which is the same as

$$v = (p''x+q''y+r''z)f(u)+\Delta f'(u),$$

when we make $$p'^2+q'^2+r'^2 = 1.$$

The other forms investigated will be the same as those given by Sommerfeld in a paper "On Diffraction," in the *Mathematische Annalen* (t. XLVII., 1896), in the form of contour integrals, where they are deduced by a stereographic projection.

The Irreducible Concomitants of any Number of Binary Quartics.

By A. Young. Received and read February 9th, 1899.

The irreducible system is here arrived at by first finding the irreducible system of types and then the number of independent forms belonging to each type for a system of N quartics. Two concomitants are said to be of the same type when they can be obtained from the same form by polarization. For the purpose of discussing the system of types, a type is regarded as being of the first degree in the coefficients of each of the quantics concerned. The finiteness of the irreducible system of types has been established by Prof. Peano.* He proves that the complete system of concomitants for any number of binary n-ics may be obtained from the system for n n-ics by polarization alone; with the one possible exception of invariants of the type

$$\begin{vmatrix} A_0 & A_1 & \dots & A_n \\ B_0 & B_1 & \dots & B_n \\ \dots & \dots & & \dots \\ K_0 & K_1 & \dots & K_n \end{vmatrix}.$$

In other words, every type of a binary n-ic which furnishes no irreducible form for n n-ics is reducible, with the possible exception just mentioned. It was with the help of this proposition that some of the reductions for the quartic were first arrived at; however, other

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* *Atti di Torino*, t. XVII., p. 580.

methods have proved shorter. The latter part of his paper is devoted to the discovery of the cubic types. From the fact that

$$\begin{vmatrix} A_0 & A_1 & A_2 & A_3 \\ B_0 & B_1 & B_2 & B_3 \\ C_0 & C_1 & C_2 & C_3 \\ D_0 & D_1 & D_2 & D_3 \end{vmatrix}$$

is reducible, it is shown that all the types occur in the system for two cubics. His results are—

The irreducible system for N cubics belongs to 10 types, as follows:—

	Type.	Simplest Form.	Number of Forms.
I.	${}_3C_1$	One of the cubics	N
II.	${}_4C_2$	Jacobian of two cubics	$\binom{N}{2}$
III.	${}_2C_2$	Hessian of one cubic	$\binom{N+1}{2}$
IV.	I_2	Third transvectant of two cubics	$\binom{N}{2}$
V.	${}_3C_3$	Covariant order 3 of one cubic	$\binom{N+2}{3}$
VI.	${}_1C_3$	Second transvectant of I. and III.	$2\binom{N+1}{3}$
VII.	I_4	Discriminant of one cubic	$\binom{N+3}{4}$
VIII.	${}_2C_4$	Jacobian of two forms III.	$3\binom{N+2}{4}$
IX.	${}_1C_5$	First transvectant of III. and VI.	$4\binom{N+3}{5}$
X.	I_6	Resultant of two forms VI.	$\binom{N+4}{6}$

For the quartic, I have first expressed the types in symbols based on the quadratic. To do this, it is proved that the types of a binary mn-ic can be expressed in symbols based on the n-ic; the symbolical factors being of the form of n-ic types; just as, in ordinary symbolical algebra, the concomitants of the m-ic are expressed in symbols based on the linear form. It is easy then to show that there is only one type to be considered, of given degree and order. Writing this $(abc \dots k)$, the fundamental identities give relations of the form

$$(1+S_1+S_2+\dots+S_k)(abc \dots k) = R,$$

where R stands for reducible terms, and $S_1, S_2, \dots, S_k$ are certain substitutions.

The chief advantage obtained from quadratic symbols lies in the possibility of using symbolical operators, with the help of which relations between forms of one degree and order may be obtained from relations between forms of the same order but of one degree lower.

The invariant type of highest degree I_6 has been expressed in terms of determinants of five rows and columns; by means of this a number of syzygies may be at once written down, in fact $I_6 P$ equals a sum of products of forms, there being at least three forms in each product, where P is an irreducible form of any type except I_2 and I_3.

1. Consider any simultaneous system of binary nm-ics,

$$(A_0, A_1, \dots, A_{mn} \between x_1, x_2)^{mn} \equiv a^n_{x^m},$$

$$(B_0, B_1, \dots, B_{mn} \between x_1, x_2)^{mn} \equiv b^n_{x^m},$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

where $$a_{x^m} \equiv (a_0, a_1, \dots, a_m \between x_1, x_2)^m,$$

and the identities are taken to define the relations between the symbolical letters $a_0, a_1, \dots,$ and the actual coefficients. Let $f(A, B, \dots, K)$ be a type belonging to this system; writing in this for $A_0, \dots$ their values in terms of the symbolical letters, $f(A, B, \dots, K)$ takes the form $\phi(a, b, \dots, k)$, say. Now make any linear transformation, and denote by dashed letters the coefficients of the transformed quantics; then

$$f(A', B', \dots, K') = \mu f(A, B, \dots, K),$$

where μ is a power of the determinant of transformation; hence also

$$\phi(a', b', \dots, k') = \mu\phi(a, b, \dots, k).$$

Therefore ϕ is a concomitant of the m-ics $a_{x^m}, b_{x^m}, \dots, k_{x^m}$; and hence ϕ can be expressed as a sum of products each factor of which is of the same form as an irreducible type of the m-ic.

