ON A CLASS OF ANALYTIC FUNCTIONS

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I.

1. The distinction between integral functions whose increase is regular (*fonctions à croissance régulière*) and those whose increase is irregular was first explicitly formulated by M. Borel.† According to him a function is *à croissance régulière* if

$$\lim_{r=\infty} \frac{\log \log M(r)}{\log r}$$

is determinate, $M(r)$ denoting the greatest value of the modulus of the function on a circle of radius r. A function is then *à croissance irrégulière* if we can determine two constants α, β $(\alpha < \beta)$ such that for an infinity of values of r tending to infinity

$$M(r) < e^{r^\alpha},$$

and for a similar infinity of values of r

$$M(r) > e^{r^\beta}.$$

The numerous extensions which have been given lately to the theory of integral functions render it necessary to give a greater degree of precision to this definition. We shall say that $f(x)$ is a function whose increase is irregular if we can determine β_ν, γ_ν $(\beta_\nu < \gamma_\nu)$ so that for two infinities of values of r tending to infinity $M(r) < e^{V_1(r)}$ and $M(r) > e^{V_2(r)}$ respectively, where

$$V_1(r) = A_1 r^{\alpha_0} (\log_1 r)^{\alpha_1} \dots (\log_{\nu-1} r)^{\alpha_{\nu-1}} (\log_\nu r)^{\beta_\nu}.$$

and

$$V_2(r) = A_2 r^{\alpha_0} (\log_1 r)^{\alpha_1} \dots (\log_{\nu-1} r)^{\alpha_{\nu-1}} (\log_\nu r)^{\gamma_\nu},$$

$\log_1 r$, $\log_2 r$, ... denoting as usual $\log r$, $\log \log r$, ... It is evident that this definition applies only to functions of finite order (*genre*), but it is easy to frame similar definitions for functions of higher order, the general principle being obvious.

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* The contents of the paper have been considerably altered in revision, and the title has been changed.

† *Leçons sur les fonctions entières*, pp. 107 *et seq.*

In this paper I propose to consider certain classes of series of the general type

(1) $$\sum_{\nu=1}^{\infty} \frac{c_\nu}{\sin \nu\pi\lambda} x^\nu,$$

where λ is any quantity which is not *real and rational.** The peculiar interest of these functions lies in the fact that for the same set of values of the numbers c_ν the series may

(i.) be convergent for all values of x, and represent an integral function whose increase is either regular or irregular;

(ii.) have a finite circle of convergence which is a critical line for the function represented by the series;

(iii.) diverge for all values of x.

All these peculiarities, for example, present themselves for different values of λ in the case in which $c_\nu = 1/\nu!$, to which I shall devote particular attention.

I wish to point out that the series (1) is not an instance of a series artificially constructed in order to provide an illustration of certain theoretical possibilities. On the contrary, the series

(2) $$\Sigma \frac{x^\nu}{\nu! \sin \nu\lambda\pi}$$

presents itself naturally and inevitably when we attempt to determine the behaviour of the function represented by the simple definite integral

(3) $$\int_0^\infty \frac{e^{-u^\lambda} du}{u+x},$$

and is therefore (in some cases) an instance of a *fonction à croissance irrégulière* quite unlike those devised by M. Borel.†

2. There is another point of view from which the series (1) may be regarded. Let us suppose that

(4) $$f(x) = \Sigma c_\nu x^\nu,$$

and that λ has such a value that (1) is convergent for at any rate some values of x. The function defined by (1) is then the most obvious solu-

* The series is obviously meaningless if λ is real and rational.

† *Loc. cit.*

tion of the difference equation

$$F(xe^{\lambda\pi i})-F(xe^{-\lambda\pi i}) = 2if(x), \tag{5}$$

an equation which is fundamentally the same as the classical difference equation

$$\Phi(x+a)-\Phi(x) = \phi(x), \tag{6}$$

to which it may easily be reduced by means of the substitutions

$$\begin{cases} x = e^{(\omega+\lambda)\pi i}, & F(e^{\omega\pi i}) = \Phi(\omega), \\ 2if\{e^{(\omega+\lambda)\pi i}\} = \phi_{\cdot}'(\omega), & 2\lambda = a. \end{cases} \tag{7}$$

The Series $\sum_{\nu=1}^{\infty} \frac{c_\nu x^\nu}{\sin \nu\lambda\pi}$.

3. I shall now consider directly what is the nature of the function defined by the series (1) in those cases in which it has a radius of convergence other than zero. I shall suppose for simplicity that

$$|c_\nu| \geqslant |c_{\nu+1}| \tag{8}$$

for all values of ν, and that $\lim c_\nu^{1/\nu}$ is determinate.

4. (i.) If λ is complex and equal to $\lambda_1+i\lambda_2$,

$$|\sin \nu\lambda\pi| = \tfrac{1}{2}e^{\nu\pi|\lambda_2|}(1+\epsilon), \tag{9}$$

where ϵ is small when ν is large. The radius of convergence of (1) is then $\rho e^{\pi|\lambda_2|}$, ρ being the radius of convergence of (4). This case possesses no particular feature of interest.

(ii.) If λ is real, we may without loss of generality suppose it *positive, irrational, and less than unity.* The region of convergence will then depend upon the arithmetical nature of λ. The radius of convergence of (1) is certainly not greater than ρ. I shall prove first *that, if* λ *is an algebraic number, the radius of convergence of* (1) *is equal to* ρ.

For suppose that λ is algebraic and of degree m, *i.e.*, the root of an irreducible equation

$$x^m+a_1x^{m-1}+\ldots+a_m = 0, \tag{10}$$

where $a_1, \ldots, a_m$ are integers. Then, by a well known property of algebraic numbers,*

$$\left| \frac{\mu}{\nu} - \lambda \right| > \frac{K_m}{\nu^{m+1}}, \tag{11}$$

where K_m is a constant (depending only on m) for all integral values of μ and ν. Hence

$$| \mu - \nu\lambda | > K_m \nu^{-m}, \tag{12}$$

from which it obviously follows that

$$| \operatorname{cosec} \nu\lambda\pi | < K'_m \nu^m \tag{13}$$

for all values of ν. Hence the radius of convergence of (1) is not less than that of $\Sigma c_\nu \nu^m x^\nu$, *i.e.*, than ρ, and therefore it is equal to ρ. In particular, if $\rho = \infty$, the series (1) represents an integral function of x.

