On a Class of Integers expressible as the Sum of Two Integral Squares. By Thomas Muir, M.A., F.R.S.E.

[*Read March 8th*, 1877.]

It is known from Lagrange, that any integer whose square root, when expressed in the ordinary way as a continued fraction, has an odd number of terms in its cycle of partial denominators, must be the sum of two integral squares: *e.g.*,

$$\sqrt{365} = 19 + \cfrac{1}{\overset{*}{9} + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{9 + \cfrac{1}{\overset{*}{38} + \ddots}}}}}$$

and $365 = 2^2 + 19^2 = 13^2 + 14^2$;

$$\sqrt{(2500m^2+1500m+325)} = (50m+15) + \cfrac{1}{\overset{*}{m} + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{m + \cfrac{1}{\overset{*}{100m+30} + \ddots}}}}}$$

and
$$\begin{aligned} 2500m^2+1500m+325 &= (50m+15)^2 + (10)^2 \\ \text{or} \quad &= (40m+18)^2 + (30m+1)^2. \end{aligned}$$

Further, it is known† that the general expression for such integers is

$$\text{(A)} \quad \left\{ \begin{array}{l} \left\{\tfrac{1}{2}K(a_1, a_2, \dots a_2, a_1)\,\text{M} + \tfrac{1}{2}K(a_1, a_2, \dots a_2)\,K(a_2 \dots a_2)\right\}^2 \\ \qquad + K(a_1, a_2, \dots a_2)\,\text{M} + K(a_2 \dots a_2)^2, \end{array} \right.$$

where $a_1, a_2, \dots a_n, a_n, \dots a_2, a_1$ is the symmetric portion of the cycle of partial denominators, and the continuant notation K ( ) is explained by the example

$$K(a_1, a_2, a_3, a_4) = \begin{vmatrix} a_1 & 1 & 0 & 0 \\ -1 & a_2 & 1 & 0 \\ 0 & -1 & a_3 & 1 \\ 0 & 0 & -1 & a_4 \end{vmatrix}$$

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* The asterisks are used like the upper dots in $\frac{1}{7} = \cdot\dot{1}4285\dot{7}$.

† Proceedings Roy. Soc. Edin., Sess. 1873–4, p. 234.

In other words, reversing the order of the propositions, and denoting for shortness' sake

$$\tfrac{1}{2}\mathrm{K}(a_1, a_2, \ldots a_2, a_1)\,\mathrm{M}+\tfrac{1}{2}\mathrm{K}(a_1, a_2, \ldots a_2)\,\mathrm{K}(a_2 \ldots a_2) \text{ by } \mathrm{R},$$

and

$$\mathrm{K}(a_1, a_2, \ldots a_2)\,\mathrm{M}+\mathrm{K}(a_2 \ldots a_2)^2 \text{ by } \mathrm{S},$$

it is known that

$$\sqrt{(\mathrm{R}^2+\mathrm{S})} = \mathrm{R}+\cfrac{1}{\overset{*}{a_1}+\cfrac{1}{a_2+\cdots+\cfrac{1}{a_n+\cfrac{1}{a_n+\cdots+\cfrac{1}{a_2+\cfrac{1}{a_1+\cfrac{1}{\overset{*}{2\mathrm{R}}+\cdots}}}}}}} \qquad \ldots\ldots\ldots\ (a),$$

and that, on account of there being $2n+1$ terms in the cycle of partial denominators, $\mathrm{R}^2+\mathrm{S}$ must be expressible as the sum of two squares.

There is thus suggested a problem of considerable importance in the Theory of Numbers, viz., to find the two squares into which $\mathrm{R}^2+\mathrm{S}$ (*i.e.* A) may be partitioned. A solution of this has been obtained, and the object of the present paper is to place on record the rather remarkable result arrived at, and to give a verification of it.†

The expression for (A) as the sum of two squares is

$$(\mathrm{B})\begin{cases} \Big[\tfrac{1}{2}\{\mathrm{K}(a_1 \ldots a_n)^2-\mathrm{K}(a_1 \ldots a_{n-1})^2\}\,\mathrm{M}+\tfrac{1}{2}\mathrm{K}(a_1 \ldots a_n)\,\mathrm{K}(a_2 \ldots a_n)^2 \\ \quad -\tfrac{1}{2}\mathrm{K}(a_1 \ldots a_{n-1})\,\mathrm{K}(a_2 \ldots a_{n-1})^2+(-1)^n\tfrac{1}{2}\mathrm{K}(a_2 \ldots a_{n-1})\,\mathrm{K}(a_2 \ldots a_n)\Big]^2 \\ + \\ \Big[\{\mathrm{K}(a_1 \ldots a_n)\,\mathrm{K}(a_1 \ldots a_{n-1})\}\,\mathrm{M}+\mathrm{K}(a_1 \ldots a_n)\,\mathrm{K}(a_2 \ldots a_{n-1})^2 \\ \qquad\qquad +\mathrm{K}(a_1 \ldots a_{n-1})\,\mathrm{K}(a_2 \ldots a_n)^2\Big]^2. \end{cases}$$