It is necessary now to show how to proceed from a symbolical product P to the form F in the actual coefficients, which it represents. Let

$$a_{x^m} \equiv \alpha_x^m, \quad b_{x^m} \equiv \beta_x^m, \; \dots;$$

then the same result will be obtained by writing in F the coefficients of the first quantic in terms of $a_0, a_1, \dots, a_m$, and then putting

$$a_r = \alpha_1^{m-r}\alpha^r$$

as by writing

$$A_r = \alpha_1^{mn-r}\alpha_2^r$$

directly in that type. Hence the result of writing in P

$$a_r = \alpha_1^{m-r}\alpha_2^r, \; \dots, \; b_r = \beta_1^{m-r}\beta_2^r, \; \dots$$

is the same mn-ic type expressed in linear symbols; the step from these to the actual coefficients presents no difficulty. As a matter of fact, an mn-ic type, when expressed in m-ic symbols, will rarely consist of a single symbolical product; still, given any symbolical product P—of the right degree in the symbols—a type F of the mn-ic may be arrived at, in general, with the help of linear symbols, which is such that when expressed in m-ic symbols it becomes $P+Q$, where the effect of substituting linear for m-ic symbols is to make Q vanish.

Hence products which vanish when the change is made to linear symbols may be ignored, and every other symbolical product of the right degree in the symbols may be regarded as giving a type of the mn-ic.

Let $(a, b, c, \dots)$ be an m-ic type; then among the factors which may occur are forms like $(a, a, c, \dots)$; a factor of this kind may be reducible, or when linear symbols are introduced it may vanish; in either case there is no need to consider it.

The lineo-linear type (a, b) is a case in point; a product with a factor (a, a) may always be ignored.

2. The types of a quadratic are—

$$a_0b_2+a_2b_0-2a_1b_1 \equiv [ab], \quad a_0x_1^2+2a_1x_1x_2+a_2x_2^2 \equiv a_{x^2},$$

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{vmatrix} \equiv |\, abc \,|, \quad \begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ x_2^2 & -x_1x_2 & x_1^2 \end{vmatrix} \equiv |\, abx^2 \,|;$$

no symbolical factor in quadratic symbols need be considered in which the same letter occurs twice.

The fundamental identities are

$$2\,|\,abc\,|\;|\,def\,| = \begin{vmatrix} [ad] & [ae] & [af] \\ [bd] & [be] & [bf] \\ [cd] & [ce] & [cf] \end{vmatrix}, \qquad \text{I.}$$

$$[ae]\,|\,bcd\,| + [be]\,|\,cad\,| + [ce]\,|\,abd\,| = [de]\,|\,abc\,|, \qquad \text{II.}$$

$$\begin{vmatrix} [ab] & [ad] & [af] & [ah] \\ [cb] & [cd] & [cf] & [ch] \\ [eb] & [ed] & [ef] & [eh] \\ [gb] & [gd] & [gf] & [gh] \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & a_2 & 0 \\ c_0 & c_1 & c_2 & 0 \\ e_0 & e_1 & e_2 & 0 \\ g_0 & g_1 & g_2 & 0 \end{vmatrix} \begin{vmatrix} b_2 & -2b_1 & b_0 & 0 \\ d_2 & -2d_1 & d_0 & 0 \\ f_2 & -2f_1 & f_0 & 0 \\ h_2 & -2h_1 & h_0 & 0 \end{vmatrix} = 0. \qquad \text{III.}$$

The identities for the forms a_{x^2}, $|\,abx^2\,|$ may be obtained at once from these, since $[ad]$, $|\,abd\,|$ become respectively a_{x^2}, $|\,abx^2\,|$ if x_2^2, $-x_1x_2$, x_1^2 are written for d_0, d_1, d_2.

From I. it follows that no product need be discussed in which more than one factor of either of the forms $|\,abc\,|$, $|\,abx^2\,|$ occurs. In quartic types each symbolical letter must occur twice in a product; hence, if there is a factor $|\,abc\,|$, there is also a factor e_{x^2}; but, from II.,

$$|\,abc\,|\,e_{x^2} = [ae]\,|\,bcx^2\,| + [be]\,|\,cax^2\,| + [ce]\,|\,abx^2\,|;$$

hence, for the quartic, factors $|\,abc\,|$ need not be considered. The invariant forms then are

$$[ab]^2,$$
$$[ab][bc][ca],$$
$$[ab][bc][cd][da],$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots.$$

We shall find it convenient to use the notation

$$(abca) \equiv [ab][bc][ca],$$
$$(abcda) \equiv [ab][bc][cd][da],$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots;$$

then $$(abcd \ldots ka) \equiv (bcd \ldots kab) \equiv (ak \ldots dcba).$$

The general form of covariant type of order 2 is

$$[ab][bc]\ldots[hk]\ |\ kax^2\ |,$$

which will be denoted by $(ab\ldots hk)$, and here

$$(ab\ldots hk) = -(kh\ldots ba).$$

Covariants order 4 are of the form

$$[ab][bc]\ldots[hk]\,h_{x^2}k_{x^2} \equiv (abc\ldots hkx^2a);$$

and so on.

