

*Theorems concerning Spheres.* By SAMUEL ROBERTS.

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1. I must first of all mention some results relative to plane space, which are suggestively analogous to those referred to in the heading of this paper.

The following theorem was the subject of a question by Professor Mannheim (*Educational Times*, Quest. 10145, *Reprint*, Vol. LII, p. 48), and has been discussed at considerable length by the late M. Eugène Catalan (*Memorie della Pontificia Accademia dei Nuovo Lincei*, Vol. VI., pp. 223-233, 1890).

Let  $A, B, C$  be the vertices of a given triangle (Fig. 1). Through  $A$  let a circle be drawn meeting the side  $AB$  a second time in a

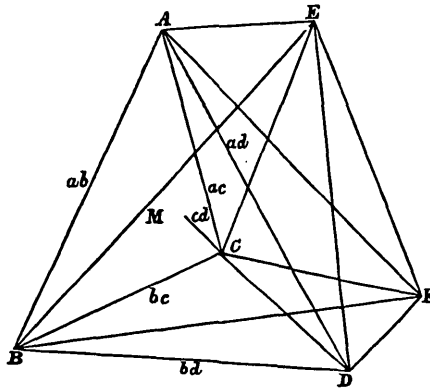


FIG. 1.

point taken at will thereon, which may be conveniently denoted by  $ab$ , and meeting the side  $AC$  a second time at a point taken at will thereon, and similarly denoted by  $ac$ .

Through  $B$  let a second circle be drawn meeting the side  $AB$  a second time in the point  $ab$ , the side  $BC$  again in the point  $bc$ , and the circle first drawn in a point  $M$ ; then the points  $C, ac, bc, M$  are concyclic. Take, further, an arbitrary point  $D$  in the plane of the triangle  $ABC$ , and draw the straight lines,  $AD$  meeting the circle through  $A$  again in a point  $ad$ ,  $BD$  meeting the circle through  $B$  again in a point  $bd$ , and  $CD$  meeting the circle through  $C$  again in  $cd$ ; then the points  $D, ad, bd, cd, M$  are concyclic.

M. Catalan and others have proved this theorem simply enough by means of the condition that a quadrilateral may be inscribable in a circle.

The method is not available for the establishment of analogues in three-dimensioned space. However, we readily arrive at the same results by the repeated application of the first part of the theorem which may be stated in the familiar form—"If an arbitrary point be taken on each side of a given triangle, and through each vertex and the points on the adjacent sides a circle be drawn, these three circles intersect in a point."

Assuming the truth of this theorem as to the triangle  $ABC$ , and the points  $ab, ac, bc$ , we can next apply it to the triangle  $ABD$ , so that  $D, ad, bd, M$  lie on one circle, and then to the triangle  $ACD$ , so that  $D, ad, cd, M$  lie on one circle, and consequently the five points  $D, ad, bd, cd, M$  lie on one circle.

2. In like manner we may take another point  $E$  at will in the plane of the triangle, and, forming the linear connexions  $EA, EB, EC, ED$ , and denoting the intersections of these with the four previously constructed circles in their order by  $ae, be, ce, de$ , we conclude that the six points  $E, ae, be, ce, de, M$  also lie on one circle. Continuing the process, we arrive at a system of  $n$  circles, and  $\frac{n \cdot n-1}{1 \cdot 2}$  lines connecting two-and-two together  $n$  points, so that there are  $n$  intersections of  $n-1$  straight lines and one circle,  $\frac{n \cdot n-1}{1 \cdot 2}$  intersections of one straight line and two circles, and one common intersection of the  $n$  circles. On each line will lie two multiple points of the first class, and one of the second, while on each circle will lie the common intersection of the circle, one point of the first class and  $n-1$  points of the second class. (Otherwise regarded, the conclusion is that, if the system of straight lines is given, and also  $n-1$  of the circles,  $n+1$  points are determined of the  $n^{\text{th}}$  circle.

The foregoing result is one with the theorem that, if  $n-1$  circles intersect in one point, and a polygon be constructed so that each side not being an extreme one passes through a single intersection of two circles, and the two vertices terminating the side lie one on each of the two circles, and if the two extreme sides pass through fixed points on the two final circles, then, the polygon being varied subject to the conditions stated, the locus of the last vertex is a circle through the common point and the fixed points. If we connect any point on

this locus with the vertices of the polygon, there will be  $n-1$  points determined on the locus by the intersections of the connexion with the corresponding circles. Thus, including the common point and the point selected on the locus,  $n+1$  points are determined. We may suppress in a variety of ways all the  $n-1$  circles but two, and all the lines but three, and obtain the same locus by the variation of the triangle formed by the three lines under the conditions (*Quart. Journal of Math.*, Vol. iv., p. 361, 1861).