In the second place, *values of* λ *can be found such that the radius of convergence of* (1) *is any quantity* R, *where*

$$0 \leqslant R < \rho. \tag{14}$$

In order to prove this and the further results which we shall establish later on, we must consider certain properties of simple continued fractions.†

5. Let us suppose that λ is expressed as a simple continued fraction

$$\frac{1}{a_0 +} \frac{1}{a_1 +} \frac{1}{a_2 + \dots}, \tag{15}$$

and that p_ν / q_ν $(\nu = 0, 1, \dots)$ are the successive convergents, while

$$\frac{p_0}{q_0} = \frac{0}{1}, \quad \frac{p_1}{q_1} = \frac{1}{a_0}, \quad \frac{p_2}{q_2} = \frac{a_1}{a_0 a_1 + 1}; \tag{16}$$

and let us denote the complete quotient

$$a_\nu + \frac{1}{a_{\nu+1} +} \frac{1}{a_{\nu+2} + \dots} \tag{17}$$

by a'_ν. Let m be any positive integer such that

$$q \; < m < q_{\nu+1}. \tag{18}$$

* Borel, *Leçons*, I., p. 27. This inequality is employed in a very similar manner by H. J. S. Smith, "On some Discontinuous Series considered by Riemann," *Mess. of Math.*, Vol. XI., pp. 1-11.

† The purpose of §§ 5, 6 is simply to establish the equations (30) and (31) at the end of § 6.

We can form successively the equations

$$(19)\quad \begin{cases} m = k_1 q_\nu + m_1 & (0 \leqslant k_1 \leqslant a_\nu, \quad 0 \leqslant m_1 < q_\nu), \\ m_1 = k_2 q_{\nu-1} + m_2 & (0 \leqslant k_2 \leqslant a_{\nu-1}, \quad 0 \leqslant m_2 < q_{\nu-1}, \\ \cdots & \cdots \quad \cdots \quad \cdots \\ m_{\nu-2} = k_{\nu-1} q_2 + m_{\nu-1} & (0 \leqslant k_{\nu-1} \leqslant a_2, \quad 0 \leqslant m_{\nu-1} < q_2), \\ m_{\nu-1} = k_\nu q_1 + m_\nu & (0 \leqslant k_\nu \leqslant a_1, \quad 0 \leqslant m_\nu < q_1), \end{cases}$$

and in one way only. Then

$$(20)\quad m = \sum_{s=1}^{\nu} k_s q_{\nu-s+1} + m_\nu .$$

Now

$$q_{\nu-s+1}\lambda = q_{\nu-s+1} \frac{a_{\nu-s+1} p_{\nu-s+1} + p_{\nu-s}}{a'_{\nu-s+1} q_{\nu-s+1} + q_{\nu-s}} = p_{\nu-s+1} + \frac{(-1)^{\nu-s+1}}{a_{\nu-s+1} q_{\nu-s+1} + q_{\nu-s}},$$

since
$$p_{\nu-s} q_{\nu-s+1} - q_{\nu-s} p_{\nu-s+1} = (-1)^{\nu-s+1}.$$

Hence

$$(21)\quad m\lambda = I_m + m_\nu \lambda + (-)^\nu S_m,$$

where

$$(22)\quad I_m = \sum_{s=1}^{\nu} k_s p_{\nu-s+1},$$

which is integral, and

$$(23)\quad S_m = \sum_{s=1}^{\nu} \frac{(-)^{s-1} k_s}{a'_{\nu-s+1} q_{\nu-s+1} + q_{\nu-s}} .$$

In what follows I shall suppose ν *odd*. The work in the case in which ν is even is strictly analogous. And I shall write

$$q'_{r+1} = a'_r q_r + q_{r-1};$$

so that

$$(23')\quad S_m = \sum_{s=1}^{\nu} \frac{(-)^{s-1} k_s}{q_{\nu-s+2}} .$$

In the first place we can prove that, if k_r is the last k which does not vanish, S_m has the sign of $(-)^{r-1}$, *i.e.*, the sign of its last term. In fact the last term is numerically greater than the sum of all preceding terms of opposite sign, that is to say,

$$(24)\quad \frac{k_r}{q'_{\nu-r+2}} > \frac{k_{r-1}}{q'_{\nu-r+3}} + \frac{k_{r-3}}{q'_{\nu-r+5}} + \dots .$$

Suppose, for example, that r is even. The most unfavourable case is obviously that in which

$$k_r = 1, \quad k_{r-1} = a_{\nu-r+2}, \quad k_{r-3} = a_{\nu-r+4}, \quad \dots, \quad k_1 = a_\nu,$$

and all the other k's vanish. This case cannot actually occur. For, if it could, we should have

$$m-m_\nu = a_\nu q_\nu + a_{\nu-2} q_{\nu-2} + \ldots + a_{\nu-r+2} q_{\nu-r+2} + q_{\nu-r+1},$$

i.e., $= q_{\nu+1}$. But, even if it could, the inequality (24) would still be valid. For we should have

$$q_{\nu+1}\lambda = I_m - S_m = p_{\nu+1} - S_m.$$

Now, as ν is odd, $$p_{\nu+1} < q_{\nu+1}\lambda.$$

Hence $S_m < 0$; *i.e.*, S_m has the sign of $(-)^{r-1}$. *A fortiori* this is true in any case which can actually occur. Moreover, since at least one k must differ from the value assigned to it in the most unfavourable case, the excess of the left-hand side of (24) over the right-hand side must be at least

$$\frac{1}{q'_{\nu+1}}, \tag{25}$$

and therefore the modulus of S_m must be at least equal to the same quantity.