3. *The Invariants.*—Multiply III. by $[bc][de][fg][ha]$, and expand; then

$$(abcdefgha)+(afgbcdeha)+(adefgbcha)+(abcfgdeha)+(afgdebcha)$$
$$+(adebcfgha) = R,$$

where R is used to express reducible terms, and terms which have factors of the form $[aa]$. The above may be conveniently written

$$[a,\ bc,\ de,\ fg,\ ha] = R.$$

From the identity

$$\begin{vmatrix} [ab] & [ac] & [ad] & [aa] \\ [bb] & [bc] & [bd] & [ba] \\ [cb] & [cc] & [cd] & [ca] \\ [db] & [dc] & [dd] & [da] \end{vmatrix} = 0,$$

we deduce

$$[a\ldots a] \equiv (abcda)+(adbca)+(acdba)+(abdca)+(adcba)+(acbda)$$
$$= R.$$

The other relations obtainable may be written

$$[a\ldots ea] = R,\quad [a,\ bc,\ d,\ e,\ a] = R,\quad [a,\ bc,\ de,\ f,\ a] = R,\ \&c.$$

The types I_2 and I_3, viz., $[ab]^2$ and $(abca)$, are unaffected by the fundamental identity; hence for N quartics there are $\binom{N+1}{2}$ and $\binom{N+2}{3}$ independent irreducible invariants respectively of these types.

For I_4 there is one equation,

$$[a\ldots a] = R,$$

or

$$2\,(abcda)+2\,(adbca)+2\,(acdba) = R.$$

There are then only two independent forms having the same letters. Let $(A^pB^q \ldots)\, I_k$ denote an invariant of the type I_k which is of degree p in the coefficients of the quartic A, of degree q in the coefficients of the quartic B, and so on. Then there are two independent irreducible forms $(ABCD)\, I_4$, one form $(ABC^2)\, I_4$, and one form $(A^2B^2)\, I_4$; in other cases the invariant is reducible. Hence for N quartics there are $\binom{N}{2}+3\binom{N}{3}+2\binom{N}{4}=\binom{N+2}{4}+\binom{N+1}{4}$ invariants of the type I_4.

There are twelve possible forms $(ABCDE)\, I_5$, and ten relations between them, all of which are of the form

$$[ab \ldots a] = R.$$

Simpler relations are obtained thus:—

$$[ab \ldots a]+[ae \ldots a]-[ac \ldots a]-[ad \ldots a]$$
$$\equiv 2\,(abcdea)+2\,(abdcea)-2\,(acbeda)-2\,(acebda) = R. \qquad \text{IV.}$$

In connexion with this type consider the form

$$|\, ABCDE \,| \equiv \begin{vmatrix} A_0 & A_1 & A_2 & A_3 & A_4 \\ B_0 & B_1 & B_2 & B_3 & B_4 \\ C_0 & C_1 & C_2 & C_3 & C_4 \\ D_0 & D_1 & D_2 & D_3 & D_4 \\ E_0 & E_1 & E_2 & E_3 & E_4 \end{vmatrix}$$
$$= (\alpha\beta)(\alpha\gamma)(\beta\gamma)(\alpha\delta)(\beta\delta)(\gamma\delta)(\alpha\epsilon)(\beta\epsilon)(\gamma\epsilon)(\delta\epsilon).$$

where $\quad a_x^4 \equiv (A_0,\, A_1,\, A_2,\, A_3,\, A_4 \between x_1,\, x_2)^4 \equiv \alpha_x^4$, &c.

Using the identity

$$3\,(\alpha\beta)(\alpha\gamma)(\beta\gamma)(\alpha\delta)(\beta\delta)(\gamma\delta) = -(\alpha\beta)^3(\gamma\delta)^3-(\alpha\gamma)^3(\delta\beta)^3-(\alpha\delta)^3(\beta\gamma)^3,$$

we obtain $\qquad 6\,|\, ABCDE \,|$

$$= -2\,(\alpha\beta)^3(\gamma\delta)^3(\alpha\epsilon)(\beta\epsilon)(\gamma\epsilon)(\delta\epsilon)-\ldots$$
$$= -(\alpha\beta)^2(\gamma\delta)^2 \begin{vmatrix} (\alpha\gamma)^2 & (\alpha\delta)^2 & (\alpha\epsilon)^2 \\ (\beta\gamma)^2 & (\beta\delta)^2 & (\beta\epsilon)^2 \\ (\epsilon\gamma)^2 & (\epsilon\delta)^2 & 0 \end{vmatrix} - \ldots$$
$$= (abedca)+(abdcea)+(acebda)+(acbdea)+(adecba)+(adcbea)$$
$$-(abecda)-(abcdea)-(acedba)-(acdbea)-(adebca)$$
$$-(adbcea)+R,$$

since $(abcdea)$ and $(\alpha\beta)^2(\beta\gamma^2)(\gamma\delta^2)(\delta\epsilon^2)(\epsilon\alpha)^2$ represent the same type.

