

will best prepare our pupils to work with a degree of accuracy and understanding the simple problems he is to meet in his science work.

LIMITS IN GEOMETRIC FORMS.

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The current doctrine of the text-book world regards the straight line and the circle as two essentially different things. The straight line is regarded as the limit toward which the circle tends, but which it never reaches. So also the circle is regarded as the limit toward which the regular polygon tends, but which it *never* reaches. The theorems regarding the circumference and the area of a circle are derived on the supposition that the circle is the limit which the regular polygon almost but never quite reaches, and that the error is negligible. But we always have the reservation that the circle is *not* a polygon, say what you will, and that there *is* an error, however small it may be; less than any assignable quantity, but yet an error after all. The difference between the circle and polygon is so small that for all practical purposes we may consider them as one; but, of course, they are not one, and never can be, etc., etc.

And through all this array of verbiage, we feel that there is a fallacy somewhere; it is and it is not, all in the same breath; the error is inexpressible and yet the forms do not coincide. We can push the polygon *almost* to the circle; what is that invisible barrier which keeps it back?

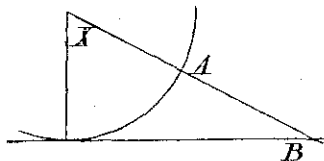
There is no barrier except our own narrow definitions and methods. The straight line is a circle of special form, not the limit of a circle; the circle is a polygon, not the limit of a polygon. There is no residual error. The circle straightens out into a straight line and sweeps over it into a circle on the other side. The inscribed polygon merges into the circle and sweeps over it into a polygon again on the outside.

Let us see what is meant by a limit, and why forms have limits. A limit is that constant value (or form) which a variable value (or form) approaches indefinitely near but never quite

reaches. The test of a limit is $r - x = 0$, and $r - x < i$, where i is any assignable infinitely small quantity.

The subject of limits as taught in the elementary text-books is very crude, and fogged with lack of perspective. In the first place, no distinction is made between the limit in the case of geometric forms (continua) and in numbers (discreta). The two cases are quite different, and the distinction must be recognized.

It seems to be a rule that geometric forms have or have not a limit, dependent entirely upon the method of generation; one method of generation having a limit and the other not, for the same variable. For example, if the angle X is generated by the motion of the intersection A , it has no limit; but if by the movement of the intersection B , its limit is a right angle.



So likewise, if we generate the arc x by the swelling of a cart-wheel rim, the limit is a straight line. But if we generate it by the tracing point of a Peaucellier linkage, it has no limit; it straightens out into a straight line and then curves the other way. In both these cases it is the same variable, a line of constant curvature. The elementary text-books blindfold their readers with a, not necessarily faulty, but narrow definition: A circle is a line which always changes its direction, and a straight line is one which does not change, etc. And then triumphantly ask how one can be the other. Throw away the blinders and get a broader view by taking a broader definition; viz., a line of constant curvature, and the contradiction ends.

The old contradiction between a tangent and a secant has begun its evanescence, by considering the tangent as a secant cutting in two coincident points, one double point. But when it comes to swelling an inscribed polygon into a circle, then, they say, the law laid down above fails, since there is no instrument to do the swelling, and however far you continue the process, there are points of the circumference yet unoccupied by the vertices of the polygon. The same objection would have been made in the case of the circle and the straight line previous to 1864, when Peaucellier invented his linkage, showing that the question of a limit does

not depend upon the inventiveness of man; but only our appreciation. Previous to 1864, such an instrument could have been imagined *in nubibus*, and the same argument used as here, and the argument would have been just as sound. The only difference would have been its effect upon the hearers.

Let us look at some examples of limits. (a.) A point moving half the remaining distance between it and its goal each second, when will it reach the goal? Never, because between it and the goal will ever remain the half of some distance. (b.) A point moving away half the distance between it and a pursuing point each instant of time, when will the pursuing point catch the other? Never, because the pursued point is always the half of some distance ahead. But this is nonsense, for a pursuing point moving twice as fast as the pursued can overtake it, as witness the minute hand of a clock and the hour hand.

Now where is the fallacy?

In (a) we have an infinite number of operations stretching out over an infinite number of seconds and therefore never ended. In (b) we have an infinite number of operations crowded into a limited time and therefore completed some time. In (a) the succession of events is regular but the speed of the moving point is decreasing to infinite slowness. In (b) the speed of the moving point is regular, but the succession of events is increasing to infinite rapidity.

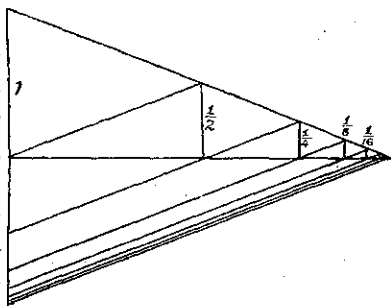
This shows that the same variable (the distance passed over by a point moving one-half the preceding distance at each operation) may or may not have a limit according to the special law governing its generation. The introduction of a *timed succession* of events produces a decreasing speed and a limit. A *timed* (constant or finite) speed produces an increasing rapidity of succession of events and no limit.