If, on the other hand, r is odd, the most unfavourable case is given by taking

$$k_r = 1, \quad k_{r-1} = a_{\nu-r+2}, \quad \ldots, \quad k_2 = a_{\nu-1},$$

$$m-m_\nu = a_{\nu-1} q_{\nu-1} + a_{\nu-3} q_{\nu-3} + \ldots + a_{\nu-r+2} q_{\nu-r+2} + q_{\nu-r+1},$$

i.e., $= q_\nu$. But then we should have

$$q_\nu \lambda = p_\nu - S_m,$$

and, as $p_\nu > q_\nu \lambda$, S_m would be positive, from which we can draw the same conclusions. Thus an inferior limit for $|S_m|$ is given by the quantity (25).

We shall also require a superior limit for $|S_m|$. If S_m is positive, it is certainly less than S'_m, deduced from S_m by taking

$$k_1 = a_\nu, \quad k_2 = 0, \quad k_3 = a_{\nu-2}, \quad \ldots, \quad k_\nu = a_1,$$

$$m-m_\nu = a_\nu q_\nu + a_{\nu-2} q_{\nu-2} + \ldots + a_1 q_1 = q_{\nu+1} - q_0 = q_{\nu+1} - 1.$$

Then $$q_{\nu+1}\lambda = I'_m - S'_m + \lambda,$$

and, as $$q_{\nu+1}\lambda = p_{\nu+1} + \frac{1}{q_{\nu+2}}$$

and $$I'_m = p_{\nu+1},$$

$$S'_m = \lambda - \frac{1}{q'_{\nu+2}}, \tag{26}$$

which is less than λ, and nearly equal to λ when ν is large.

Now S_m can be equal to S'_m if, and only if, $m = q_{\nu+1} - 1$, $m_\nu = 0$. In this case $S_m < \lambda$; in all other cases

(26′) $$S_m < \lambda - \frac{1}{q'_{\nu+1}},$$

since at least one k must differ from the value assigned to it in the formation of S'_m.

If, on the other hand, S_m is negative, it is numerically less than $-S''_m$, deduced from S_m by taking

$$k_1 = 0, \quad k_2 = a_{\nu-1}, \quad k_3 = 0, \quad \dots, \quad k_\nu = 0,$$

$$m - m_\nu = a_{\nu-1} q_{\nu-1} + a_{\nu-3} q_{\nu-3} + \dots + a_2 q_2 = q_\nu - q_1 = q^\nu - a_0.$$

Then $$q_\nu \lambda = I''_m - S''_m + a_0 \lambda,$$

and, as $$q_\nu \lambda = p_\nu - \frac{1}{q'_{\nu+1}}$$

and $$I''_m = p_\nu - 1,$$

(27) $$-S''_m = 1 - a_0 \lambda - \frac{1}{q'_{\nu+1}},$$

which is less than $$1 - a_0 \lambda = \frac{1}{a_0 a'_1 + 1},$$

and nearly equal to it when ν is large.

To sum up, we have obtained the following limits for S_m :—

(i.) If $S_m > 0$,

(28) $$\frac{1}{q_{\nu+1}} < S_m < \lambda - \frac{1}{q'_{\nu+1}}.*$$

(ii.) If $S_m < 0$,

(28′) $$\frac{1}{q'_{\nu+1}} < -S_m < 1 - a_0 \lambda - \frac{1}{q'_{\nu+1}}.$$

6. Now

(29) $$|\sin m\lambda\pi| = |\sin (m_\nu \lambda - S_m)\pi|.$$

There are two cases to consider:

(*a*) Suppose $m_\nu = 0$. Then
$$|\sin m\lambda\pi| = |\sin S_m \pi| > \frac{K}{q'_{\nu+1}},$$
where K is a constant.

(*b*) Suppose $1 \leqslant m_\nu \leqslant a_0 - 1$. If $S_m > 0$, it is clear that
$$m_\nu \lambda - S_m < (a_0 - 1)\lambda < 1 - \lambda.$$

On the other hand, $$m_\nu \lambda - S_m \geqslant \lambda - S_m > \frac{1}{q'_{\nu+1}}.$$

* One exception to the second of these inequalities was noted above. In this exceptional case the inequality $S_m < \lambda$ will be sufficient for our purpose.

If $S_m < 0$, $m_\nu\lambda - S_m > \lambda$, and, on the other hand,

$$m_\nu\lambda - S_m \leqslant (a_0-1)\lambda - S_m < 1-\lambda.$$

And *in all cases*

(30) $$|\sin m\lambda\pi| > \frac{K}{q'_{\nu+1}} \quad (q_\nu < m < q_{\nu+1}).$$

On the other hand, $|\sin q_{\nu+1}\lambda\pi| = \sin\frac{\pi}{q_{\nu+2}}$,

so that for large values of ν

(31) $$|\sin q_{\nu+1}\lambda\pi| = \frac{\pi(1+\epsilon_\nu)}{q'_{\nu+2}},$$

where ϵ_ν is small when ν is large. The two formulæ (30) and (31) will form the basis of the succeeding argument.

7. We return now to the series (1). In all that follows I shall suppose that

$$a_1 < a_2 < a_3 < \dots;$$

so that for large values of ν

$$a_\nu, \quad q_\nu, \quad q_{\nu+1}/q_\nu$$

are large and

$$a'_\nu/a_\nu, \quad q'_\nu/q_\nu$$

nearly equal to unity.

Suppose first that the radius of convergence of $\Sigma c_\nu x^\nu$ is finite; we may without loss of generality suppose it equal to unity. And let $R = 1/a < 1$.

It is easy to see that, if we are given any sequence of ascending integers Q_ν such that $\lim Q_{\nu+1}/Q_\nu = \infty$, we can find a continued fraction such that

$$\lim q_\nu/Q_\nu = 1.$$

We can therefore find a value of λ such that

(32) $$\lim q_{\nu+1}/a^{q_\nu} = 1.$$

The radius of convergence of the series

(33) $$\Sigma\frac{c_{q_\nu}x^q}{\sin q_\nu\lambda\pi}$$

is the same as that of

$$\Sigma q_{\nu+1}c_{q_\nu}x^{q_\nu},$$

and is evidently R.