Hence, from equation IV.,

$$
\begin{aligned}
&(abdcea)-(acbeda)+R = (acebda)-(abcdea)+R \\
&= (acbdea)-(adceba)+R = (adecba)-(acdbea)+R \\
&= (adcbea)-(abdeca)+R = (abedca)-(adbcea)+R \\
&= |\,ABCDE\,| \\
&= \tfrac{1}{3}\{(abdcea)+(acebda)+(acbdea)+(adecba)+(adcbea) \\
&\qquad\qquad +(abedca)\}+R, \qquad \text{V.}
\end{aligned}
$$

this last in virtue of

$$[ab\ldots a]+[ac\ldots a]+[ad\ldots a]+[ae\ldots a]=R.$$

These relations include all those from which we started; further they are independent; hence, there are six independent irreducible invariants $(ABCDE)\ I_5$. For N quartics the number of independent irreducible forms I_5 is

$$6\binom{N}{5}+8\binom{N}{4}+3\binom{N}{3}=3\binom{N+2}{5}+2\binom{N+1}{5}+\binom{N}{5}.$$

For I_6 there are two systems of equations, viz.,

$$[abc\ldots a]=R \quad\text{and}\quad [a,\ bc,\ d,\ e,\ fa]=R.$$

A third equation, which, though not independent of these, is useful, may be found thus:

$$2\,|\,abc\,|\ |\,def\,| = \begin{vmatrix} [ad] & [ae] & [af] \\ [bd] & [be] & [bf] \\ [cd] & [ce] & [cf] \end{vmatrix};$$

expand the determinant and square both sides

$$
\begin{aligned}
4\,|\,abc\,|^2\ |\,def\,|^2 &= R+(adbecfa)+(afbdcea)+(aebfcda) \\
&\quad +(adbfcea)+(afbecda)+(aebdcfa) \\
&= R+[a-b-c-a], \text{ say;}
\end{aligned}
$$

then $\qquad [a-b-c-a]=R.$

This last is the sum of two equations, for, if S be the sum of the sixty possible forms $(ABCDEF)\ I_6$,

$$[ad, be, c, f, a]+[af, bd, c, e, a]+[ae, bf, c, d, a]$$
$$+[a, d, be, cf, a]+[a, f, bd, ce, a]+[a, e, bf, cd, a]$$
$$+[ad, b, e, cf, a]+[af, b, d, ce, a]+[ae, b, f, cd, a]$$
$$+[abc \dots a]+[acb \dots a]+[ab \dots ca]$$
$$-S+[a-b-c-a]$$
$$= 6\left[(adbecfa)+(afbdcea)+(aebfcda)\right] = R, \qquad \text{VI.}$$

since S is reducible, it being the sum of the ten expressions $[a-b-c-a]$.

The results of operating with $[ef]\left[f_0\frac{\partial}{\partial e_0}+f_1\frac{\partial}{\partial e_1}+f_2\frac{\partial}{\partial e_2}\right]$ on $[ea]$, $[ee]$ and $(abcdea)$ are $[ef][fa]$, $[ef]^2$, and $(abcdefa)+(abcdfea)$ respectively. Hence from any relation $\Sigma I_5 = R$ there may be at once deduced one of the form $\Sigma I_6 = R$. The above operation will be found equivalent to substituting the coefficients of that covariant of two quartics which is of the same type as the Hessian for the coefficients of a single quartic.

Applying this operator to V., we obtain

$$[ef]\left[f_0\frac{\partial}{\partial e_0}+f_1\frac{\partial}{\partial e_1}+f_2\frac{\partial}{\partial e_2}\right] \mid ABCDE \mid$$
$$= \mid ABCD,\ EF \mid$$
$$= (abdcefa)+(abdcfea)-(acbefda)-(acbfeda)+R$$
$$= (abefdca)+(abfedca)-(adbcefa)-(adbcfea)+R$$
$$= \tfrac{1}{2}\left\{-(afdbeca)-(aedbfca)+(afbdeca)+(aebdfca)\right\}+R,$$

using equations of the form VI.

Hence $\quad (abdcefa)+(abdcfea)+(adbcefa)+(adbcfea)+R$

$$= (abefdca)+(abfedca)+(acbefda)+(acbfeda)+R.$$

If b and e, f and d be interchanged in this, the first line is unaltered; it therefore

$$= (aebdfca)+(aedbfca)+(acebdfa)+(acedbfa)+R = R,$$

as is seen by adding the three lines and using equations of the form VI.; therefore

$$
\begin{aligned}
| ABCD, EF | \\
&= \tfrac{1}{2}\{-(afdbeca)-(aedbfca)+(afbdeca)+(aebdfca)\}+R \\
&= -(afdbeca)-(aedbfca)+R = (afbdeca)+(aebdfca)+R \\
&= | ABCD, FE |.
\end{aligned}
$$

It follows that the type I_6 may be expressed in the determinant forms $| ABCD, EF |$; for

$$
\begin{aligned}
(afdbeca)+(aedbfca) &= | ABDC, EF | + R, \\
-(aedbfca)-(aecbfda) &= | EFAB, DC | + R, \\
(aecbfda)+(afdbeca) &= | DCEF, AB | + R.
\end{aligned}
$$

Therefore

$$2(afdbeca) = | ABDC, EF | + | EFAB, DC | + | DCEF, AB | + R, \qquad \text{VII.}$$

There are fifteen possible forms $| ABCD, EF |$. Written in full,

$$
| ABCD, EF | = 2 \begin{vmatrix}
A_0 & B_0 & C_0 & D_0 & E_0F_2-2E_1F_1+E_2F_0 \\
A_1 & B_1 & C_1 & D_1 & \tfrac{1}{2}(E_0F_3-E_1F_2-E_2F_1+E_3F_0) \\
A_2 & B_2 & C_2 & D_2 & \tfrac{1}{6}(E_0F_4+2E_1F_3-6E_2F_2+2E_3F_1+E_4F_0) \\
A_3 & B_3 & C_3 & D_3 & \tfrac{1}{2}(E_1F_4-E_2F_3-E_3F_2+E_4F_1) \\
A_4 & B_4 & C_4 & D_4 & E_2F_4-2E_3F_3+E_4F_2
\end{vmatrix}
$$

