Table of Complex Multiplication Moduli. *By* A. G. Greenhill.

[*Read May 8th*, 1890.]

A complete catalogue of all the numerical results in the solution of the modular equations for Complex Multiplication, when the square of the ratio of the periods is an integer, or a rational fraction, is of course impossible; but a list of those cases which have been solved up till now will be found useful as a verification of the form and coefficients of the modular equations, and also in some cases will serve for the determination of the coefficients when the form of the modular equation can be inferred from independent considerations.

It is proposed, then, to give here a list of these numerical results, as far as is known at present, embodying the values obtained by Abel, Jacobi, Kronecker, Hermite, Joubert, and Weber, and more recently of R. Russell and Mathews.

The numerical results are also presented in more than one shape, for verification in case of a misprint.

Denoting the ratio of the periods of the elliptic functions, K'/K, by $\sqrt{\Delta}$, and taking Δ as integral, it is convenient to draw up the list in four classes, according to the form of Δ (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 301); thus, in

Class A, $\Delta \equiv 3$, mod. 8;

Class B, $\Delta \equiv 7$, mod. 8;

Class C, $\Delta \equiv 1$, mod. 4;

Class D, $\Delta \equiv 2$, mod. 4;

the class for $\Delta \equiv 0$, mod. 4, not requiring much separate consideration.

Class A: $\Delta \equiv 3$, mod. 8.

Here the simplest numerical result to tabulate is Klein's *absolute invariant* J (*Math. Ann.*, Vol. XIV., p. 111), or Hermite's α (*Equations modulaires*), connected with J by the relation

$$J = -\frac{4}{27}\alpha;$$

in terms of Legendre's moduli κ and κ', we may take

$$J = -\frac{(1-16\kappa^2\kappa'^2)^3}{108\kappa^2\kappa'^2}, \quad \alpha = \frac{(1-16\kappa^2\kappa'^2)^3}{16\kappa^2\kappa'^2}.$$

It is convenient also to denote $\sqrt[3]{J}$ by $-\gamma_2$, and to put

$$J-1 = -27\gamma_3^2;$$

so that

$$\gamma_2^3 - 27\gamma_3^2 = -1;$$

and γ_2 and γ_3 are then the invariants in the *normalized* integral.

$\Delta = 3$; $\quad J = 0, \quad \alpha = 0, \quad 2\kappa\kappa' = \frac{1}{2} = \sin\frac{1}{6}\pi.$

$\Delta = 11$; $\quad J = -2^9 \div 3^3, \quad \gamma_2 = \frac{8}{3}, \quad \gamma_3 = 7\sqrt{11} \div 27.$

$\Delta = 19$; $\quad J = -2^9, \quad \gamma_2 = 8, \quad \gamma_3 = \sqrt{19}, \quad \gamma_2 + 1 = 3^2,$

$$\gamma_2^2 - \gamma_2 + 1 = 3 \times 19.$$

$\Delta = 27$; $\quad J = -2^9 \times 5^3 \div 3^2, \quad \alpha = 2^7 \times 3 \times 5^3,$

$$\gamma_2 = -2^3 \times 5 \times \sqrt[3]{9}.$$

$\Delta = 35$; $\quad J = -\gamma_2^3, \quad \gamma_2 = \frac{8}{3}\sqrt{5}\,\{\tfrac{1}{2}(\sqrt{5}+1)\}^4,$

$$\gamma_2 + 1 = \tfrac{7}{3}\left(\frac{\sqrt{5}+1}{2}\right)^6, \quad \gamma_2^2 - \gamma_2 + 1 = \tfrac{1}{9}(63+26\sqrt{5})^2,$$

$$\gamma_3 = (256+115\sqrt{5})\sqrt{7} \div 27.$$

$\Delta = 43$; $\quad J = -2^{12} \times 5^3, \quad \gamma_2 = 80, \quad \gamma_3 = 21\sqrt{43},$

$$\gamma_2 + 1 = 3^4, \quad \gamma_2 - \gamma_2 + 1 = 3 \times 7^2 \times 43.$$

$\Delta = 51$; $\quad J = -64(5+\sqrt{17})^3(\sqrt{17}+4)^2,$

$$\gamma_3 = 7\sqrt{3}(128+31\sqrt{17}) \div 3^2 \text{ (Kiepert).}$$

$\Delta = 59$; α, and therefore J, given by a cubic equation.

Mr. R. Russell finds that the equation for $z = \sqrt[12]{(4\kappa\kappa')}$ is

$$z^9 - 2\sqrt[3]{2}\,z^8 + \sqrt[3]{4}\,z^7 - z^6 + 2\sqrt[3]{2}\,z^5 - \sqrt[3]{4}\,z^4 + z^3 - 2\sqrt[3]{2}z^2 + 2\sqrt[3]{4}z - 1 = 0;$$

and thence that the cubic equation for

$$\sqrt[3]{\alpha} = (1-z^{24})/z^8$$

is

$$\alpha - 392\sqrt[3]{2}\,\alpha^{\frac{2}{3}} + 1072\sqrt[3]{4}\,\alpha - 2816 = 0$$

and therefore

$$27J + 7056J^{\frac{2}{3}} + 12864J^{\frac{1}{3}} + 2816 = 0.$$

$\Delta = 67$; $J = -2^9 \times 5^3 \times 11^3$, $\gamma_2 = 2^3 \times 5 \times 11$, $\gamma_3 = 217\sqrt{67}$,

$$\gamma_2 + 1 = 3^2 \times 7^2, \quad \gamma_2^2 - \gamma_2 + 1 = 3 \times 7^2 \times 31^2 \times 67.$$

$\Delta = 75$; $J = -64\sqrt{5}\,(31\sqrt{5} + 69)$, $\gamma_2 = 4 \times 5^3\,(69 + 31\sqrt{5})$,

$$\gamma_3 = \tfrac{1}{9}\sqrt{3}\,(4352\sqrt{5} + 9729).$$

Or, (Russell) with $z = \sqrt[12]{(4\kappa\kappa')}$,

$$z^6 + \sqrt[3]{4}\,z^5 - 2\sqrt[3]{2}\,z + 1 = 0.$$

$\Delta = 83$; α given by a cubic equation (Russell),

$$\alpha - 1740\sqrt[3]{2}\,\alpha^{\frac{2}{3}} + 2000\sqrt[3]{4}\,\alpha^{\frac{1}{3}} - 32000 = 0.$$