The remainder of the series (1) is $\sum_{\nu=1}^{\infty} u_\nu$, where

$$u_\nu = \sum_{q_\nu+1}^{q_{\nu+1}-1}\frac{c_n x^n}{\sin n\lambda\pi}.$$

Now $$|u_\nu| < K_{q_{\nu+1}} \, | c_{q_\nu} x^{q_\nu} | / (1 - | x |),$$

and so the radius of convergence of the series Σu_ν is *at least R*. Hence the radius of convergence of (1) is R.

In this case the circle $| x | = R$ is a *coupure* for the function

$$F(x) = \Sigma \frac{c_\nu x^\nu}{\sin \nu\lambda\pi}. \tag{34}$$

For $$F(xe^{\lambda\pi i}) - F(xe^{-\lambda\pi i}) = 2if(x),$$

where $f(x) = \Sigma c_\nu x^\nu$. There is at least one singularity of $F(x)$ of the type $x = Re^{i\theta}$, and it is plain that, if k is any integer,

$$x = Re^{i(\theta + 2k\lambda\pi)}$$

is also a singularity; and these points are everywhere dense on the circle $| x | = R$.

If ρ is infinite, we can still without loss of generality suppose $R = 1/a < 1$, and we can determine λ so that

$$\lim \, (c_{q_\nu} q_{\nu+1})^{1/q_\nu} = a.$$

The argument is then substantially the same. And it is obvious that by supposing $q_{\nu+1}$ a function of q_ν whose increase is sufficiently rapid, we can in any case ensure that the series (1) shall be divergent for all values of x other than zero.

8. In order to justify the assertions made in § 1, it remains only to prove that, when $f(x)$ is an integral function, values of λ can be found for which $F(x)$ is an integral function and its increase irregular.

I shall consider, for simplicity, the particular case in which $c_\nu = 1/\nu!$, which enables us to illustrate adequately the different cases which may occur.

In the first place, the increase of $F(x)$ *may* be regular. Suppose, for example, that λ is an algebraic number of degree m. By a well known theorem $M(r)$ is greater than the modulus of the greatest term in the series (1), and so certainly $$> Ke^r r^{-\frac{1}{2}}.^*$$

On the other hand, the terms of (1) are less than those of the series

$$K \Sigma \nu^m \frac{r^\nu}{\nu!}$$

* Since, by Stirling's theorem, $\dfrac{n^n}{n!} = \dfrac{e^n}{\sqrt{2\pi n}} (1 + \epsilon_n)$ when n is large.

(by § 4), and so $$M(r) < Ke^r r^m.$$

Thus, if $$M(r) = e^{V(r)},$$

$$r - \tfrac{1}{2}\log r + K < V(r) < r + m\log r + K,$$

and therefore the increase of $F(x)$ is regular.

9. I propose now to find inferior and superior limits for $M(r)$ when $r = q_\nu$, and to deduce that $F(x)$ may be an integral function whose increase is irregular. I shall write w_n for the general term of the series (1), and U_ν for
$$\sum_{q_\nu}^{q_{\nu+1}-1} w_n.^*$$

Assuming for the moment that $F(x)$ is an integral function, an *inferior* limit to $M(r)$ is given by the fact that $M(r)$ is greater than the modulus of the numerically greatest term in the series. Selecting the q_ν-th term, we find that

(36) $$M(r) > q_{\nu+1} q_\nu^{q_\nu}/q_\nu!.$$

Again, since $|x^n/n!|$ is greatest when $n = q_\nu$,

(37) $$\left|\sum_1^{q_\nu - 1} w_n\right| < Kq_\nu^2 \frac{q_\nu^{q_\nu}}{q_\nu!},$$

and, if the q_ν's are so chosen that

(38) $$\text{im}\; q_{\nu+1}/q_\nu^2 = \infty,$$

this is certainly small in comparison with (36).

On the other hand, if $\mu \geqslant \nu$,

(39) $$|U_\mu| < Kq_{\mu+1}^2 \frac{q_\nu^{q_\mu}}{q_\mu!} = U_{\nu,\mu},$$

say. Now $$\frac{U_{\nu,\nu+1}}{U_{\nu,\nu}} < K\frac{q_{\nu+2}^2}{q_{\nu+1}^2}\frac{q_\nu!}{q_{\nu+1}!}q_\nu^{q_{\nu+1}-q_\nu} < K\left(\frac{eq_\nu}{q_{\nu+1}}\right)^{q_{\nu+1}} q_{\nu+2}^2$$

by an easy application of Stirling's theorem, and so, by (38),

(40) $$< Kq_{\nu+2}^2 q_{\nu+1}^{-\frac{1}{2}q_{\nu+1}}.$$

If the q_ν's are such that

(41) $$\lim q_{\nu+1}^2/q_\nu^{\frac{1}{2}q_\nu} = 0,$$

the quantity on the right hand of (39) will be exceedingly small.

Similarly we can prove that $U_{\nu,\mu+1}/U_{\nu,\mu}$ is exceedingly small for any $\mu \geqslant \nu$. From this fact, in conjunction with (39) and (37), it follows

* In the notation of § 7, $U_\nu = w_{q_\nu} + u_\nu$

that the series is convergent for $x = q_\nu$ and that the modulus of its sum is less than

$$K^2_{q_{\nu+1}} \frac{q_\nu^{q_\nu}}{q_\nu!}.$$

The series (1) therefore represents an integral function $F(x)$, and we have the inequalities

$$(42) \qquad q_{\nu+1} \frac{q_\nu^{q_\nu}}{q_\nu!} < M(r) < K q_{\nu+1}^2 \frac{q_\nu^{q_\nu}}{q_\nu!}.$$

But, since $q_\nu = r$, $$\frac{q_\nu^{q_\nu}}{q_\nu!} \sim \frac{e^r}{\sqrt{2\pi r}}.$$

Hence

$$(43) \qquad q_{\nu+1} \frac{e^r}{\sqrt{2\pi r}} < M(r) < K q_{\nu+1}^2 \frac{e^r}{\sqrt{2\pi r}},$$

for $r = q_\nu$, provided the conditions (38) and (41) are satisfied.