3. The diagram of Fig. 1 may be regarded as representing straight lines and planes in general space. Viewing it so, let  $ABCD$  be a tetrahedron. On each of the edges  $AB, AC, AD, BC, BD, CD$ , in their order, let there be taken a point at will represented according to the previous notation by  $ab, ac, ad, bc, bd, cd$ . It is known that, if a sphere be constructed through the vertex  $A$  and the points  $ab, ac, ad$ , a second through the vertex  $B$  and the points  $ab, bc, bd$ , and a third through the vertex  $C$  and the points  $ac, bc, cd$ , then the points  $D, ad, bd, cd$  and a triple intersection of the three so constructed spheres lie on one sphere, *i.e.*, the four spheres meet in a point  $M$  (*Proc.*, Vol. xii.).

Take another arbitrary point in space  $E$ , and connect linearly with  $A, B, C, D$  by  $AE, BE, CE, DE$ , meeting the four spheres through  $A, B, C, D$ , respectively, in the points  $ae, be, ce, de$ ; the six points  $E, ae, be, ce, de, M$  lie on one sphere. For we can apply the previous result to the tetrahedron  $ABCE$ , showing that  $E, ae, be, ce, M$  are on one sphere, and next to the tetrahedron  $BCDE$ , showing that the points  $E, be, ce, de, M$  are on one sphere. Again, we may take any other point  $F$ , and connecting as before with  $A, B, C, D, E$ , determine seven points  $F, af, \&c., M$  on a sphere. In this way, we arrive at a system of  $n$  spheres and  $n$  points connected two-and-two by  $\frac{n \cdot n-1}{1 \cdot 2}$  edges, formed by  $\frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3}$  planes. There are  $n$  intersections of  $n-1$  planes and one sphere,  $\frac{n(n-1)}{1 \cdot 2}$  intersections of two planes and two spheres,  $n$  intersections of one plane and three spheres, and one common intersection of  $n$  spheres.

4. According to the foregoing reckoning we shall determine  $n+1$  points on the  $n^{\text{th}}$  sphere. But, in reality, there are determined  $\frac{n^2-n+4}{2}$  multiple intersections on each sphere, *viz.*, one intersection of  $n-1$  planes and one sphere,  $n-1$  intersections of two planes and

two spheres,  $\frac{n-1 \cdot n-2}{2}$  intersections of one plane and three spheres, and the common point of  $n$  spheres,

A certain number of triple intersections and simple intersections are not here taken into account.

For example, in the case of a tetrahedron, the intersection of the sphere through a vertex with the opposite face and all the triple intersections depending on it are left out. We omit, in fact, twenty-four intersections of two planes and one sphere, and twenty-four intersections of one plane and two spheres.

It is not necessary to work out the numbers generally. Moreover, the multiplicities of the omitted intersections may be increased in special cases, and their numbers will be consequently modified.

5. We will examine a little more in detail the case of the tetrahedron ( $n = 4$ ). The arbitrary points taken on the several edges determine more than at first appears.

There are eight quadruple intersections on each of the four spheres, determining in each case a hexahedron with plane quadrilateral faces. The tetrahedron is thus formed exteriorly or interiorly into four hexahedra each inscribable in a sphere. The diagram (Fig. 2) will give a fairly good idea of the arrangement when the figure is divided

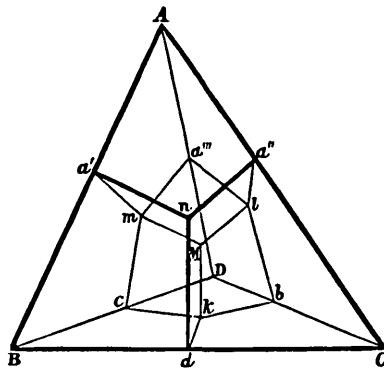


FIG. 2.

interiorly. The hexahedra are, of course,  $Aa'a''a'''lmnM$ ,  $Ba'cdkmmM$ ,  $Ca''bdklnM$ ,  $Da'''bcklmM$ .