$\Delta = 91 = 7 \times 13$; $\gamma_2 = 908 + 252\sqrt{13}$,

$$\gamma_3 = 11\sqrt{7}\,(2\sqrt{13} + 7)(5\sqrt{13} + 18), \quad \gamma_2 + 1 = 9\,(2\sqrt{13} + 7)^2,$$

$$\gamma_2^2 - \gamma_2 + 1 = 3 \times 7 \times 11^2 \left\{\tfrac{1}{2}\,(\sqrt{13} + 3)\right\}^6.$$

$\Delta = 99 = 3^2 \times 11$;

$$\alpha = 2^6\,(4591804316 + 799330532\sqrt{33}) \quad \text{(Weber)}$$

$$= (\sqrt{33} + 1)(55 + \sqrt{33})^3\,(23 + 4\sqrt{33})^3.$$

$\Delta = 107$, a prime; Mr. Russell obtains a cubic for α,

$$\alpha - 79 \times 80\,\sqrt[3]{2}\,\alpha^{\frac{2}{3}} - 69 \times 800\,\sqrt[3]{4}\,\alpha^{\frac{1}{3}} - 17 \times 16000 = 0.$$

$\Delta = 115 = 5 \times 23$;

$$J = -2^6 \times 5\sqrt{5}\,(157\sqrt{5} + 351)^3, \quad \gamma_2 + 1 = 3^2\,(6\sqrt{5} + 13)^2,$$

$$\gamma_3 = (6\sqrt{5} + 13)(378 + 169\sqrt{5})\,\sqrt{23}.$$

$\Delta = 123 = 3 \times 41$; not yet solved.

$\Delta = 131$, and $\Delta = 139$, both primes; and requiring, according to Russell, quintic and cubic equations for α, respectively.

$\Delta = 147 = 3 \times 7^2$. Then (Russell), with $z = \sqrt[12]{(4\kappa\kappa')}$,

$$z^8 - 2z^7 + 7z^4 - 4z + 1 = 0.$$

$\Delta = 155 = 5 \times 31$; not yet solved.

$\Delta = 163$; $\alpha = 2^{10} . 3^3 . 5^3 . 23^3 . 29^3$ (Hermite),

$$J = -\gamma_2^3, \quad \gamma_2 = 53360, \quad \gamma_3 = 185801\sqrt{163},$$

$$\gamma_2 + 1 = 3^2 \times 7^2 \times 11^2, \quad \gamma_2^2 - \gamma_2 + 1 = 3 \times 19^2 \times 127^2 \times 163.$$

Also, $$2s^3-4s^2+6s-1=0$$

gives $$s=\sqrt[12]{(\tfrac{1}{4}\kappa\kappa')}.$$

$\Delta=171=3^2\times 19$; not yet solved.

$\Delta=179$; not yet solved.

$\Delta=187=11\times 17$; $\quad \beta=3+\sqrt{17}$,

where $$\beta=\frac{1-2s^3}{s-s^3},$$

and $$\sqrt[3]{2}\,s=\sqrt[12]{(4\kappa\kappa')} \quad \text{(Russell)}.$$

$\Delta=211$; $\quad \beta^3-3\beta^2+\beta-2=0$,

where $$\beta=\frac{1-2s^3}{2s+2s^3} \quad \text{(Russell)}.$$

$\Delta=283$; $\quad \beta^3-4\beta^2-1=0$ (Russell).

$\Delta=331$; $\quad \beta^3-7\beta^2+9\beta-4$, $\quad \beta=\dfrac{1+2s^2-2s^3}{2s}$ (Russell).

$\Delta=427=7\times 61$;

$$2s^3+2s^2+(7+\sqrt{61})\,s-1=0 \quad \text{(Russell)}.$$

$\Delta=907$; $\beta^3-29\beta^2+85\beta-66=0$, $\quad \beta=\dfrac{1+4s^2-2s^3}{2s}$ (Russell).

Class B. $\Delta\equiv 7$, mod. 8.

Here the simplest function to tabulate seems to be $\kappa\kappa'$, or rather $\sqrt[4]{16\kappa\kappa'}$, and sometimes $\sqrt[12]{16\kappa\kappa'}$, when Δ is not a multiple of 3.

$\Delta=7$; $\sqrt[4]{\kappa\kappa'}=\frac{1}{2}$, $\sqrt[4]{16\kappa\kappa'}=1$, $\sqrt[12]{16\kappa\kappa'}=1$.

$\Delta=15=3\times 5$; $\sqrt[4]{\kappa\kappa'}=\sin 18^\circ$, or $\sin 54^\circ=\frac{1}{4}(\sqrt{5}\pm 1)$; or

$$\sqrt[4]{16\kappa\kappa'}=\tfrac{1}{2}(\sqrt{5}\pm 1), \quad (16\kappa\kappa')^{-\frac{1}{4}}-(16\kappa\kappa')^{\frac{1}{4}}=1.$$

$\Delta=23$; $x=\sqrt[12]{16\kappa\kappa'}$, given by the cubic equation $x^3+x^2-1=0$.

$\Delta=31$; $x=\sqrt[12]{16\kappa\kappa'}$, given by the cubic equation $x^3+x-1=0$.

$\Delta = 39$; $x = \sqrt[4]{16\kappa\kappa'}$, given by the equation

$$x^4+2x^3+4x^2+3x-1 = 0,$$

or

$$2x = -1+\{\tfrac{1}{2}(\sqrt{13}-1)\}^{\frac{3}{2}}$$

(Joubert, *Comptes Rendus*, t. 50).

$\Delta = 47$; $y = \sqrt[6]{16\kappa\kappa'}$ is given by the quintic equation

$$y^5+3y^2+2y-1 = 0;$$

and $x = \sqrt[12]{16\kappa\kappa'}$ by $\quad x^5-2x^4+2x^3-x^2+1 = 0.$

This quintic is soluble by radicals; the real root, according to Prof. Paxton Young, being $u_1+u_2+u_3+u_4$, where, as simplified by Cayley (*American Journal of Maths.*, Vol. XIII., p. 53),

$$10^5u_1 = 2600\,(15+7\sqrt{5})+40\,(79+33\sqrt{5})\sqrt{\{\tfrac{1}{2}(5+\sqrt{5})\}}\sqrt{47},$$

$$10^5u_2 = 2600\,(15-7\sqrt{5})-40\,(79-33\sqrt{5})\sqrt{\{\tfrac{1}{2}(5-\sqrt{5})\}}\sqrt{47},$$

$$10^5u_3 = 2600\,(15-7\sqrt{5})+40\,(79-33\sqrt{5})\sqrt{\{\tfrac{1}{2}(5-\sqrt{5})\}}\sqrt{47},$$

$$10^5u_4 = 2600\,(15+7\sqrt{5})-40\,(79+33\sqrt{5})\sqrt{\{\tfrac{1}{2}(5+\sqrt{5})\}}\sqrt{47}.$$