Now let us suppose that when ν is odd

$$q_{\nu+1} \sim q_\nu^p \quad (p > 2),$$

and when ν is even $$q_{\nu+1} \sim q_\nu^{a q_\nu} \quad (a < \tfrac{1}{2}).$$

Then when $r = q_\nu$ and ν is odd

$$M(r) < K r^{2p-\frac{1}{2}} e^r,$$

and when $r = q_\nu$ and ν is even

$$M(r) > K r^{-\frac{1}{2}} e^{ar \log r + r}.$$

Hence the increase of $F(x)$ is irregular.

10. In a precisely similar manner we could, by taking

$$c_\nu = 1/(\nu!)^\beta \quad \text{or} \quad c_\nu = 1/\Gamma(\beta\nu + 1),$$

construct functions $F(x)$ such that, for an infinity of values of r, $M(r)$ is (roughly) of order $e^{r^{1/\beta}}$ and, for another infinity of values of r, $M(r)$ is (roughly) of order $$e^{ar^{1/\beta} \log r}.$$

We can also find functions of *infinite* order which possess similar peculiarities. For example, by taking

$$c_\nu = \log 2 . \log 3 \ldots \log \nu,$$

we can define a function whose maximum modulus is of the order of

$$e^{e^r/r}$$

for one infinity of values of r, and of order

$$e^{ae^r \log r}$$

for another. And the whole of the preceding analysis might be made considerably more precise, as we have generally left a considerable margin in our inequalities.

The preceding method, however, does not (as might at first sight be expected) enable us, for a *given* set of coefficients c_ν, such as $c_\nu = 1/\nu!$, to determine λ so that the increase of $F(x)$ shall be *arbitrarily* great. For, since we must for convergence have $q_{\nu+1} < q_\nu!$, $|w_{q_\nu}|$ is never, when $r = q_\nu$, of order substantially greater than $e^{r \log r}$.

II.

11. It might well be thought that functions such as those which I have considered in the first part of this paper were merely examples of an artificial character constructed in order to illustrate theoretical possibilities. This is far from being the case, as I shall proceed to show.

Let us consider the function

$$(1) \qquad F_{\lambda, a}(x) = \int_0^\infty \frac{e^{-u^\lambda} u^{a-1}}{u+x}\, du,$$

x being a complex variable whose variation is restricted by a cut along the negative real axis, λ and a being any real or complex quantities subject to certain restrictions which will be defined later, and u^λ and u^{a-1} having their principal values. This function includes as particular cases a number of well known functions. For instance,

(i.) if $\lambda = a = 1$, and we suppose for a moment that x is real and positive,

$$(2) \quad F_{1,1}(x) = e^x \int_x^\infty \frac{e^{-w}}{w}\, d\omega = -e^x\, li(e^{-x}) = e^x \left\{ \sum_1^\infty \frac{(-)^{n-1} x^n}{n.n!} - \gamma - \log x \right\},$$

this expansion defining the behaviour of $F_{1,1}(x)$ for all values of x.

(ii.) If $\lambda = 1$, $a = \frac{1}{2}$, and $x = \xi^2$ is real and positive,

$$(3) \qquad F_{1,\frac{1}{2}}(x) = 2\int_0^\infty \frac{e^{-\omega^2} d\omega}{\omega^2+\xi^2} = \frac{2\sqrt{\pi}}{\xi}\, e^{\xi^2} \int_\xi^\infty e^{-t^2} dt\,^* $$
$$= \pi \left\{ x^{-\frac{1}{2}} e^x - \sum_0^\infty \frac{x^n}{\Gamma(n+\frac{3}{2})} \right\},$$

after some transformations which will easily be supplied. This expansion again defines the behaviour of $F_{1,\frac{1}{2}}(x)$ for all values of x.

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* G. F. Meyer's edition (1871) of Dirichlet's *Lectures on Definite Integrals*, § 98, p. 285; Arendt's edition (1904), p. 208.

(iii.) If $\lambda = \frac{1}{2}$, $a = \frac{1}{2}$, and $x = \xi^2$ as before,

$$F_{\frac{1}{2},\frac{1}{2}}(x) = 2\int_0^\infty \frac{e^{-\omega}\,d\omega}{\omega^2+\xi^2} = \frac{2}{\xi}\left(\cos\xi\int_1^\infty \frac{\sin\xi t}{t}\,dt - \sin\xi\int_1^\infty \frac{\cos\xi t}{t}\,dt\right).^*$$

From this we can deduce, by some simple transformations,

$$(4) \qquad F_{\frac{1}{2},\frac{1}{2}}(x) = \frac{2\cos\sqrt{x}}{\sqrt{x}}\left\{\tfrac{1}{2}\pi - \sqrt{x}\sum_0^\infty \frac{(-)^n x^n}{(2n+1)(2n+1)!}\right\}$$
$$+ \frac{2\sin\sqrt{x}}{\sqrt{x}}\left\{\gamma + \tfrac{1}{2}\log x + \sum_1^\infty \frac{(-)^n x^n}{2n \, . \, 2n!}\right\},$$

a formula which again defines the behaviour of $F_{\frac{1}{2},\frac{1}{2}}(x)$ for all values of x.

12. Let us now consider for what values of λ and a the general integral is convergent.

(i.) If λ is *complex*, we may without loss of generality suppose its real part positive, since the transformation of the integral by the substitution $u = \dfrac{1}{v}$ gives

$$F_{\lambda, a}(x) = \frac{1}{x} F_{-\lambda, 1-a}\left(\frac{1}{x}\right).$$

If $\lambda = \mu + i\nu$, u^λ has its principal value, and so

$$|e^{-u^\lambda}| = e^{-u^\mu \cos(\nu \log u)},$$

and it is easy to see that the integral is divergent.