We may consider the spheres as given, while the tetrahedron is altered by displacement in accordance with the conditions imposed. The figure  $lmnM$ , forming three trihedral angles whose sum is measured by  $4\pi$ , will remain fixed.

That such variation of the tetrahedron may be effected appears as follows.

Let  $ABCD$  be the original tetrahedron (Fig. 3).

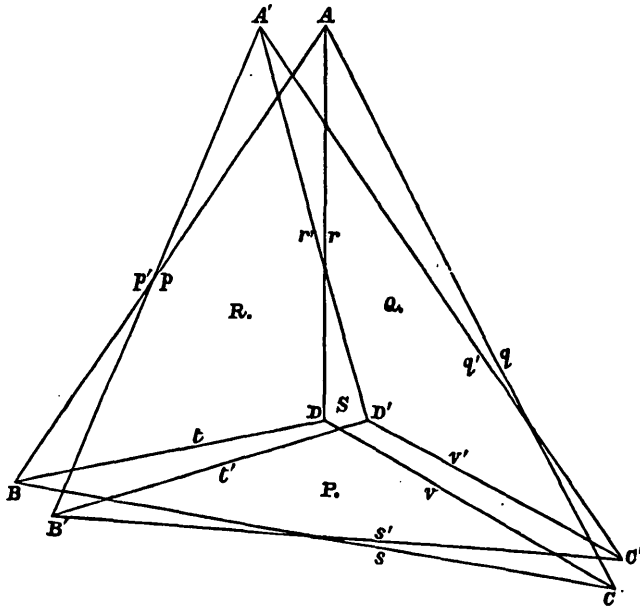


FIG. 3.

Let  $A'p'B'$  be drawn so that  $A'$  is on the sphere  $A$ , and  $B'$  on the sphere  $B$ , and  $p'$  is on the circle of intersection of  $A$  and  $B$ . Here I use the letters of the vertices to denote the respective spheres passing through them. Then, draw  $A'D'$  in the plane  $A'B'R$  and meeting in  $r'$  and  $D'$  the sphere  $D$ ,  $R$  being a triple intersection of  $A$ ,  $B$ ,  $D$  and  $r'$  on the intersection of  $A$ ,  $D$ . Join  $B'D'$ . The sphere  $A$  determines a circle through  $A'$ ,  $p'$ ,  $R$ , meeting  $A'D'$  in  $r'$ . The sphere  $B$  determines a circle through  $B'$ ,  $p'$ ,  $R$ , meeting  $B'D'$  in  $t'$ . The points  $D', r', t', R$  lie on one circle. But the points  $D', r', R$  determine a circle on the sphere  $D$ , so that  $t'$  lies on  $D$ .

Let  $M$  be the intersection of  $A, B, C, D$ , and let  $P, Q, S$  be the intersections of the triads of spheres  $(B, C, D)$ ,  $(A, C, D)$ ,  $(A, B, C)$ . Planes through the edge  $A'B'$  and the point  $S$ , through the edge  $B'D'$  and the point  $P$ , and through the edge  $A'D'$  and the point  $Q$  determine the edges  $A'C', D'C', B'C'$ . The sphere  $A$  will pass through  $Q$  and also meet  $A'C'$  in  $q'$ , the sphere  $B$  will pass through  $P$  and also meet  $B'C'$  in  $s'$ , the sphere  $D$  will pass through  $P$  and meet  $D'C'$

in  $v'$ . Hence a sphere will pass through  $C, q', s', v', M, P, Q, S$ ; but this is the sphere  $D$ , which passes through  $C, q, s, v, M, P, Q, S$ ; so that, as we make further displacements, the locus of the last vertex is the sphere  $C$ .

6. There are a few particular inferences which may be noticed :  
 (a) When a vertex of the figure (Fig. 1) is considered as generating a sphere, and the number of director spheres is greater than three, it becomes unnecessary to retain all the plane faces of the figure, just as in the plane analogue we may suppress certain of the double chords; in fact, we obtain in a variety of ways the same locus when all but three of the spheres are suppressed.

(b) We may regard the diagram of Fig. 1 as a flat evanescent figure in solid space. The triple intersections of three circles in each face coalesce, so that we fall back upon M. Mannheim's theorem, when we regard only the sections of the spheres.