$\Delta = 55 = 5\times 11$; then, for

$$K'/K = \sqrt{55}, \quad \Lambda'/\Lambda = \sqrt{(11\div 5)},$$

$$\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'} = \frac{7-\sqrt{5}}{8},$$

$$-\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'} = \frac{\sqrt{5}\sqrt{(10\sqrt{5}-18)}}{8},$$

obtained by the combination of the modular equations of the 5th and 11th orders (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 322), and then

$$\sqrt[4]{16\kappa\kappa'} = \{\sqrt{(6\sqrt{5}-2)}-(5-\sqrt{5})\}\div 4,$$

$$\sqrt[4]{16\lambda\lambda'} = \{\sqrt{(6\sqrt{5}-2)}+(5-\sqrt{5})\}\div 4;$$

and then (Russell)

$$\sqrt[12]{16\kappa\kappa'} = \{\sqrt{(6\sqrt{5}-2)}-\sqrt{5}+1\}\div 4,$$

$$\sqrt[12]{16\lambda\lambda'} = \{\sqrt{(6\sqrt{5}-2)}+\sqrt{5}-1\}\div 4;$$

so that $\sqrt[12]{16\kappa\kappa'}$ and $-\sqrt[12]{16\lambda\lambda'}$ are the real roots of the biquadratic equation

$$z^4-z^3+2z-1 = 0.$$

These results were obtained by Mr. Russell, by putting

$$\sqrt[4]{\lambda} = -\sqrt[4]{\kappa'}, \quad \sqrt[4]{\lambda'} = \sqrt[4]{\kappa}$$

in his modular equation of the 71st order.

$\Delta = 63 = 3^2 \times 7$; $x = \sqrt[4]{16\kappa\kappa'}$, given by

$$(x^2 - x + 5)^2 - 21\,(x-1)^2 = 0$$

(Joubert, *Comptes Rendus*, t. 50).

$\Delta = 71$. Employing Russell's modular equation

$$P^3 + 4\,(-P^2 + Q)(4R)^{\frac{1}{3}} + 2P\,(4R)^{\frac{2}{3}} - 4R = 0,$$

where $\quad P = \sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} - 1, \quad Q = \sqrt[4]{\kappa\lambda\kappa'\lambda'} - \sqrt[4]{\kappa\lambda} - \sqrt[4]{\kappa'\lambda'},$

$$R = -\sqrt[4]{\kappa\lambda\kappa'\lambda'};$$

and putting $\kappa' = \lambda$, $\kappa = \lambda'$, and $x = \sqrt[12]{16\kappa\kappa'}$, then

$$(x+1)^2 (x^7 + x^6 - x^5 - x^4 - x^3 + x^2 + 2x - 1) = 0,$$

the equation of the 7th degree, being solvable by radicals, according to Abel.

$\Delta = 79$. The equation for $x = \sqrt[4]{16\kappa\kappa'}$ is, from Fiedler's modular equation,

$$(x^2 - x - 1)^2\,(x^5 - 4x^4 + 4x^3 - 5x^2 + 12x - 1) = 0,$$

the quintic equation being solvable by radicals. This is derived from Fiedler's modular equation for $n = 79$.

Then, according to Russell, the quintic factor reduces to

$$z^5 - z^4 + z^3 - 2z^2 + 3z - 1 = 0,$$

if

$$z = x^{\frac{1}{3}} = \sqrt[12]{16\kappa\kappa'}.$$

$\Delta = 87 = 3 \times 29$; not yet solved.

$\Delta = 95 = 5 \times 19$. (G. B. Mathews.) With $y^{12} = 256\,\kappa\lambda\kappa'\lambda'$,

$$y^{12} - 7y^8 - 2y^7 - 7y^6 - (y-1)^5 = 0,$$

or $\quad (y^3 + y^2 + 1)(y^9 - y^8 + y^7 - 2y^6 - 4y^5 + y^4 - 6y^3 + 9y^2 - 5y + 1) = 0,$

or $\quad (y^3 + y^2 + 1)\,\{(2y^2 + 2y + 1)^2 - 5\}\,(y^5 - 3y^4 + 5y^3 - 7y^2 + 4y - 1) = 0;$

and then, according to Russell, the equation for

$$x = \sqrt[12]{16\kappa\kappa'},$$

is $$(2x^4 - x^3 + x^2 + 4x + 1)^2 = 5\,(x^3 + x^2 + 1)^2.$$

Then $$\left(2x^2 - \frac{\sqrt{5}+1}{2}\,x + 1\right)^2 = (2\sqrt{5}-1)\left(\frac{\sqrt{5}+1}{2}\,x - 1\right)^2,$$

$$2x^2 + \{\sqrt{(2\sqrt{5}-1)} - 1\}\,\frac{\sqrt{5}+1}{2}\,x - \sqrt{(2\sqrt{5}-1)} + 1 = 0,$$

$$\left[2x + \{\sqrt{(2\sqrt{5}-1)} - 1\}\,\frac{\sqrt{5}+1}{4}\right]^2 = \frac{3\sqrt{5} - 3 + (5-\sqrt{5})\sqrt{(2\sqrt{5}-1)}}{16},$$

whence $$x = \sqrt[12]{16\kappa\kappa'} \text{ or } \sqrt[12]{16\lambda\lambda'},$$

for $$K'/K = \sqrt{95}, \quad \Lambda'/\Lambda = \sqrt{(19 \div 5)}.$$

$\Delta = 103$; $111 = 3 \times 37$; $119 = 7 \times 17$; 127; $135 = 5 \times 27$; $143 = 11 \times 13$; ... not yet solved.

Class C: $\Delta \equiv 1$, mod. 4.

Here Hermite's $$\alpha = \frac{(1 - 4\kappa^2\kappa'^2)^2}{\kappa^2\kappa'^2};$$

so that, if $\beta = \tfrac{1}{2}\sqrt{\alpha}$, $\beta = (2\kappa\kappa')^{-1} - 2\kappa\kappa'$;

and $\gamma = \sqrt{(\beta^2+4)} = (2\kappa\kappa')^{-1} + 2\kappa\kappa'$, $\sqrt{(\gamma+2)} = (2\kappa\kappa')^{-\frac{1}{2}} + (2\kappa\kappa')^{\frac{1}{2}}$;

also denoting $(2\kappa\kappa')^{-\frac{1}{3}} - (2\kappa\kappa')^{\frac{1}{3}}$ by t,

$$t^3 + 3t = \beta.$$

$\Delta = 1$; $\quad \beta = 0, \quad 2\kappa\kappa' = 1, \quad J = 1, \quad \gamma_3 = 0.$