(ii.) If λ is *purely imaginary*, the integral is convergent if

$$0 < R(a) < 1,$$

and may be expanded in the form

$$(5) \qquad F_{i\nu, a}(x) = \sum_0^\infty \frac{(-)^n}{n!}\int_0^\infty \frac{u^{ni\nu+a-1}}{u+x}\,du = \pi x^{a-1}\sum_0^\infty \frac{x^{ni\nu}}{n!\sin(a+ni\nu)\pi},$$

where x^{a-1} has its principal value. This series represents an integral function of $x^{i\nu}$.

(iii.) If λ is real, we may suppose $\lambda \geqslant 0$ after what precedes. The case of $\lambda = 0$ is trivial, $F_{0,a}(x)$ reducing to a constant multiple of x^{a-1}.

(iv.) We need, therefore, only consider the case in which λ is *real and positive*. In this case the integral is convergent, if $R(a) > 0$, for all values of x save real and negative or zero values, and it represents a branch of a function of x whose only singularities are 0 and ∞.†

* Dirichlet (Arendt), p. 209. The transformation really dates from Cauchy's "Mémoire sur les Intégrales Définies" (*Œuvres*, t. I., p. 377).

† Although the integral ceases to converge when x is real and negative, it is easy to see that no finite negative value of x is really a singularity.

13. I shall now consider the integral

$$(6)\qquad P\int_{-\infty i}^{\infty i} \Gamma(-t)e^{-at}\,dt,$$

the contour of integration being the imaginary axis in the plane of $t = \xi + i\eta$, and P denoting the principal value.* Since

$$|\Gamma(-i\eta)| = e^{-\frac{1}{2}\pi|\eta|}\,|\eta|^{-\frac{1}{2}}\sqrt{2\pi}(1+\epsilon_\eta),$$

the integral is convergent, provided the imaginary part of a is numerically less than $\frac{1}{2}\pi$.

By a simple application of Cauchy's theorem we obtain the formula

$$(7)\qquad P\int_{-\infty i}^{\infty i} \Gamma(-t)\,e^{-at}\,dt = 2\pi i\,(e^{-e^{-a}} - \tfrac{1}{2}).$$

This may also be put in the forms

$$(8)\qquad P\int_{-\infty}^{\infty} \Gamma(i\eta)\,e^{ai\eta}\,d\eta = 2\pi\,(e^{-e^{-a}} - \tfrac{1}{2}),$$

$$(9)\qquad P\int_{-\infty}^{\infty} \Gamma(i\eta)\,u^{-\lambda i\eta}\,d\eta = 2\pi\,(e^{-u^{\lambda}} - \tfrac{1}{2}).$$

If $u = re^{i\theta}$, we must, in (9), have $|\lambda\theta| < \frac{1}{2}\pi$. The formulæ (7) and (9) are particular and slightly exceptional cases of the formulæ

$$\int_{-\infty i-\kappa}^{\infty i-\kappa} \Gamma(-t)e^{-at}\,dt\dagger = 2\pi i e^{-e^{-a}} \quad (\kappa > 0),$$

$$\int_{-\infty}^{\infty} \Gamma\left(i\eta + \frac{s-1}{\lambda}\right) u^{-\lambda i\eta}\,d\eta = 2\pi u^{s-1}e^{-u^{\lambda}},$$

where λ and $s-1$ are real and positive.

14. Now multiply (9) by $\dfrac{u^{a-1}}{x+u}$, where $0 < R(a) < 1$, and integrate from $u = 0$ to $u = \infty$. Since

$$\int_0^{\infty} \frac{u^{a-1-\lambda i\eta}}{x+u}\,du = x^{a-1-\lambda i\eta}\,\frac{\pi}{\sin(a-\lambda i\eta)\,\pi},$$

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* $\Gamma(-t)$ becomes infinite for $t = 0$ like $-1/t$. I have considered the theory of "principal values" in great detail in three papers in these *Proceedings* (Vol. XXXIV., p. 16; Vol. XXXIV., p. 55; and Vol. XXXV., p. 81).

† This formula is due to Mellin, *Acta Soc. Fennicæ*, Vol. XXIX., 4, p. 41.

where $x^{a-1-\lambda i\eta}$ has its principal value, we find*

$$(10) \qquad 2F_{\lambda,a}(x) = \frac{\pi x^{a-1}}{\sin a\pi} + x^{a-1} P \int_{-\infty}^{\infty} \frac{\Gamma(i\eta)\, x^{-\lambda i\eta}}{\sin(a-\lambda i\eta)\pi}\, d\eta.$$

This is our fundamental equation. It has been proved on the assumptions that $\lambda > 0$, $-\pi < am.x < \pi$, and $0 < R(a) < 1$.

15. Another easy application of Cauchy's theorem shows that

$$(11) \qquad P\int_{-\infty}^{\infty} \frac{\Gamma(i\eta)\, x^{-\lambda i\eta}}{\sin(a-\lambda i\eta)\pi}\, d\eta = \frac{1}{i} P \int_{-\infty i}^{\infty i} \frac{\Gamma(-t)\, x^{\lambda t}}{\sin(a+\lambda t)\pi}\, dt$$

$$= 2\pi \left\{ \tfrac{1}{2} \operatorname{cosec} a\pi - \lim_{R=\infty} S_R \right\},$$

where S_R denotes the sum of the residues of the subject of integration for which the real part of t is positive and $|t|$ less than R, and R tends to infinity in such a way that the circle $|t| = R$ never passes at less than a certain fixed distance from any pole of the subject of integration. Hence

$$(12) \qquad F_{\lambda,a}(x) = \pi x^{a-1} (\operatorname{cosec} a\pi - \lim_{R=\infty} S_R).$$

15. Let us suppose first that λ is *rational*; and we may without loss of generality suppose it *integral*. For, if $\lambda = \alpha/\beta$, α and β being integral, we find on transforming the integral which expresses $F_{\lambda,a}(x)$ by the substitution $u = w^{\beta}$, and splitting up $1/(w^{\beta}+x)$ into partial fractions, that $F_{\alpha/\beta,a}(x)$ may be expressed as the sum of a finite number of functions of the type $A_s F_{\alpha,a\beta}(x_s)$.