Amongst these forms one kind of equation exists, viz.,

$$
\begin{aligned}
| ABCD, EF | + | EABC, DF | + | DEAB, CF | \\
+ | CDEA, BF | + | BCDE, AF | = 0, \qquad \text{VIII.}
\end{aligned}
$$

as may be verified by taking the coefficients of $F_0, F_1, \ldots, F_4$ in turn. When $(abcdefa)$ is expressed in the determinant forms, $[abc \ldots a] = R$ is identically satisfied, and $[ab, cd, e, f, a] = R$ becomes the sum of two equations VIII. Hence there are no relations between the forms $| ABCD, EF |$ beyond those included in VIII. There are six relations VIII., and five are independent. Hence there are ten independent irreducible forms $(ABCDEF)\ I_6$. For a system of N quartics there are

$$10\binom{N}{6}+20\binom{N}{5}+10\binom{N}{4} = 10\binom{N+2}{6}$$

irreducible invariants of this type.

Invariant types of higher degree than 6 are reducible. For, using the operator $[fg]\left[g_0\frac{\partial}{\partial f_0}+g_1\frac{\partial}{\partial f_1}+g_2\frac{\partial}{\partial f_2}\right]$, we obtain from VII.

$$2\,(afgdbeca)+2\,(agfdbeca)$$
$$= |\,A, B, D, C, EFG\,| + |\,E, FG, A, B, DC\,| + |\,D, C, E, FG, AB\,| : \qquad \text{IX.}$$

Therefore

$$R = [a \ldots beca]$$
$$= |\,A,B,D,C,EFG\,| + |\,E,FG,A,B,DC\,| + |\,D,C,E,FG,AB\,|$$
$$+ |\,A,B,G,C,EDF\,| + |\,E,DF,A,B,GC\,| + |\,G,C,E,DF,AB\,|$$
$$+ |\,A,B,F,C,EGD\,| + |\,E,GD,A,B,FC\,| + |\,F,C,E,GD,AB\,|.$$

Interchange A and B, and add the result to the original equation

$$|\,D, C, E, FG, AB\,| + |\,G, C, E, DF, AB\,| + |\,F, C, E, GD, AB\,| = R. \qquad \text{X.}$$

Hence also

$$|\,A, B, D, C, EFG\,| + |\,A, B, G, C, EDF\,| + |\,A, B, F, C, EGD\,|$$
$$+ |\,E, FG, A, B, DC\,| + |\,E, DF, A, B, GC\,| + |\,E, GD, A, B, FC\,|$$
$$= R. \qquad \text{XI.}$$

But, from VIII.,

$$|\,A, B, D, C, EFG\,| + |\,FG, A, B, D, EC\,| + |\,C, FG, A, B, ED\,|$$
$$+ |\,D, C, FG, A, EB\,| + |\,B, D, C, FG, EA\,| = R.$$

Hence, using equations of the form X.,

$$|\,A, B, D, C, EFG\,| + |\,A, B, G, C, EDF\,| + |\,A, B, F, C, EGD\,|$$
$$= -\,|\,C, FG, A, B, ED\,| - |\,C, DF, A, B, EG\,| - |\,C, DG, A, B, EF\,|$$
$$+ R;$$

and therefore XI. becomes

$$\begin{aligned} &| E, FG, A, B, DC | + | E, DF, A, B, GC | + | E, GD, A, B, FC | \\ &= | C, FG, A, B, ED | + | C, DF, A, B, EG | + | C, DG, A, B, EF | + R \\ &= | C, FG, E, B, AD | + | C, DF, E, B, AG | + | C, GD, E, B, AF | + R \\ &= | A, FG, E, B, CD | + | A, DF, E, B, CG | + | A, GD, E, B, CF | + R \\ &= R, \end{aligned}$$

the last two equations being obtained from the first by substitutions. Interchange F and G, C and D in the last of these, and add the result to the original equation; then

$$2 \, | A, FG, E, B, CD | = R,$$

and, in virtue of the equation obtained from VIII.,

$$| A, B, C, D, EFG | = R.$$

Therefore IX. becomes

$$(afgdbeca) + (agfdbeca) = R.$$

From this it may at once be deduced that the sum of $(abcdefga)$ and any form obtained from it by a substitution formed of an odd number of transpositions is reducible.

Hence $(abcdefga) + (agfedcba) = R$ or $2\,(abcdefga) = R$,

and the type I_7 is reducible.

Operate on $(abcdefga)$ with $[gh]\left[h_0 \frac{\partial}{\partial g_0} + h_1 \frac{\partial}{\partial g_1} + h_2 \frac{\partial}{\partial g_2}\right]$; then

$$(abcdefgha) + (abcdefhga) = R. \qquad \text{XII.}$$

The equation $[ab, cd, ef, gh, a] = R$

has six terms, each obtainable from the first by means of a substitution formed of an even number of transpositions; therefore, by repeated use of XII., it gives

$$6\,(abcdefgha) = R.$$

The reduction of invariant types of higher degree follows in the same way.