$\Delta = 5$; $\quad \beta = 4, \quad (2\kappa\kappa')^{\frac{1}{3}} = \tfrac{1}{2}(\sqrt{5}-1), \quad t = 1.$

$\Delta = 9$; $\quad \beta = 8\sqrt{3}, \quad \gamma = 14, \quad \sqrt{(\gamma+2)} = 4,$

$$2\kappa\kappa' = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4;$$

$$(2\kappa\kappa')^{-\frac{1}{2}} + (2\kappa\kappa')^{\frac{1}{2}} = 4, \qquad (2\kappa\kappa')^{-\frac{1}{4}} - (2\kappa\kappa')^{\frac{1}{4}} = \sqrt{2}.$$

$\Delta = 13$; $\quad \beta = 36, \quad t = 3, \quad (2\kappa\kappa')^{\frac{1}{3}} = \tfrac{1}{2}(\sqrt{13}-3).$

$\Delta = 17$; $\quad \gamma = 10\,(\sqrt{17}+4), \quad \sqrt{(\gamma+2)} = 5 + \sqrt{17};$

$$(2\kappa\kappa')^{-\frac{1}{4}} + (2\kappa\kappa')^{\frac{1}{4}} = \tfrac{1}{2}(\sqrt{17}+1) \quad \text{(Weber)},$$

but immediately derivable from Mr. Russell's modular equation.

$$\Delta = 21\,; \qquad \beta = 3\,(\sqrt{3}+1)^4, \quad \gamma = 32\sqrt{7}+18\sqrt{21}\,;$$

$$2\kappa\kappa' = \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3 \left(\frac{3-\sqrt{7}}{\sqrt{2}}\right)^2, \quad \text{for } \mathrm{K}'/\mathrm{K} = \sqrt{(21)}\,;$$

$$2\lambda\lambda' = \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3 \left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^2, \quad \text{for } \Lambda'/\Lambda = \sqrt{(7\div 3)}.$$

The surds in the brackets are written so that the reciprocal is the complementary surd.

$$\Delta = 25 = 5^2; \quad \beta = 2^4\times 3^2 \times \sqrt{5}, \quad \gamma = 322 = 2\times 7\times 23,$$

$$\sqrt{(\gamma+2)} = 18, \quad \sqrt[12]{2\kappa\kappa'} = \tfrac{1}{2}\,(\sqrt{5}-1)\,;$$

so that $\quad (2\kappa\kappa')^{-\frac{1}{12}}-(2\kappa\kappa')^{\frac{1}{12}} = 1, \quad (2\kappa\kappa')^{-\frac{1}{12}}+(2\kappa\kappa')^{\frac{1}{12}} = \sqrt{5}.$

$\Delta = 29.$ Here β is given by the cubic equation

$$\beta^3-588\beta^2-976\beta-3136 = 0,$$

and t by the cubic equation

$$t^3-9t^2+8t-20 = 0\,;$$

so that $x = \sqrt[3]{2\kappa\kappa'}$ is given by the reciprocal sextic

$$x^6+9x^5+5x^4+2x^3-5x^2+9x-1 = 0 \quad \text{(Weber)},$$

or by $\qquad 2x^3+9x^2+8x+5 = \sqrt{(29)}(x+1)^2.$

This is immediately derivable from Russell's modular equation for $n = 29$ (*Proc. Lond. Math. Soc.*, Vol. XXI., a paper not yet published),

$$\begin{aligned} P^5 &+ (4R)^{\frac{1}{3}}\,(-94P^4+2^8\,.\,3^2PQ-2^{13}Q^2) \\ &+ (4R)^{\frac{2}{3}}\,(2^2\,.\,505P^3-2^9\,.\,13PQ) \\ &+ 4R\,(-2^3\,.\,745P^2+2^{10}\,.\,15PQ)+(4R)^{\frac{4}{3}}\,P \quad -(4R)^{\frac{5}{3}} = 0\,; \end{aligned}$$

or $\qquad P^{\frac{1}{2}}\,\{P^2+34P\,(4R)^{\frac{1}{3}}-36\,(4R)^{\frac{2}{3}}\}$

$$\pm\sqrt{2}\,(4R)^{\frac{1}{6}}\,\{9P^2-64Q-26P\,(4R)^{\frac{1}{3}}+60\,(4R)^{\frac{2}{3}}\} = 0\,;$$

where $\qquad P = \kappa\lambda+\kappa'\lambda'-1, \quad Q = \kappa\lambda\,\kappa'\lambda'-\kappa\lambda-\kappa'\lambda',$

$$R = -\,\kappa\lambda\,\kappa'\lambda'\,;$$

on putting $\kappa' = \lambda$, $\kappa = \lambda'$, and $x = \sqrt[3]{2\kappa\kappa'}$.

$\Delta = 33$; $\qquad a = 2^4 \times 3\,(75+13\sqrt{33})^2,$

$$\gamma = 10\,(52+1\sqrt{33}), \quad \sqrt{(\gamma+2)} = 3\,(\sqrt{33}+5),$$

$$2\kappa\kappa' = \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^2 \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6, \quad K'/K = \sqrt{33};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{11}+3}{\sqrt{2}}\right)^2 \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6, \quad \Lambda'/\Lambda = \sqrt{(11 \div 3)}.$$

$\Delta = 37$; $\qquad a = 2^6 \times 3^4 \times 7^4, \qquad \beta = 2^3 \times 3^2 \times 7^2,$

$$\gamma = 2 \times 5 \times 29 \times \sqrt{37},$$

$$2\kappa\kappa' = (\sqrt{37}-6)^3 \quad \text{(Kronecker)},$$

$$(2\kappa\kappa')^{-\frac{1}{3}} - (2\kappa\kappa')^{\frac{1}{3}} = 12.$$

$\Delta = 41$. Weber's equation (*Acta Math.*, XI., p. 388) for

$$x = (2\kappa\kappa')^{-\frac{1}{6}} + (2\kappa\kappa')^{\frac{1}{6}},$$

is

$$x^2 - \tfrac{1}{2}\,(\sqrt{41}+5)^2 + \tfrac{1}{2}\,(7+\sqrt{41}) = 0,$$

or

$$x^4 - 5x^3 + 3x^2 + 3x + 2 = 0.$$

$\Delta = 45 = 3^2 \times 5$. Here

$$a = 2^6\,(17+10\sqrt{3})^4, \qquad \beta = 2^2\,(17+10\sqrt{3})^2,$$

$$\gamma = 2\sqrt{5}\,(527+304\sqrt{3});$$

$$2\kappa\kappa' = \left(\frac{\sqrt{5}-1}{2}\right)^6 \left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^4, \quad K'/K = \sqrt{(45)};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{5}-1}{2}\right)^6 \left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)^4, \quad \Lambda'/\Lambda = \sqrt{(9 \div 5)}.$$