We may remark further that the formulæ of § 14 were proved on the assumption $0 < R(a) < 1$. The case in which $R(a) = 1$ will be considered later. Those cases in which $R(a) > 1$ may be reduced to these two cases by means of the formula

$$(13) \qquad F_{\lambda,a}(x) = \frac{1}{\lambda} \Gamma\left(\frac{a-1}{\lambda}\right) - xF_{\lambda,a-1}(x).$$

We shall therefore at present confine ourselves to the case in which λ is an integer and $0 < R(a) < 1$.

16. There can be no *double* poles of the subject of integration in (11), for this would require $0 < a < 1$ and $p = a + \lambda q$ for integral values of p

* The inversion of the order of integration is easily justified.

and q, which is manifestly impossible. Hence we find that

$$(14) \qquad S_R = \Sigma \frac{(-)^{m-1} x^{\lambda m}}{m! \sin(a+\lambda m)\pi} + \frac{1}{\lambda\pi} \Sigma (-)^n \Gamma\left(-\frac{n-a}{\lambda}\right) x^{n-a},$$

and the two series are clearly convergent separately when prolonged to infinity. Hence

$$(15) \quad F_{\lambda, a}(x) = \pi x^{a-1} \sum_{m=0}^{\infty} \frac{(-)^m x^{\lambda m}}{m! \sin(a+\lambda m)\pi} + \frac{1}{\lambda} \sum_0^{\infty} (-)^n \Gamma\left(-\frac{n+1-a}{\lambda}\right) x^n.$$

Here $-\pi < am.x < \pi$, and x^{a-1} has its principal value. It can easily be verified that, if $\lambda = \frac{1}{2}$, $a = \frac{1}{2}$, this equation reduces to (3) of § 11.

17. It is easy to see that, if $a = 1-\epsilon+i\sigma$ $(\sigma \neq 0)$, each of the two sides of (15) is continuous for $\epsilon = 0$. Hence (15) holds for $a = 1+i\sigma$.

If $\sigma = 0$, this is still true, but we can no longer simply substitute the limiting value of a in the right-hand side. We find after a little reduction that

$$(16) \qquad F_{\lambda, 1}(x) = -\log x \,.\, e^{(-)^{\lambda+1} x^{\lambda}} + \frac{1}{\lambda} \Sigma' (-)^n \Gamma\left(-\frac{n}{\lambda}\right) x^n$$
$$- \frac{1}{\lambda} \sum_0^{\infty} \frac{(-)^{(\lambda+1)p} x^{\lambda p}}{p!} \left(\gamma - 1 - \frac{1}{2} - \dots - \frac{1}{p}\right),$$

the dash over the sign of summation denoting that it extends to all values of n except multiples of λ.

If, *e.g.*, $\lambda = 1$,

$$(17) \qquad F_{1,1}(x) = -e^x \log x - \sum_0^{\infty} \frac{x^p}{p!} \left(\gamma - 1 - \frac{1}{2} - \dots - \frac{1}{p}\right).$$

Comparing this with (17), we are led to the conclusion that

$$(18) \qquad \sum_0^{\infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{p}\right) \frac{x^p}{p!} = e^x \sum_1^{\infty} \frac{(-)^{n-1} x^n}{n.n!},$$

which is easily verified with the help of the well-known identity*

$$^pC_1 - \tfrac{1}{2}\, {}^pC_2 + \tfrac{1}{3}\, {}^pC_3 - \dots = 1 + \frac{1}{2} + \dots + \frac{1}{p}.$$

18. We may now proceed to consider the much more interesting case in which λ is *irrational*. Before doing so we may summarize our conclusions as regards the rational case as follows:—

(i.) If λ is positive and rational, and the real part of a positive,

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* Chrystal's *Algebra*, Vol. II., p. 19, Ex. 18.

the function $F_{\lambda, a}(x)$, which is partially represented by the integral (1), has two singular points, viz., 0 and ∞, and no others.

(ii.) If the x-plane is cut along the negative real axis, then in the rest of the plane

$$F_{\lambda, a}(x) = x^{a-1} G_1(x^\lambda) + G(x),$$

where G and G_1 are integral functions, and x^{a-1} and x^λ have their principal values, except in the special case in which $\lambda = \alpha/\beta$ and $a\beta$ is a positive integer, when the expansion contains logarithmic terms. In this equation $F_{\lambda, a}(x)$ is of course the principal branch of the function partially represented by the integral (1).

We may observe further that, if ξ is positive and η small, the values of $F_{\lambda, a}(-\xi+i\eta)$ and $F_{\lambda, a}(-\xi-i\eta)$ differ by a quantity whose limit for $\eta = 0$ is

$$2\pi i e^{-\xi^\lambda} \xi^{a-1},$$

and the value of

$$P\int_0^\infty \frac{e^{-u^\lambda} u^{a-1}}{u-\xi}\, du$$

is the arithmetic mean of these two values; and, finally, that every branch of the function tends to zero when x approaches infinity by any path which does not wind an infinite number of times round the origin.

19. When λ is irrational the condition for double poles is as before $p = a+\lambda q$, and this is in general impossible: it is always impossible, for example, if a is complex or real and rational, and in any case cannot occur for more than one pair of values of p and q.* We shall therefore exclude this possibility; the modifications necessary if it should occur present no difficulty.

Using Cauchy's theorem precisely as before, we find

$$(19)\quad F_{\lambda, a}(x) = \lim_{R=\infty} \left\{ \pi x^{a-1} \sum_0^{[R]} \frac{(-)^m x^{\lambda m}}{m! \sin(a+\lambda m)\pi} + \frac{1}{\lambda} \sum_0^{[\lambda R + R(a) - 1]} (-)^n \Gamma\left(-\frac{n+1-a}{\lambda}\right) x^n \right\},$$

where $[k]$ denotes as usual the greatest integer contained in k. The limit on the right-hand side is certainly determinate, but it will be clear from the first part of this paper that the series

$$(20)\qquad \sum^\infty \frac{(-)^m x^{\lambda m}}{m! \sin(a+\lambda m)\pi}, \qquad \sum^\infty (-)^n \Gamma\left(-\frac{n+1-a}{\lambda}\right) x^n$$

are not necessarily convergent.