For, putting

$$\mathrm{K}(a_1, \ldots, a_{n-1}) = \alpha, \quad \mathrm{K}(a_1, \ldots, a_n) = \beta,$$
$$\mathrm{K}(a_2, \ldots, a_{n-1}) = \gamma, \quad \mathrm{K}(a_2, \ldots, a_n) = \delta,$$

so that

$$\alpha\delta-\beta\gamma = (-1)^{n+1};$$

then

$$\mathrm{K}(a_1, \ldots, a_1) = \alpha^2+\beta^2,$$
$$\mathrm{K}(a_1, \ldots, a_2) = \alpha\gamma+\beta\delta,$$
$$\mathrm{K}(a_2, \ldots, a_2) = \gamma^2+\delta^2,$$

† I have chosen, for shortness' sake, to state the result and then verify it, in preference to following the process by which it was originally obtained from Lagrange's theorem on the subject. The verification here given is not, however, that at first presented to the Society, but a recast made in accordance with a suggestion of Professor Cayley.

and the identity to be established is

$$\begin{aligned}&\{\tfrac{1}{2}(\alpha^2+\beta^2)\,\mathrm{M}+\tfrac{1}{2}(\alpha\gamma+\beta\delta)(\gamma^2+\delta^2)\}^2\\&\qquad+(\alpha\gamma+\beta\delta)\,\mathrm{M}+(\gamma^2+\delta^2)^2\\&=\{-\tfrac{1}{2}(\alpha^2-\beta^2)\,\mathrm{M}+\tfrac{1}{2}\beta\delta^3-\tfrac{1}{2}\alpha\gamma^3+(-1)^n\,\tfrac{3}{2}\,\gamma\delta\}^2\\&\qquad+\{\alpha\beta\mathrm{M}+\beta\gamma^3+\alpha\delta^3\}^2.\end{aligned}$$

Making use of the relation $\alpha\delta-\beta\gamma=(-1)^{n+1}$, this may be changed into the form

$$\begin{aligned}&[\tfrac{1}{2}(\alpha^2+\beta^2)\,\mathrm{M}+\tfrac{1}{2}(\alpha\gamma^3+\beta\delta^3+\alpha\gamma\delta^2+\beta\gamma^2\delta)]^2\\&\qquad-[-\tfrac{1}{2}(\alpha^2-\beta^2)\,\mathrm{M}+\tfrac{1}{2}(-\alpha\gamma^3+\beta\delta^3-3\alpha\gamma\delta^2+3\beta\gamma^2\delta)]^2\\&=\{\alpha\beta\mathrm{M}+\beta\gamma^3+\alpha\delta^3\}^2-(\alpha\delta-\beta\gamma)^2\{(\alpha\gamma+\beta\delta)\,\mathrm{M}+(\gamma^2+\delta^2)^2\},\end{aligned}$$

and, expressing the left-hand side of this as the product of two factors and expanding the right-hand side, it becomes

$$\begin{aligned}&[\beta^2\mathrm{M}+\delta\{2\beta\gamma^2-\delta(\alpha\gamma-\beta\delta)\}]\,[\alpha^2\mathrm{M}+\gamma\{2\alpha\delta^2+\gamma(\alpha\gamma-\beta\delta)\}]\\&\quad=\ \mathrm{M}^2.\alpha^2\beta^2\\&\qquad+\mathrm{M}\{2\alpha\beta(\beta\gamma^3+\alpha\delta^3)-(\alpha\delta-\beta\gamma)^2(\alpha\gamma+\beta\delta)\}\\&\qquad+\quad(\beta\gamma^3+\alpha\delta^3)^2-(\alpha\delta-\beta\gamma)^2(\gamma^2+\delta^2)^2.\end{aligned}$$