4. The equations for covariants order 2 are:

(i.) Those obtained from the factors $[ab]$ only, viz.,

$$[a \ldots e] = R, \quad [a, bc, d, e, f] = R, \;\&c.$$

(ii.) Those obtained from III. by writing in that identity $h_1x_1^2+h_2x_1x_2$ for h_2, $h_0x_1^2-h_2x_2^2$ for $2h_1$, and $-h_0x_1x_2-h_1x_2^2$ for h_0, and therefore $|\,ahx^2\,|$ for $[ah]$; these are of the forms

$$[a\,...] = R,\quad [a,\, bc,\, d,\, e,] = R,\ \&c.$$

(iii.) A system of equations obtained from II., thus:

$$\begin{aligned}[ef]\,|\,abc\,|\,d_{x^2} = d_{x^2}\,&\{[af]\,|\,bce\,| + [bf]\,|\,cae\,| + [cf]\,|\,abe\,|\,\} \\ = \ &[af]\{[bd]\,|\,cex^2\,| + [cd]\,|\,ebx^2\,| + [ed]\,|\,bcx^2\,|\,\} \\ &+[bf]\{[cd]\,|\,aex^2\,| + [ad]\,|\,ecx^2\,| + [ed]\,|\,cax^2\,|\,\} \\ &+[cf]\{[ad]\,|\,bex^2\,| + [bd]\,|\,eax^2\,| + [ed]\,|\,abx^2\,|\,\}.\end{aligned}$$

Multiply this identity by $[ef][bd][ca]$; then

$$[bd,\, ca,\, fe] \equiv (bdcafe)+(cafedb)+(efbdca)+(cadbfe)+(efcadb)+(bdefca) = R.$$

Similarly the following may be deduced:—

$$[bd,\, ca,\, e] \equiv (bdcae)+(caedb)+(ebdca)+(cadbe)+(ecadb)+(bdeca) = R,$$

$$[b,\, ca,\, e] \equiv (bcae)+(caeb)+(ebca)+(cabe)+(ecab)+(beca) = R.$$

For the types ${}_2C_2$, ${}_2C_3$ there are no equations; and for a system of N quartics there are $\binom{N}{2}$ and $2\binom{N+1}{3}+\binom{N}{3}$ irreducible covariants of these two types respectively.

There are six equations like

$$[b,\, ca,\, d] = R$$

amongst the twelve forms $(ABCD)_2C_4$, all of which are independent; the equations

$$[a\,...] = R$$

are, however, deducible from these. Hence there are six independent irreducible forms $(ABCD)_2C_4$; and for N quartics there are $3\binom{N+2}{4}+3\binom{N+1}{4}$ irreducible covariants of this type.

As regards higher degrees, consider the form obtained by operating with $\epsilon_x\left[x_1\dfrac{\partial}{\partial\epsilon_2}-x_2\dfrac{\partial}{\partial\epsilon_1}\right]$ on $|\,ABCDE\,|$; where $\epsilon_x^4\equiv e_{x^2}^2$; from V.,

$$|\,ABCDE\,| = R+(\alpha\beta)^2(\beta\delta)^2(\delta\gamma)^2(\gamma\epsilon)^2(\epsilon\alpha)^2-(\alpha\gamma)^2(\gamma\beta)^2(\beta\epsilon)^2(\epsilon\delta)^2(\delta\alpha)^2;$$

therefore

$$\tfrac{1}{2}\epsilon_x\left[x_1\frac{\partial}{\partial\epsilon_1}-x_2\frac{\partial}{\partial\epsilon_1}\right] \mid ABCDE \mid \equiv \tfrac{1}{2} \mid ABCD, Ex^2 \mid \equiv \tfrac{1}{2}\Delta_E \text{ (say)}$$

$$= R+(\alpha\beta)^2(\beta\delta)^2(\delta\gamma)^2(\gamma\epsilon)(\epsilon\alpha)\left\{\gamma_x(\epsilon\alpha)-\alpha_x(\gamma\epsilon)\right\}\epsilon_x$$

$$-(\alpha\gamma)^2(\gamma\beta)^2(\beta\epsilon)(\epsilon\delta)(\delta\alpha)^2\left\{\beta_x(\epsilon\delta)-\delta_x(\beta\epsilon)\right\}\epsilon_x$$

$$= R+(eabdc)-(abdce)-(edacb)+(dacbe) \equiv R+[eabdc] \text{ (say)}$$

$$= R+[ebdac] = R+[eacbd] = R+[ecbad] = R+[eadcb]$$

$$= R+[edcab]. \qquad \text{XIII.}$$

Now $\quad [dc, ba, e]+[ab, cd, e]+[a, bed, c]+[b, cea, d]$

$$= [bacde]-[dcabe]-[abdce]+[cdbae]+3\,(cbeda)+3\,(bcead).$$

Therefore $\quad (cbeda)+(bcead)$

$$= \tfrac{1}{6}\left\{ \mid BCDE, Ax^2 \mid + \mid EABC, Dx^2 \mid - \mid CDEA, Bx^2 \mid - \mid DEAB, Cx^2 \mid \right\} + R$$

$$= \tfrac{1}{6}\left\{\Delta_A-\Delta_B-\Delta_C+\Delta_D\right\}+R = \Sigma\Delta+R. \qquad \text{XIV.}$$

Again,

$$[cd, ab, e]-[b, aed, c]+[a, bed, c]-[cdabe]+[badce]-[abdce]+[cdbae]$$

$$= 3\,(caedb)-3\,(cbeda)+(aedcb)+(cbaed)-(cabed)-(bedca)-(cebad)$$

$$-(aecdb)+(becda)+(ceabd)$$

$$= 3\,(caedb)-3\,(cbeda)+(cbdae)+(edacb)-(edbca)-(cadbe)-(adbce)$$

$$-(dbcae)+(dacbe)+(bdace)+R+\Sigma\Delta \text{ (by XIV.)}$$

$$= 5\,(caedb)-5\,(cbeda)+[ad, cb, e]+[bc, da, e]-[ac, db, e]$$

$$-[bd, ca, e]+R+\Sigma\Delta.$$

Therefore $\quad (caedb)-(cbeda) = R+\Sigma\Delta. \qquad$ XV.