$\Delta = 49 = 7^2$. Here

$$a = 2^5 \times 3^4\,(3+\sqrt{7})^3\,\sqrt{7},$$

$$\gamma = 2\,(3^4 \times 23 + 2^6 \times 11\sqrt{7}), \quad \sqrt{(\gamma+2)} = 4\,(\sqrt{7}+2)^3;$$

and

$$2\kappa\kappa' = \left(\frac{\sqrt{7}+1-\sqrt{2}\,\sqrt[4]{7}}{2\sqrt{2}}\right)^{12},$$

so that

$$(2\kappa\kappa)^{\frac{1}{12}} + (2\kappa\kappa')^{-\frac{1}{12}} = \frac{\sqrt{7}+1}{\sqrt{2}}.$$

$\Delta = 53$. According to Russell and Mathews, the equation for t is

$$t^3 - 23t^2 + 12t - 121 = 0,$$

or for $x = \sqrt[3]{2\kappa\kappa'}$ is the reciprocal sextic

$$x^6 + 23x^5 + 12x^4 + 75x^3 - 12x^2 + 23x - 1 = 0.$$

$\Delta = 57 = 3 \times 19$. Here

$$2\kappa\kappa' = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6 \left(\frac{3\sqrt{19}-13}{\sqrt{2}}\right)^2, \quad K'/K = \sqrt{57};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^6 \left(\frac{3\sqrt{19}-13}{\sqrt{2}}\right)^2, \quad \Lambda'/\Lambda = \sqrt{(19 \div 3)}.$$

$\Delta = 61$. According to Mathews' result (*Proc. Lond. Math. Soc.*, Dec., 1889, Vol. XXI.), the equation for

$$t = (2\kappa\kappa')^{-\frac{1}{4}} - (2\kappa\kappa')^{\frac{1}{4}},$$

is
$$t^3 - 30t^2 + 9t - 108 = 0,$$

or
$$m^3 - 27m^2 - 5m - 1 = 0,$$

for
$$a = 2^6 . 3^4 . m^4.$$

For $x = \sqrt[3]{2\kappa\kappa'}$, the equation

$$x^6 + 30x^5 + 6x^4 + 48x^3 - 6x^2 + 30x - 1 = 0.$$

$\Delta = 65 = 5 \times 13$. (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 332). On combining the modular equations of the 5th and 13th orders, and putting

$$4\kappa\lambda\, \kappa'\lambda' = x^6,$$

then
$$x^{-1} + x = \tfrac{1}{2}(\sqrt{65}+5), \quad x^{-\frac{1}{2}} + x^{\frac{1}{2}} = \tfrac{1}{2}(\sqrt{13}+\sqrt{5}),$$

and
$$\kappa\kappa' + \lambda\lambda' = x^3(2x^{-2} - 5 + 2x^2),$$

and then
$$(2\kappa\kappa')^{-\frac{1}{12}} - (2\kappa\kappa')^{\frac{1}{12}} = (2\lambda\lambda')^{-\frac{1}{12}} + (2\lambda\lambda')^{\frac{1}{12}}$$
$$= \sqrt[4]{(\sqrt{65}+8)} \quad \text{(Russell)},$$

for
$$K'/K = \sqrt{(65)}, \quad \Lambda'/\Lambda = \sqrt{(13 \div 5)}.$$

$\Delta = 69 = 3 \times 23$; proceeding in a similar manner with

$$4\kappa\lambda\, \kappa'\lambda' = x^{12},$$

we find $x^{-1}+x = \frac{1}{2}(3\sqrt{2}+\sqrt{6}), \quad \kappa\kappa'+\lambda\lambda' = \sqrt{2}\,x^3(1-x^6)$

(*Proc. Lond. Math. Soc.*, Vol. XIX., p. 332); and then (Russell) the equation for

$$\beta = (2\kappa\kappa')^{-1}-2\kappa\kappa',$$

when $K'/K = \sqrt{69}$, or $\sqrt{(23\div 3)}$; is

$$\beta^2-12\sqrt{3}(2+\sqrt{3})(374+216\sqrt{3})\beta+144(31+18\sqrt{3})^2 = 0.$$

$\Delta = 73$. We find, by approximate numerical calculation (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 363),

$$(2\kappa\kappa')^{-\frac{1}{4}}+(2\kappa\kappa')^{\frac{1}{4}} = \tfrac{1}{2}(\sqrt{73}+5),$$

so that $\gamma = 4930\sqrt{73}+42120, \quad \sqrt{(\gamma+2)} = 17\sqrt{73}+145.$

$\Delta = 77 = 7\times 11$; (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 334) with

$$4\kappa\lambda\,\kappa'\lambda' = x^{12},$$

we find $\qquad x^{-1}+x = y = \frac{1}{2}(\sqrt{22}+\sqrt{2});$

and $\qquad \kappa\kappa'+\lambda\lambda' = \sqrt{2}\,x(1-4x^2+4x^4-x^6)(2x-3x^3+2x^5).$

Then (Russell) $\qquad z^2-(23+8\sqrt{11})z+4 = 0$

gives $\qquad z = (2\kappa\kappa')^{-\frac{1}{2}}-(2\kappa\kappa')^{\frac{1}{2}},$

for $\qquad K'/K = \sqrt{(77)}$ or $\sqrt{(11\div 7)}$.

$\Delta = 81 = 3^4$; not yet solved.

$\Delta = 85 = 5\times 17$. Here

$$2\kappa\kappa' = \left(\frac{\sqrt{5}-1}{2}\right)^{12}\left(\frac{\sqrt{85}-9}{2}\right)^3, \quad K'/K = \sqrt{85};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{5}+1}{2}\right)^{12}\left(\frac{\sqrt{85}-9}{2}\right)^3, \quad \Lambda'/\Lambda = \sqrt{(17\div 5)};$$

so that $\qquad (2\kappa\kappa')^{-\frac{1}{2}}-(2\kappa\kappa')^{\frac{1}{2}} = \frac{3}{2}(21+5\sqrt{17}),$

$$(2\lambda\lambda')^{-\frac{1}{2}}-(2\lambda\lambda')^{\frac{1}{2}} = \tfrac{3}{2}(21-5\sqrt{17}).$$

$\Delta = 89$. Here $p = 6$ (Gauss); a 12-ic in $\sqrt[6]{2\kappa\kappa'}$ must be expected (Russell); not yet calculated.