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* If, e.g., $\lambda = \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)$, $a = \frac{1}{\sqrt{2}}$, it occurs for $p = 1$, $q = 2$, and no other pair of values.

20. If one series converges, so does the other. We can prove, as in the first part, that—

(i.) each series is convergent if a is complex, or if a is real and rational and λ *algebraic* ;

(ii.) that for suitably chosen values of λ the series may have any radius of convergence, and represent functions for which the circle of convergence is a critical line;

(iii.) that, if the series represent integral functions of x^λ and x respectively, the increase of these functions may be regular or irregular.

21. If $a = 1$, the form of the equation (19) must be modified much as in § 17. We find

$$(21)\quad F_{\lambda,1}(x) = -\left(\log x + \frac{\gamma}{\lambda}\right) + \lim_{R=\infty}\left\{\pi \sum_1^{[R]} \frac{(-)^m x^{\lambda m}}{m!\sin\lambda m\pi} + \frac{1}{\lambda}\sum_1^{[\lambda R]}(-)^n\Gamma\left(-\frac{n}{\lambda}\right)x^n\right\}.$$

The series
$$\Sigma\frac{(-)^m x^{\lambda m}}{m!\sin\lambda m\pi}$$
is one of those to which the first part of this paper was devoted. The case in which its radius of convergence is finite is particularly interesting, it being not a little curious to find that the sum of two functions of x and x^λ respectively, whose region of existence is bounded by the circle $|x| = R$, should be a function of x, which exists all over the plane.

Many other pairs of series can be constructed possessing similar properties. For example,
$$\Sigma\frac{(-)^m x^{m\lambda}}{\sin m\lambda\pi} + \frac{1}{\lambda}\Sigma\frac{(-)^n x^n}{\sin\dfrac{m\pi}{\lambda}}$$
is such a pair. In this case the radius of convergence of either series cannot possibly be greater than 1, and is equal to 1 if λ is algebraic; and the circle of convergence (whatever its radius) is a *coupure* for the function represented by either series. But the two series taken together in a manner analogous to that indicated by (21) in any case converge throughout the interior of the circle $r = 1$ (supposed cut along the negative real axis).

22. When the series in (21)—or in (19)—are not separately convergent, we have an interesting example of the phenomenon of a "pair

of series" which only converge when taken together in the manner specified in those equations. Such "pairs of series" have been considered before by Prof. Lerch and myself,* but, so far as I know, not in connection with the question of the representation of an analytic function in the neighbourhood of a singular point.

When the series diverge separately for *all* values of x the equation (21) serves none the less to determine the behaviour of $F_{\lambda,1}(x)$ near the origin.† Thus $F_{\lambda,1}(x)$ behaves like

$$-\left(\log x + \frac{\gamma}{\lambda}\right), \tag{22}$$

and we can determine the way in which any given differential coefficient for $F_{\lambda,1}(x)$ behaves near $x=0$. For instance, $F^{(n)}_{\lambda,1}(x)$ behaves like

$$\frac{(-)^n n!}{\lambda}\Gamma\left(-\frac{n}{\lambda}\right)+\frac{(-)^n(n-1)!}{x^n} + \pi\sum_1^p \frac{(-)^m}{m!}\,\frac{\lambda m(\lambda m-1)\ldots(\lambda m-n+1)}{\sin m\lambda\pi}\,x^{\lambda m-n}, \tag{23}$$

where p is the greatest integer such that $\lambda p - n < 0$. In fact the information furnished in this case is really not less complete than in the case in which the series are separately convergent.

22. We can write (21) in the form

$$F_{\lambda,1}(x) = -\left(\log x+\frac{\gamma}{\lambda}\right) + \lim\left\{\pi\sum_1 \frac{(-)^m x^{\lambda m}}{m!\sin\lambda m\pi} - \frac{\pi}{\lambda}\sum_1 \frac{(-)^n x^n}{\Gamma\left(1+\frac{n}{\lambda}\right)\sin\frac{n\pi}{\lambda}}\right\}$$

$$(-\pi \leqslant am.\,x \leqslant \pi).$$

The function $F_{\lambda,1}(x)$ therefore satisfies the two difference equations

$$\begin{cases} F(\xi e^{\pi i}) - F(\xi e^{-\pi i}) = -2\pi i\, e^{-\xi^\lambda}, \\ F(\xi e^{\pi i/\lambda}) - F(\xi e^{-\pi i/\lambda}) = -\dfrac{2\pi i}{\lambda} E_{1/\lambda}(-\xi). \end{cases} \tag{24}$$

* See a paper "On certain Series of Discontinuous Functions connected with the Modular Functions" (*Quart. Jour. of Math.*, Vol. XXXVI., p. 93), where references are given.

† Our previous conclusions as to its behaviour elsewhere are still valid.

where $E_a(\xi)$ is Mittag-Leffler's function*

$$E_a(\xi) = \sum_0^\infty \frac{\xi^n}{\Gamma(an+1)}.$$

The latter equation may be verified by means of the expression of $E_a(\xi)$ as a contour-integral given by Mittag-Leffler.†

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* *Acta Math.*, t. xxix., p. 101.

† A full investigation of the properties of this most interesting function will be found in Mittag-Leffler's memoir quoted above and in two memoirs by A. Wiman in the same volume of the *Acta Mathematica*. I may mention incidentally (though it has no connection with the subject of this paper) that the function $E_a(x)$ gives (for suitable values of a and s) an interesting generalisation of Heine's contour integral for the gamma function, viz.,

$$\frac{i}{2\pi}\int E_a(-u)(-u)^{s-1}\,du = \frac{1}{\Gamma(1-as)},$$

the contour being the same as in Heine's formula, to which the above equation reduces for $a = 1$.