Now on the right-hand side here the term independent of M is

$$\begin{aligned}&\{\beta\gamma^3+\alpha\delta^3+(\alpha\delta-\beta\gamma)(\gamma^2+\delta^2)\}\\&\{\beta\gamma^3+\alpha\delta^3-(\alpha\delta-\beta\gamma)(\gamma^2+\delta^2)\},\end{aligned}$$

which

$$\begin{aligned}&=\delta\{2\alpha\delta^2+\gamma(\alpha\gamma-\beta\delta)\}\\&\quad\gamma\{2\beta\gamma^2-\delta(\alpha\gamma-\beta\delta)\},\end{aligned}$$

as it should be; and as regards the coefficients of M, we have to show that

$$\begin{aligned}&2\alpha\beta\gamma\delta(\alpha\gamma+\beta\delta)-(\alpha^2\delta^2-\beta^2\gamma^2)(\alpha\gamma-\beta\delta)\\&\qquad=2\alpha\beta(\beta\gamma^3+\alpha\delta^3)-(\alpha\delta-\beta\gamma)^2(\alpha\gamma+\beta\delta),\end{aligned}$$

or $\quad(\alpha^2\delta^2+\beta^2\gamma^2)(\alpha\gamma+\beta\delta)-(\alpha^2\delta^2-\beta^2\gamma^2)(\alpha\gamma-\beta\delta)=2\alpha\beta(\beta\gamma^3+\alpha\delta^3)$,

which is the case.

Two particular cases of the general theorem thus established are of even greater interest.

When $\mathrm{M}=0$, the continued fraction we have been considering becomes

$$\tfrac{1}{2}\mathrm{K}(a_1, a_2, \ldots a_2)\,\mathrm{K}(a_2\ldots a_2)+\cfrac{1}{\underset{*}{a_1}+\cfrac{1}{a_2+\cdots+\cfrac{1}{a_n+\cfrac{1}{a_n+\cdots+\cfrac{1}{a_2+\cfrac{1}{a_1+\cfrac{1}{\mathrm{K}(a_1\ldots a_2)\,\underset{*}{\mathrm{K}}(a_2\ldots a_2)}}}}}}},$$

and we learn that the integer of which it is the square root is expressible in *two* ways as the sum of two integral squares, viz.,

$$\left\{\tfrac{1}{2}\mathrm{K}(a_1 \dots a_2)\,\mathrm{K}(a_2 \dots a_2)\right\}^2 + \left\{\mathrm{K}(a_2 \dots a_2)\right\}^2,$$

and
$$\left\{\tfrac{1}{2}\mathrm{K}(a_1 \dots a_n)\,\mathrm{K}(a_2 \dots a_n)^3 - \tfrac{1}{2}\mathrm{K}(a_1 \dots a_{n-1})\,\mathrm{K}(a_2 \dots a_{n-1})^3 + (-1)^n\tfrac{3}{2}\mathrm{K}(a_2 \dots a_{n-1})\,\mathrm{K}(a_2 \dots a_n)\right\}^2$$
$$+\left\{\mathrm{K}(a_1 \dots a_n)\,\mathrm{K}(a_2 \dots a_{n-1})^3 + \mathrm{K}(a_1 \dots a_{n-1})\,\mathrm{K}(a_2 \dots a_n)^3\right\}^2.$$

As an instance of this, we find

$$\begin{aligned}
&[16ab^4+32ab^3+8b^4+24ab^2+20b^3+18b^2+8ab+10b+2]^2 \\
&+[16ab^3+24ab^2+8b^3+16ab+12b^2+4a+8b+3]^2 \\
=\ &[16ab^4+32ab^3+8b^4+32ab^2+20b^3+22b^2+16ab+12b+4a+3]^2 \\
&+[4b^2+4b+2]^2 \\
=\ &\mathrm{R}^2+s^2, \text{ say,}
\end{aligned}$$

and $\sqrt{(\mathrm{R}^2+s^2)} = \mathrm{R} + \cfrac{1}{\dot{2a+1} + \cfrac{1}{2b+1 + \cfrac{1}{2b+1 + \cfrac{1}{2a+1 + \cfrac{1}{\dot{2\mathrm{R}} + \ddots}}}}}$

When $\mathrm{M} = \mathrm{K}(a_1 \dots a_2)$, the integer whose square root is equivalent to the continued fraction is expressible as the sum of two squares, and also as the sum of *three* squares. These expressions are apparent from (A) and (B), but a particular case with figures may be taken, for the purpose of giving greater definiteness to the general subject, and of exhibiting the calculus connected with it.

Let 1, 1, 1, 1, 1, 1 be the symmetric portion of the cycle of partial denominators, then

$$\begin{aligned}
\mathrm{K}(a_1 \dots a_1) &= \mathrm{K}(1,1,1,1,1,1) = 13, \\
\mathrm{K}(a_1 \dots a_2) &= \mathrm{K}(1,1,1,1,1) = 8 = \mathrm{M}, \\
\mathrm{K}(a_2 \dots a_2) &= \mathrm{K}(1,1,1,1) = 5, \\
\mathrm{K}(a_1 \dots a_n) &= \mathrm{K}(1,1,1) = 3, \\
\mathrm{K}(a_2 \dots a_n) = \mathrm{K}(a_1 \dots a_{n-1}) &= \mathrm{K}(1,1) = 2, \\
\mathrm{K}(a_2 \dots a_{n-1}) &= \mathrm{K}(1) = 1,
\end{aligned}$$

and thus, from (α), we have

$$\sqrt{5283} = 72 + \cfrac{1}{\dot{1} + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\dot{144} + \ddots}}}}}}}$$

which may be readily verified,

and $$5283 = 28^2+67^2, \text{ from (B)},$$
and $$= 72^2+8^2+5^2, \text{ from (A)}.$$

I hope to be able to communicate to the Society at some future time the solution of the general problem which is related to Göpel's theorem (*v.* Jacobi's "*Notiz über A. Göpel,*" in Crelle, xxxv., p. 314) as the general problem of this paper is related to Legendre's theorem.