(c) In Fig. 1 suppose that the vertex  $F$  is removed to an infinite distance. It follows that, if we draw from the vertices  $A, B, C, D, E$  parallel straight lines, they will again meet the respective spheres in points which lie on a plane passing through the common point of intersection  $M$ .

7. A more important particular case is the figure of five planes, of which a form is given in Fig. 4. Here the six arbitrary points on

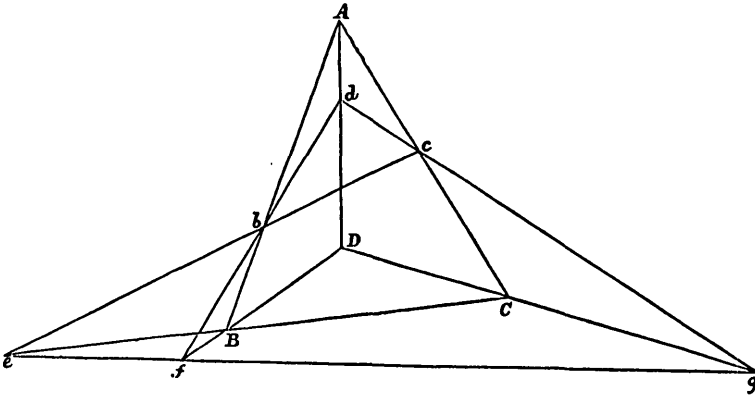


FIG. 4.

the edges of the tetrahedron  $ABCD$  lie in one plane. In the figure,  $b, c, d, e, f, g$  lie in the fifth plane, and there are formed five tetrahedra. Of the five spheres circumscribing these, four meet each plane in

circles circumscribing the triangles formed by the intersections of four straight lines. These circles meet in a quadruple intersection of the four spheres, and the centres of the circles are concyclic with that point. Thus the spheres about  $ABC$  and  $D$ ,  $Abc$  and  $d$ ,  $Bbe$  and  $f$ ,  $Ddf$  and  $g$  meet in the plane  $ABD$ ; the spheres about  $ABC$  and  $D$ ,  $Abc$  and  $d$ ,  $Bbe$  and  $f$ ,  $Cce$  and  $g$  meet in the plane  $ABC$ . The two multiple intersections are therefore the triple intersections of the three spheres common to both sets. If we select a tetrahedron, and omit the sphere circumscribing it, the four spheres through the vertices intersect in the fifth plane.

If we add another plane, the sections of the spheres circumscribing the tetrahedron by a plane will consist of circles circumscribing the triangle formed by five straight lines, and, if there are  $n$  planes, the sections of the spheres by a plane will be circles circumscribing the triangles formed by the  $n-1$  intersections of the plane with the remainder of the planes. To these systems of lines Miquel's theorem and the extensions by Clifford, Longchamps, &c., apply, and we need not occupy ourselves further with them in the present connexion.

We may consider a figure representing the intersections of five planes as reduced in the limit to one plane. Thus, by inspection of Fig. 5, we see that, if  $bcd$ ,  $BCD$  are homologous triangles,  $A$  their

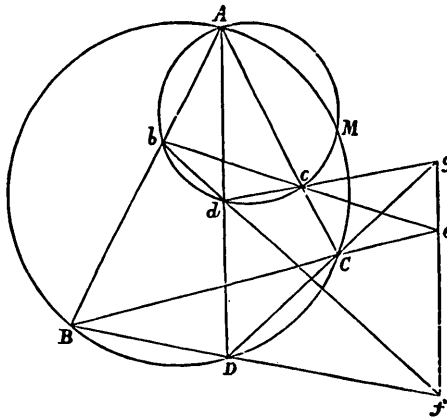


FIG. 5.

centre of homology, and  $efg$  the axis, and if  $Abcd$ ,  $ABCD$  are inscribable in circles, then the points  $cCeg$ ,  $bBef$ ,  $dDfg$  are sets of concyclic points and the five circles meet in a point  $M$ . The symmetry of the figure shows that each intersection of three lines is

the centre of homology with respect to two of the triangles, and each circle circumscribing a triangle passes through its centre of homology with regard to another triangle.

8. The inversion by reciprocal radii vectores of Fig. 1 in the simplest case, that of the tetrahedron, introduces more symmetry.