Again, using XV.,

$$[acedb]+[bdeca]-[e \ldots a]$$

$$= (ecdab)+(edacb)-(edbca)-(ecdba)+R+\Sigma\Delta.$$

Subtract the sum of the results of interchanging c and a, c and b, respectively in this equation from its original form; then, with the help of XV., we obtain

$$(edacb)+(edcba)+(edbac)-(edcab)-(edabc)-(edbac) = R+\Sigma\Delta.$$

But $\quad [ed\ldots] = R;$

therefore $\quad (edacb)+(edcba)+(edbac) = R+\Sigma\Delta. \qquad$ XVI.

The equations XV. and XVI. contain all the equations for $(abcde)$. It follows from XV. and XVI. that any form

$$_2C_5 = \Sigma\,(a\,\ldots) + \Sigma\Delta + R\,;$$

further they prove that forms having a definite letter b in the second or fifth place can be expressed in terms of those having b in the third or fourth place, and at the same time a in the first place; XV. leaves only six of these forms to be discussed, and amongst these we have one relation, viz.,

$$(acbde) + (acebd) + (aebcd) + (adcbe) + (adbec) + (aedbc) = R + \Sigma\Delta.$$

Hence there are five independent irreducible forms $(abcde)$, and five forms $|\,ABCD,\ Ex^2\,|$ which have yet to be discussed. These determinant forms prove to be reducible. Thus

$$[abcde] + [bcdea] + [cdeab] + [deabc] + [eabcd] \equiv 0\,;$$

therefore $$\Delta_A + \Delta_B + \Delta_C + \Delta_D + D_E = R. \qquad \text{XVII.}$$

Now, from XVI., XV., and $[e,\ da,\ c,\ b,] = R$ we obtain

$$(edacb) + (ecdab) + (ebcda) = R - (edabc) - (ebdac) - (ecbda)$$
$$= \Sigma\Delta + R = \lambda_1\Delta_A + \lambda_2\Delta_B + \lambda_3\Delta_C + \lambda_4\Delta_D + \lambda_5\Delta_E + R \text{ (say)}$$
$$= \lambda_1\Delta_A + \lambda_3\Delta_B + \lambda_2\Delta_C + \lambda_4\Delta_D + \lambda_5\Delta_E + R,$$

since the interchange of B and C merely changes the sign of each expression. Hence $\lambda_2 = \lambda_3$, and, in virtue of XVII., they may be each taken to be zero. Interchange e and b, a and d in the above result; then $$(badce) + (bcade) + (becad) = \lambda_4\Delta_A + \lambda_5\Delta_B + \lambda_1\Delta_D + R\,; \qquad \text{XVIII.}$$

adding,

$$(ebcda) + (becad) = (\lambda_1 + \lambda_4)(\Delta_A + \Delta_D) + \lambda_5(\Delta_B + \Delta_E) + R$$
$$= \tfrac{1}{6}\left\{ -\Delta_A - \Delta_D + \Delta_B + \Delta_E \right\} + R,$$

in virtue of XIV. Therefore

$$\lambda_5 = \tfrac{1}{6} = -(\lambda_1 + \lambda_4).$$

Substituting symmetrically the results of XVIII.,

$$9\Delta_E = 3\,[eabdc] + 3\,[ebdac] + 3\,[eacbd] + 3\,[ecbad] + 3\,[eadcb] + 3\,[edcab] + R$$
$$= 2\left\{ -12\lambda_5\Delta_E - 3\,(\lambda_1 + \lambda_4)(\Delta_A + \Delta_B + \Delta_C + \Delta_D) \right\} + R = -5\Delta_E + R.$$

Therefore these determinant forms are reducible.

Hence for N quartics there are

$$5\binom{N}{5}+12\binom{N}{4}+9\binom{N}{3}+2\binom{N}{2}=2\binom{N+3}{5}+3\binom{N+2}{5}$$

independent irreducible covariants of the type ${}_2C_5$.

All covariants of order 2 and of degree higher than 5 are reducible; for

$$(abcdefa) = \Sigma \mid ABCD,\ EF \mid + R.$$

Operate with $a_x\left[x_1\frac{\partial}{\partial a_2}-x_2\frac{\partial}{\partial a_1}\right]$; then

$$(abcdef)-(bcdefa) = R+\Sigma\Delta.$$

The determinant $\mid Ax^2,\ B,\ C,\ D,\ EF \mid$ is obtainable from $\mid Ax^2, B, C, D, E \mid$ by an operation $[ef]\left[f_0\frac{\partial}{\partial e_0}+f_1\frac{\partial}{\partial e_1}+f_2\frac{\partial}{\partial e_2}\right]$, and is therefore reducible.