$\Delta = 93 = 3 \times 31$;

$$2\kappa\kappa' = \left(\frac{\sqrt{31} - 3\sqrt{3}}{2}\right)^3 \left(\frac{39 - 7\sqrt{31}}{\sqrt{2}}\right)^2, \quad K'/K = \sqrt{93};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{31} + 3\sqrt{3}}{2}\right)^3 \left(\frac{39 - 7\sqrt{31}}{\sqrt{2}}\right)^2, \quad \Lambda'/\Lambda = \sqrt{(31 \div 3)}.$$

$\Delta = 97$. By approximate calculation (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 364),

$$(2\kappa\kappa')^{-\frac{1}{6}} + (2\kappa\kappa')^{\frac{1}{6}} = \tfrac{1}{2}(\sqrt{97} + 9);$$

$$\sqrt{(\gamma + 2)} = (2\kappa\kappa')^{-\frac{1}{2}} + (2\kappa\kappa')^{\frac{1}{2}} = 41\sqrt{97} + 405.$$

$\Delta = 101$; a prime. Since Gauss's $p = 7$, no simple result can be expected.

$\Delta = 105 = 3 \times 5 \times 7$; (Kronecker and Weber; also *Proc. Lond. Math. Soc.*, Vol. XIX., p. 338),

$$2\kappa\kappa' = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6 \left(\frac{\sqrt{5}-1}{2}\right)^6 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^6 \left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}}\right)^2,$$

$$K'/K = \sqrt{105};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6 \left(\frac{\sqrt{5}+1}{2}\right)^6 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^6 \left(\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}}\right)^2,$$

$$\Lambda'/\Lambda = \sqrt{(35 \div 3)};$$

$$2\kappa\kappa' = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^6 \left(\frac{\sqrt{5}-1}{2}\right)^6 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^6 \left(\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}}\right)^2,$$

$$K'/K = \sqrt{(21 \div 5)};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^6 \left(\frac{\sqrt{5}+1}{2}\right)^6 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^6 \left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}}\right)^2,$$

$$\Lambda'/\Lambda = \sqrt{(15 \div 7)}.$$

$\Delta = 109, 113, 117$, not yet solved. But 113 should have a result of same form as 41.

$\Delta = 121$; Russell puts $\lambda = \lambda' = \frac{1}{2}\sqrt{2}$,

in the modular equation of the 11th order, and obtains for

$$x = (2\kappa\kappa')^{-\frac{1}{12}} - (2\kappa\kappa')^{\frac{1}{12}},$$

$$x^3 - 2\sqrt{2}x^2 - x - \sqrt{2} = 0.$$

$\Delta = 129$; not yet solved.

$\Delta = 133 = 7 \times 19$. According to Weber (*Math. Ann.*, Vol. XXXIII., p. 402),

$$2\kappa\kappa' = \left(\frac{3-\sqrt{7}}{\sqrt{2}}\right)^6 \left(\frac{5\sqrt{7}-3\sqrt{19}}{2}\right)^8, \quad K'/K = \sqrt{133},$$

$$2\lambda\lambda' = \left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^6 \left(\frac{5\sqrt{7}-3\sqrt{19}}{2}\right)^8, \quad \Lambda'/\Lambda = \sqrt{19 \div 7}.$$

$\Delta = 137, 141, 145, 149, 153, 157$, not yet solved.

$\Delta = 161 = 7 \times 23$. Putting $4\kappa\lambda\kappa'\lambda' = x^{12}$, and $x^{-1}+x = \sqrt{2}\,y$, then (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 379), the equation for y is found, by combination of the modular equations of the 7th and 23rd orders, to be

$$4y^6-32y^5+100y^4-160y^3+113y^2-24y+9 = 0,$$

or $$(2y^2-6y+9)\left\{(25y^2-y+1)^2-23y^4\right\} = 0,$$

or $$(2y^2-6y+9)\left\{(2y^2-5y+5)^2-23(y-1)^2\right\} = 0.$$

$\Delta = 165 = 3 \times 5 \times 11$; according to Weber (*Math. Ann.*, XXXIII.),

$$2\kappa\kappa' = \left(\frac{\sqrt{5}-1}{2}\right)^6 \left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^6 (3\sqrt{5}-2\sqrt{11})^2 \left(\frac{\sqrt{15}-\sqrt{11}}{2}\right)^8,$$

$$K'/K = \sqrt{(165)} = \sqrt{(3 \times 5 \times 11)};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{5}-1}{2}\right)^6 \left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)^6 (3\sqrt{5}-2\sqrt{11})^2 \left(\frac{\sqrt{15}+\sqrt{11}}{2}\right)^8,$$

$$\Lambda'/\Lambda = \sqrt{(55 \div 3)};$$

$$2\kappa\kappa' = \left(\frac{\sqrt{5}+1}{2}\right)^6 \left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^6 (3\sqrt{5}-2\sqrt{11})^2 \left(\frac{\sqrt{15}+\sqrt{11}}{2}\right)^8,$$

$$K'/K = \sqrt{(33 \div 5)};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{5}+1}{2}\right)^6 \left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)^6 (3\sqrt{5}-2\sqrt{11})^2 \left(\frac{\sqrt{15}-\sqrt{11}}{2}\right)^8,$$

$$\Lambda'/\Lambda = \sqrt{(15 \div 11)}.$$

$\Delta = 169 = 13^2$. According to Russell, when $\Delta = n^2$, and n is not a multiple of 3, then we obtain an equation for $z = \sqrt[12]{2\kappa\kappa'}$ by putting $\lambda = \lambda' = \frac{1}{2}\sqrt{2}$ in the modular equation of the nth degree; in this case he obtains

$$x^3-4x^2+x-12 = 0,$$

for $$x = (2\kappa\kappa')^{-\frac{1}{12}}-(2\kappa\kappa')^{\frac{1}{12}}.$$

$\Delta = 173$; not yet solved.

$\Delta = 177 = 3 \times 59$. According to Weber (*Math. Ann.*, XXXIII.),

$$2\kappa\kappa' = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{18} \left(\frac{3\sqrt{59}-23}{\sqrt{2}}\right)^{2}, \quad K'/K = \sqrt{177},$$

$$2\lambda\lambda' = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^{18} \left(\frac{3\sqrt{59}-23}{\sqrt{2}}\right)^{2}, \quad \Lambda'/\Lambda = \sqrt{(59 \div 3)}.$$

$\Delta = 193$; $(2\kappa\kappa')^{-\frac{1}{2}} + (2\kappa\kappa')^{\frac{1}{2}} = \sqrt{193}+13$ (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 363). Obtained by approximate calculation.