Taking the centre of inversion at an arbitrary point in space, we get, for the four faces, four spheres passing through the centre and intersecting in six circles which have four triple intersections. These form a tetrahedron with circular edges and spherical sides. We have also four spheres, meeting in a point. Each of these passes through the inverses of the arbitrary points on the edges of the original tetrahedron adjacent to the vertex through which the sphere passes. These inverses may themselves be regarded as arbitrary points, one on each circular edge. This is the direct interpretation of the original theorem, but does not fully express the symmetry.

There are eight spheres intersecting in sixteen quadruple points, the radical centres of sets of four spheres. Let us say the spheres are  $A, B, C, D, a, b, c, d$ . We have to take no account of the intersections of  $A$  and  $a, B$  and  $b, C$  and  $c, D$  and  $d$ .

The quadruple intersections may be denoted as in the following scheme:—

$$\begin{array}{cccc} Abcd, & ABCD, & abcd, & aBCD, \\ ABcd, & ABCd, & aBcd, & aBCd, \\ AbCd, & AbCD, & abCd, & abCD, \\ AbcD, & ABcD, & abcD, & aBcD, \end{array}$$

showing that there are eight such points on each sphere. It follows that, if we take six spheres  $B, C, D, b, c, d$ , and from the triple intersections as indicated, bearing in mind that  $Abcd, abcd$  must mean that  $A$  passes through one of the triple intersections of  $bcd$  and  $a$  passes through the other, then, if the eight points of one set form the apices of a hexahedron inscribable in the sphere  $A$ , the other eight form a hexahedron inscribable in the sphere  $a$ . The analogue in plane space is—"The circles which have for chords the four sides of a quadrilateral inscribable in a circle form by the other intersections of the same pairs of circles a quadrilateral inscribable in a circle" (Catalan, *Théorèmes et Problèmes*, sixième édition, p. 39).

The limiting case may be noted in which the six spheres meet in one point, the tangents at which are parallel to the faces of a



hexahedron with quadrilateral faces and inscribable in a sphere, and to this also there is a plane analogue.\*

9. Invert now the figure of five planes and its five associated spheres. This gives us ten spheres and sixteen points of quintuple intersections. Let the spheres derived from the five planes be denoted by  $a, b, c, d, e$ , and the other five spheres by  $A, B, C, D, E$ . The quintuple intersections will be duly represented by

$$\begin{array}{cccc} AbcdE, & ABCdE, & ABcDE, & AbCDE, \\ AbcDE, & ABcde, & AbCde, & ABCDe, \\ abCdE, & aBcdE, & abcDE, & aBCDE, \\ aBCde, & aBcDe, & abCde, & abcde. \end{array}$$

The total number of quintuple arrangements containing the first five letters of the alphabet would be  $2^5 = 32$ . But, if we take any one of the set, say  $AbcdE$ , its complementary form  $aBODE$  does not appear. There are left sixteen sets.

From the scheme it appears that eight of the quintuple intersections lie on the sphere  $A$ , and eight others on the sphere  $a$ . Also the hexahedra have not only six plane quadrilateral faces but also two diagonal planes. Thus as to the hexahedron circumscribed by the sphere  $A$ , the circular sections  $AE, AB, \&c., Ae, Ab, \&c.$ , pass severally through four intersections, and the same is the case with the circular sections  $aB, aC, \&c., ab, ac, \&c.$

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\* Having proceeded so far, I happened to refer to a paper by M. Auguste Miquel, and in the second part (Liouville, *Journal*, t. x., 1<sup>e</sup> série, 1845) I unexpectedly found the theorem of this article in the second form. Accepting Miquel's proof, we may evidently by inversion with respect to one of the quadruple points pass back to the original theorem relative to the tetrahedron, from which we set out. For we shall have four planes whose six intersections correspond to the six circles of intersection through the point. We shall have also the six arbitrary points, one on each linear edge, and finally the four spheres each passing through a vertex (intersection of three planes) and the arbitrary points on the three adjacent edges. The proposition is the last in the second part of Miquel's paper, and differs rather in character from the other contents, which relate to circles in the plane and on a spherical surface. Inversion by reciprocal "radii vectores" was at that date of recent introduction. In fact, the theory of "images" is given by Professor W. Thomson (Lord Kelvin) in the same volume. Accordingly, M. Miquel makes no use of the method which is directly applicable to some of his propositions. It appears that Mr. Stubbs employed the substitution  $\rho^{-1}$  for  $\rho$  in space equations towards the end of 1843.