Further, $a_x\left[x_1\frac{\partial}{\partial a_2}-x_2\frac{\partial}{\partial a_1}\right] \mid BCDE,\ AF \mid$ is reducible owing to VIII. Therefore

$$(abcdef)-(bcdefa) = R.$$

Now $$[a,\ bcde,\ f] = R;$$

therefore $$3\,(abcdef)+3\,(fbcdea) = R.$$

Hence the sum of $(abcdef)$ and a form obtained from it by an odd substitution is reducible; therefore

$$(abcdef)+(bcdefa) = R,$$

and hence $(abcdef)$ is reducible.

The reducibility of forms of higher degree and of the second order may be obtained in exactly the same way as it was for invariants.

5. Covariants of order higher than 2 may be obtained from those of order 4 lower, by writing, in the quadratic symbolical expressions for these, for a_0, x_2^2; for a_1, $-x_1x_2$; for a_2, x_1^2. It is at once apparent that all covariants of order higher than 6 are reducible; and that the only forms which have yet to be discussed are ${}_4C_1$, ${}_4C_2$, ${}_4C_3$, ${}_4C_4$, ${}_4C_5$, ${}_6C_2$, ${}_6C_3$, ${}_6C_4$.

For the forms order 4, the only further reductions possible are those due to products of two covariants order 2. The first form to be affected by these is ${}_4C_4$.

Take then such a product

$$2\,[ab]\mid abx^2 \mid [cd] \mid cdx^2 \mid$$

$$= [ab][cd]\begin{vmatrix} [ac] & [ad] & a_{x^2} \\ [bc] & [bd] & b_{x^2} \\ c_{x^2} & d_{x^2} & 0 \end{vmatrix}$$

$$= -(abx^2dca)+(abx^2cda)+(abcdx^2a)-(abdcx^2a).$$

But, from V., $(abx^2dca) = (adbcx^2a) + \mid ABCDx^4 \mid + R,$

$$(abdcx^2a) = (acbx^2da) + \mid ABCDx^4 \mid + R.$$

Therefore

$$2 \mid ABCDx^4 \mid = (abx^2cda)+(abcdx^2a)-(adbcx^2a)-(acbx^2da)+R.$$

Perform the substitutions (bcd) and (bdc), and add the results; then

$$6 \mid ABCDx^4 \mid = R.$$

There are then only six forms to discuss, connected by the equations

$$\begin{aligned}(abx^2cda) &+ (abcdx^2a)+R \\ &= (adbcx^2a)+(adx^2bca)+R = (acx^2dba)+(acdbx^2a)+R \\ &= \tfrac{1}{3}\{(abx^2cda)+(abcdx^2a)+(adbcx^2a)+(adx^2bca)+(acx^2dba) \\ &\qquad\qquad +(acdbx^2a)\}+R \\ &= R.\end{aligned}$$

Hence there are three independent forms $(ABCD)\,{}_4C_4$, and for N quartics there are $3\binom{N+1}{4}$ independent irreducible covariants of the type ${}_4C_4$.

Since I_0 is expressible in the forms $\mid ABCD,\ EF \mid$, which are connected by equations VIII., it follows that ${}_4C_5$ may be expressed in terms of the determinants $\mid ABCx^4,\ EF \mid$; which are reducible, since $\mid ABCDx^4 \mid$ is reducible.

Hence the type ${}_4C_5$ is reducible.

As regards the types degree 6, there are $\binom{N}{2}$ irreducible forms of the type ${}_6C_2$ for N quartics. For ${}_6C_3$ it is necessary to refer to ${}_2C_4$; the equations for ${}_2C_4$ give

$$(cax^2b)+(bx^2ca) = R \quad \text{or} \quad (bx^2ca) = R+(bx^2ac)$$

and $$(bcx^2a)+(cx^2ab)+(cx^2ba)+(acx^2b) = R.$$

Hence $$2\,(cx^2ab) = (ax^2cb)+(bx^2ca)+R,$$

and therefore the difference of any two forms $(ABC)\,{}_6C_3$ is reducible.

Hence the number of independent irreducible covariants of the type ${}_6C_3$ for N quartics is $\binom{N+2}{3}$.

Reference to the work for ${}_2C_5$ shows that forms ${}_6C_4 = \Sigma(x^2 \ldots) + R$, but $(x^2 \ldots) = 0$; hence the type ${}_6C_4$ is reducible.

6. The complete system of irreducible concomitants for a set of N quartics is as follows:—

Type.	Number of Independent Forms.
I_2	$\binom{N+1}{2}$
I_3	$\binom{N+2}{3}$
I_4	$\binom{N+1}{4} + \binom{N+2}{4}$
I_5	$\binom{N}{5} + 2\binom{N+1}{5} + 3\binom{N+2}{5}$
I_6	$10\binom{N+2}{6}$
${}_2C_2$	$\binom{N}{2}$
${}_2C_3$	$\binom{N}{3} + 2\binom{N+1}{3}$
${}_2C_4$	$3\binom{N+1}{4} + 3\binom{N+2}{4}$
${}_2C_5$	$3\binom{N+2}{5} + 2\binom{N+3}{5}$
${}_4C_1$	N
${}_4C_2$	$\binom{N+1}{2}$
${}_4C_3$	$2\binom{N+1}{3}$
${}_4C_4$	$3\binom{N+1}{4}$
${}_6C_2$	$\binom{N}{2}$
${}_6C_3$	$\binom{N+2}{3}$.