$\Delta = 253 = 11 \times 23$; (Weber, *Math. Ann.*, XXXIII., p. 402),

$$2\kappa\kappa' = \left(\frac{5-\sqrt{23}}{\sqrt{2}}\right)^{6} \left(\frac{9\sqrt{23}-13\sqrt{11}}{2}\right)^{2}, \quad K'/K = \sqrt{253},$$

$$2\lambda\lambda' = \left(\frac{5-\sqrt{23}}{\sqrt{2}}\right)^{6} \left(\frac{9\sqrt{23}+13\sqrt{11}}{2}\right)^{2}, \quad \Lambda'/\Lambda = \sqrt{(23 \div 11)}.$$

$\Delta = 273 = 3 \times 7 \times 13$; (Weber),

$$2\kappa\kappa' = \left(\frac{\sqrt{13}-3}{2}\right)^{6} \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{6} \left(\frac{15\sqrt{7}-11\sqrt{13}}{2}\right)^{2},$$

$$K'/K = \sqrt{(273)}.$$

$\Delta = 385 = 5 \times 7 \times 11$; (Weber),

$$2\kappa\kappa' = \left(\frac{\sqrt{5}-1}{2}\right)^{12} \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{11}-\sqrt{7}}{2}\right)^{6},$$

$$K'/K = \sqrt{385};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{5}-1}{2}\right)^{12} \left(\frac{\sqrt{11}+3}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{11}+\sqrt{7}}{2}\right)^{6},$$

$$\Lambda'/\Lambda = \sqrt{(77 \div 5)};$$

$$2\kappa\kappa' = \left(\frac{\sqrt{5}+1}{2}\right)^{12} \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{11}+\sqrt{7}}{2}\right)^{6},$$

$$K'K/ = \sqrt{(55 \div 7)};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{5}-1}{2}\right)^{12} \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{11}+\sqrt{7}}{2}\right)^{6},$$

$$\Lambda'/\Lambda = \sqrt{(35 \div 11)}.$$

$\Delta = 345 = 3 \times 5 \times 23$. (Weber.)

$$2\kappa\kappa' = \left(\frac{\sqrt{5}-1}{2}\right)^{12} \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{0} \left(\frac{3\sqrt{3}-\sqrt{23}}{2}\right)^{0} \left(\frac{7\sqrt{23}-15\sqrt{5}}{\sqrt{2}}\right)^{2},$$

$$K'/K = \sqrt{(345)}.$$

$\Delta = 357 = 3 \times 7 \times 17$. (Weber.)

$$2\kappa\kappa' = \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{24} \left(\frac{3-\sqrt{7}}{2}\right)^{12} \left(\frac{\sqrt{21}-\sqrt{17}}{2}\right)^{0} \left(\frac{11-\sqrt{119}}{\sqrt{2}}\right)^{4},$$

$$K'/K = \sqrt{(357)}.$$

$\Delta = 1365 = 3 \times 5 \times 7 \times 13$. (Weber, *Math. Ann.*, XXXIII., p. 402.)

$$2\kappa\kappa' = \left(\frac{\sqrt{5}-1}{2}\right)^{6} \left(\frac{3-\sqrt{7}}{\sqrt{2}}\right)^{0} \left(\frac{\sqrt{13}-3}{2}\right)^{6} \left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^{0} \left(\frac{\sqrt{7}-13}{2}\right)^{12}$$

$$\times (\sqrt{13}-2\sqrt{3})^{0} \left(\frac{\sqrt{39}-\sqrt{35}}{2}\right)^{8} \left(\frac{\sqrt{65}-3\sqrt{7}}{\sqrt{2}}\right)^{2},$$

$$K'/K = \sqrt{1365};$$

the other values of $2\kappa\kappa'$ for $K'/K = \sqrt{(1365)} \div m$, where m is a factor of 1365, being obtained by an appropriate change of sign between the surds of the factors; so also for $\Delta = 273, 345, 357$, &c.

Class D; $\Delta \equiv 2$, mod. 4.

Here Hermite's invariant α is given by

$$\alpha = -\frac{(1+\kappa^2)^4}{\kappa^2\kappa'^4};$$

and we put $\kappa'\lambda' = 2\sqrt{(\kappa\lambda)}$, equivalent to the modular equation for $n = 2$; and then introduce numbers β and γ, given by

$$\sqrt[4]{(-\alpha)} = \beta = (\kappa\lambda)^{-\frac{1}{2}} - (\kappa\lambda)^{\frac{1}{2}}, \quad \gamma = \sqrt{(\beta^2+4)} = (\kappa\lambda)^{-\frac{1}{2}} + (\kappa\lambda)^{\frac{1}{2}}.$$

Also, if $\beta = 2\cosh 2\phi$; then

$$2\sinh 2\phi = \sqrt{2}\left[\left\{\tfrac{1}{2}(\kappa^{-1}-\kappa)\right\}^{\frac{1}{2}} + \left\{\tfrac{1}{2}(\kappa^{-1}-\kappa)\right\}^{-\frac{1}{2}}\right];$$

$$\kappa+\lambda = 2\sqrt{\kappa\lambda}\cosh 2\phi; \quad \kappa^{2n}+\lambda^{2n} = 2(\kappa\lambda)^n \cosh n\phi;$$

$$-\kappa+\lambda = 2\sqrt{\kappa\lambda}\sinh 2\phi; \quad -\kappa^{2n}+\lambda^{2n} = 2(\kappa\lambda)^n \sinh n\phi;$$

and then $$e^{4\phi} = \lambda/\kappa = v^4/u^4,$$

$$\gamma = \sqrt{2}\left[\left\{\tfrac{1}{2}(\kappa^{-1}-\kappa)\right\}^{\frac{1}{2}} + \left\{\tfrac{1}{2}(\kappa^{-1}-\kappa)^{-\frac{1}{2}}\right\}\right],$$

in Jacobi's notation.

$\Delta = 2$; $\alpha = -2^4$, $\beta = 2$, $\cosh\phi = 1$, $\phi = 0$, $\kappa = \lambda = \sqrt{2}-1$.