A horse straining at his halter finds the distance between him and the door diminished one half each second. Will he get out? Never! A horse straining at his halter finds the distance between him and the door diminished one half at each instant of strain. Can he get out? Certainly! He walks right out the door, just as the minute hand passes the hour hand. In the first case there is a *timed succession* of events. In the second there is a *continuous and steady strain*, a *timed rate of progress*—finite speed.

On the other hand, when we come to the summation of the terms of a series, the introduction of the discrete terms seems to take the place of a timed succession of events, and the series has a limit, if convergent.

An illustration of the difference between the summation of *discreta* and *continua* is given in the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$. If we consider these terms as ordinates erected at finite intervals, the summation has a limit, 2.

But if we crowd the ordinates into a triangle as shown, the sum is easily seen by reason of the similar triangles to be exactly 2; and there is no unattainable limit, no residual error. In each case we have dealt with *exactly the same ordinates*; in one case *arranged* so as to have a limit to



the sum, in the other case no limit. Which result we shall get is merely a question of arrangement. In this instance the sum of an infinite converging series is a real quantity and not an elusive limit just out of reach. The limit is the *limit of the process* and not necessarily any *intrinsic property of the variable*.

If we imagine an inscribed polygon swelled toward the circle by doubling the number of sides, etc., the circle seems to be the limit of the operation, for the very process of doubling introduces the timed succession of events which results in a limit. But imagine a process which forced each center of a chord (inscribed square) into a symmetrical position (i. e. on to the circumference of a circle through the undisturbed points) and imagine this kept up at an even speed of surface change. The succession of events increases to infinite rapidity and the inscribed polygons sweep through the circle into circumscribed polygons. The newly produced vertices are arranged on the circumference of the initial circumscribed circle until the circle itself is reached, and then they arrange themselves on circles (of increasing size), the alternate vertices being forced out until the undisturbed ones evanesce on a straight line, and the polygons become of lessening number of sides until the circumscribing square is reached and the process repeats itself into a new circle around this new square, and so on.

Instead of saying "alternate vertices," etc., we might say reversal of the process which produces the circle from the circumscribing square by forcing the vertices inward symmetrically until they evanesce on a straight line, the newly produced vertices being symmetrically arranged. This process kept up at an even speed of surface change sweeps the polygons through the circle and by reversal of the swelling process, into the inscribed square, and so on through a new cycle.

In imagination we can see the polygons swelling into other polygons, the transition figuring between the sets being the circle, as the parabola is the transition curve between the ellipse and the hyperbola. If we imagine the lines to be general lines of infinite length, we can see, the plane, initially crossed by the four bundles of lines, gradually becoming more and more crossed until finally the whole plane is crossed and recrossed with lines, except the central circular portion, which is sharply delimited from the rest by the circular boundary which separates the shaded from the unshaded portion. As the process goes on we see the lines coalescing again, the plane becoming less darkened until finally we arrive again at a square, and the process begins over again.

That we have no mechanism for producing these results is of no importance. Until 1864 we had no mechanism which would enlarge the circle into a straight line. Nor have we any now for sweeping the ellipse through its transition curve, the parabola, into the hyperbola. It does not seem likely that we will have. Nor did the Peaucellier linkage seem likely at one time.

To conform to all this, the narrow definition of the polygon and circle must be enlarged. We must define a polygon as a configuration of lines; a regular polygon as a symmetrical configuration, one of the special cases of which is a circle, a symmetrical configuration of an infinite number of lines.

The gist of the matter seems to be that speaking of the limit of geometric forms is merely another way of advertising the fact that we have adopted a process which produces a limit. The limit is *the result of the process* and *not an intrinsic property of the form*. Some other process could avoid the limit. A frequent illustration of this is the historic problem of squaring the circle. In numbers this is impossible because, among other reasons, it is the limit of an infinite series of discrete terms. In geometry, with a

ruler and compass the length of the circumference is also the limit of an infinite series of operations and therefore unattainable. But change the process by using the integrator, and what was before a limit and just out of reach, becomes attainable and we get a line equal to the circumference.

THE MATHEMATICAL HANDBOOK OF AHMES.

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The Handbook of Ahmes was written about 1700 B. C.—more than a thousand years before the beginning of the classic period of Greek mathematics. It stands as an isolated peak in the history of mathematics and practically marks the beginning of this history. It seems to have been written as a compendium of useful and curious mathematical facts for the learned Egyptian priests living at about the time when the Israelites were slaves in their country.

The book is replete with facts of the greatest interest, not only to the students of mathematics, but also to those who are interested in the history of the development of the human intellect. Even the title of the book is naïve. It is as follows: "Directions for obtaining a knowledge of all dark things * * * of all secrets which are involved in the objects." This title gives evidence of the ancient belief in the power and comprehensiveness of mathematical knowledge, and is comparable with the much more recent saying of Isidorus, bishop of Seville, who expressed his admiration of number in his encyclopedia in the following words: "Take away number from all things and everything goes to destruction."

The five parts of the Handbook of Ahmes are devoted respectively to the following subjects: Arithmetic, stereometry, geometry, calculation of pyramids, collection of practical examples. The first part begins with a table in which the forty-eight fractions having two for a numerator and the odd numbers from 5 to 99 as denominators are expressed as sums of different fractions having unity for their common numerator. The table is curious that we reproduce it here, omitting only the verification which was given with each fraction.

This table is of great historical importance. It appears that no