April 12*th*, 1877.

LORD RAYLEIGH, F.R.S., President, in the Chair.

Mr. Charles Pendlebury, B.A., was elected a Member.

The following communications were made to the Society :—

"On Hesse's Ternary Operator and Applications:" by J. J. Walker.

"On the Circular Relation of Möbius;" and "On the Linear Transformation of the Integral $\int \frac{du}{\sqrt{U}}$:" Prof. Cayley.

"On some cases of Parallel Motion:" Mr. Harry Hart.

"A method of solving Partial Differential Equations, which have a general first Integral, applied to equations of the third order with two independent variables:" Prof. H. W. Lloyd Tanner. (Abstract read by Mr. Tucker.)

The following presents were received :—

"Proceedings of Physical Society of London," Vol. ii., Pt. ii., June 1876 to January 1877.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome xi., 2e sémestre, 1876, and "Table des Matières et Noms d'Auteurs ;" deuxième série, Tome i., Janvier 1877 (Tome xii. de la collection).

"Crelle," 82 Band, 3es und 4es Heft, Berlin, 1877.

"Monatsbericht," Nov. and Dec. 1876.

"Annali di Matematica," serie iia, Jan. 1877, Tomo 8o, Fasc 1o.

"Atti della R. Accademia dei Lincei anno cclxxiv," 1876–77, serie terza; "Transunti," Vol. i., Fasc. 3o, Feb., Fasc. 4o, Marzo, 1877. Roma.

"Bulletin de la Société Mathématique de France," Tome v., Janvier, No. 1, Paris, 1877.

"Proceedings of the Royal Society," Vol. xxv., No. 178.

"Nieuw Archief voor Wiskunde," Deel ii., Amsterdam, 1876.

"Investigation of Corrections to Hansen's Tables of the Moon, with

Tables for their application, by Simon Newcomb," forming Part iii. of papers published by the Commission of the Transit of Venus, Washington, 1876.

"Report on the Difference of Longitude between Washington and Ogden, Utah," by J. R. Eastman, Washington, 1876.

Address delivered by J. J. Sylvester, F.R.S., at Johns Hopkins University, on Commemoration Day (Feb. 22, 1877), Baltimore: from the Author.

"Ueber den Verlauf der Abels'chen Integrale bei den Curven vierten Grades" (zweiter Aufsatz), von Felix Klein, in München (from "Mathematische Annalen," Vol. xi., pp. 293–305.)

"Treatise on Integral Calculus," by B. Williamson, 1877: from the Author.

"Smithsonian Report," 1875, Washington, 1876.

"Memorie della R. Accademia di Scienze, Lettere ed Arti in Modena," Tomo xv., Tomo xvi., 1875.

"Transactions of the Cambridge Philosophical Society," Vol. xi., Pts. i., ii., iii.; Vol. xii., Pts. i., ii.

On the Circular Relation of Möbius. By Prof. CAYLEY.

[*Read April* 12*th*, 1877.]

In representing a given imaginary or complex quantity $u, = x+iy$, by means of the point whose coordinates are x, y, we assume in the first instance that x, y are real,—but in the results this restriction may be abandoned—for instance, if the imaginary quantities u, u', c are connected by the equation $u^2+u'^2=c^2$; then, writing $u=x+iy$, $u'=x'+iy'$, $c=a+bi$, we have $x^2-y^2+x'^2-y'^2=a^2-b^2$, $xy+x'y'=ab$, equations connecting the points U, U′, C which serve to represent the quantities u, u', c, and which (regarding C as a fixed point) establish a correspondence between the two variable points U, U′: any given value $u, = x+iy$, is represented by the point U, and corresponding hereto we have (in the present case) two points U′, viz., these are the real intersections of the curves $x'^2-y'^2=a^2-b^2-(x^2-y^2)$, $x'y'=ab-xy$, and then the coordinates x', y' of either of these give the value $x'+iy'$ of u'.

But, the two curves once arrived at, we may for other purposes be concerned with their intersections as well imaginary as real; or, still more generally, all the quantities entering into the two equations may be regarded as imaginary.