, March 1884 ; Vol. vi., Pt. 1, April—May, 1884.

"Educational Times," for June.

"Memoirs of the Manchester Literary and Philosophical Society," 3rd Series, Vol. vii. and ix.

"Proceedings of the Manchester Literary and Philosophical Society," Vols. xx., xxi., and xxii.

"On Electrical Motions in a Spherical Conductor," by Prof. H. Lamb (from "Philosophical Transactions," Pt. ii., 1883).

"Johns Hopkins University Circulars," Vol. iii., No. 30.

"Proceedings of the Canadian Institute," Vol. ii., Fasc. No. 1, 8vo ; Toronto, 1884.

"On an Unsymmetrical Law of Error in the Position of a Point in Space," by E. L. De Forest, (from "Transactions of Connecticut Academy").

"Atti del R. Istituto Veneto," Tomo i., Ser. vi., Disp. 4, 5, 6, 7, 8, 9, 10 ; Tomo ii., Ser. vi., Disp. 1, 2.

"Bulletin de la Société Mathématique de France," T. xii., No. 1.

"Bulletin des Sciences Mathématiques et Physiques," 2nd series, T. vii. (Index) ; T. viii., April and May.

"Beiblätter zu den Annalen der Physik und Chemie," B. viii., St. 4 and 5.

"Acta Mathematica," iv. 3.

"Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin," Nos. 1—17.