$\Delta = 6$;

$$\alpha = -2^4\times 3^3,\quad \beta = 2\sqrt{3},\quad \gamma = 4,\quad \cosh 2\phi = \sqrt{3},\quad \sinh 2\phi = \sqrt{2};$$

$$\kappa = (\sqrt{3}-\sqrt{2})\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2,\quad \text{for}\quad K'/K = \sqrt{6};$$

$$\lambda = (\sqrt{3}+\sqrt{2})\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2,\quad \text{for}\quad \Lambda'/\Lambda = \sqrt{(2\div 3)}.$$

$\Delta = 10$; $\beta = 6$, $\cosh 2\phi = 3$, $\sinh\phi = 1$, $\alpha = -2^4\times 3^4$,

$$\kappa = (\sqrt{2}-1)^2(\sqrt{10}-3),\quad \text{for}\quad K'/K = \sqrt{10},$$

$$\lambda = (\sqrt{2}+1)^2(\sqrt{10}-3),\quad \text{for}\quad \Lambda'/\Lambda = \sqrt{(2\div 5)}.$$

$\Delta = 14$; then

$$\alpha = -2^4(8\sqrt{2}+11)^2,\quad \gamma = 4\sqrt{2}+4,\quad \sqrt{(\gamma+2)} = 2+\sqrt{2};$$

$$\cosh 2\phi = \sqrt{(8\sqrt{2}+11)},\quad \sinh 2\phi = \sqrt{(8\sqrt{2}+10)},$$

$$\kappa = \left\{2\sqrt{2}+2-\sqrt{(8\sqrt{2}+11)}\right\}\left\{\sqrt{(8\sqrt{2}+11)}-\sqrt{(8\sqrt{2}+10)}\right\},$$

$$K'/K = \sqrt{14},$$

$$\lambda = \left\{2\sqrt{2}+2-\sqrt{(8\sqrt{2}+11)}\right\}\left\{\sqrt{(8\sqrt{2}+11)}+\sqrt{(8\sqrt{2}+10)}\right\},$$

$$\Lambda'/\Lambda = \sqrt{(2\div 7)}.$$

$\Delta = 18$; $\alpha = -2^4\times 7^4$,

$$\cosh 2\phi = 7,\quad \sinh 2\phi = 4\sqrt{3},\quad \cosh\phi = 2,\quad \sinh\phi = \sqrt{3};$$

$$\kappa = (\sqrt{2}-1)^3\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4,\quad K'/K = \sqrt{18} = 3\sqrt{2},$$

$$\lambda = (\sqrt{2}-1)^3\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^4,\quad \Lambda'/\Lambda = \sqrt{(2\div 9)} = \tfrac{1}{3}\sqrt{2}.$$

$\Delta = 22$; $\alpha = -\beta^4 = -2^4\times 3^4\times 11^2$, $\beta = 6\sqrt{11}$, $\gamma = 20$;

$$\cosh 2\phi = 3\sqrt{11},\ \sinh 2\phi = 7\sqrt{2},\ \cosh 4\phi = 197,\ \sinh 4\phi = 42\sqrt{22};$$

$$\kappa = (3\sqrt{11}-7\sqrt{2})\left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^2,\ K'/K = \sqrt{22};$$

$$\lambda = (3\sqrt{11}+7\sqrt{2})\left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^2,\ \Lambda'/\Lambda = \sqrt{(2\div 11)}.$$

$\Delta = 26$. Here $y = \sinh\phi$ is given by the cubic equation

$$y^3+3y^2+2y+2 = 0 \text{ or } (y+1)^3-y+1 = 0.$$

$\Delta = 30$;

$$\beta = 2\sqrt{3}\,(4\sqrt{2}+5),\quad \gamma = 20+12\sqrt{2},\quad \sqrt{(\gamma+2)} = 3\sqrt{2}+2,$$

$$\cosh 2\phi = \sqrt{3}\,(4\sqrt{2}+5),\quad \sinh 2\phi = \sqrt{10}\,(3+2\sqrt{2}),$$

$$e^{2\phi} = (\sqrt{6}+\sqrt{5})\left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)^2;$$

$$\kappa = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2(\sqrt{3}-\sqrt{2})^2(\sqrt{6}-\sqrt{5})\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^2,$$

$$K'/K = \sqrt{30};$$

$$\lambda = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^2(\sqrt{3}-\sqrt{2})^2(\sqrt{6}+\sqrt{5})\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^2,$$

$$\Lambda'/\Lambda = \sqrt{(10\div 3)};$$

$$\kappa = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^2(\sqrt{3}-\sqrt{2})^2(\sqrt{6}-\sqrt{5})\left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)^2,$$

$$K'/K = \sqrt{(6\div 5)};$$

$$\lambda = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2(\sqrt{3}-\sqrt{2})^2(\sqrt{6}+\sqrt{5})\left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)^2,$$

$$\Lambda'/\Lambda = \sqrt{(2\div 15)}.$$

$\Delta = 34$; $\cosh 2\phi = 3\,(\sqrt{17}+4)$,

$$\cosh\phi = \tfrac{1}{2}(\sqrt{17}+3) = \left(\frac{\kappa^{-1}-\kappa}{2}\right)^{\frac{1}{6}}+\left(\frac{\kappa^{-1}-\kappa}{2}\right)^{-\frac{1}{6}}.$$

$\Delta = 38$. Here β is given by the cubic equation

$$\beta^3-5\sqrt{2}\,\beta^2-2\beta-22\sqrt{2} = 0.$$

$\Delta = 42$. Taking Weber's value of f_1 (*Math. Ann.*, XXXIII., p. 402), which gives

$$\tfrac{1}{2}(\kappa^{-1}-\kappa) = \tfrac{1}{8}f_1^{12} = \left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^6(2\sqrt{2}+\sqrt{7})^2,$$

we find

$$\beta = 42+16\sqrt{6},\quad \gamma = 16\sqrt{7}+6\sqrt{42},$$

$$\cosh 2\phi = 21+8\sqrt{6},\quad \cosh\phi = 2\sqrt{2}+\sqrt{3},\quad \sinh\phi = \sqrt{6}+2,$$

$$\kappa = \left(\frac{3-\sqrt{7}}{\sqrt{2}}\right)^2(\sqrt{7}-\sqrt{6})(\sqrt{2}-1)\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2,$$

$$K'/K = \sqrt{42};$$

$$\lambda = \left(\frac{3-\sqrt{7}}{\sqrt{2}}\right)^2 (\sqrt{7}-\sqrt{6})(\sqrt{2}+1)\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^2,$$

$$\Lambda'/\Lambda = \sqrt{2 \div 21};$$

$$\kappa = \left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^2 (\sqrt{7}-\sqrt{6})(\sqrt{2}-1)\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^2,$$

$$K'/K = \sqrt{14 \div 3};$$

$$\lambda = \left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^2 (\sqrt{7}-\sqrt{6})(\sqrt{2}+1)\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2,$$

$$\Lambda'/\Lambda = \sqrt{6 \div 7}.$$

$\Delta = 46$; $\beta = 6\sqrt{(147+104\sqrt{2})}$, $\gamma = 4(13+9\sqrt{2})$,

$$\sqrt{(\gamma+2)} = 6+3\sqrt{2}, \quad \left(\frac{\kappa^{-1}-\kappa}{2}\right)^{\frac{1}{2}} + \left(\frac{\kappa^{-1}-\kappa}{2}\right)^{-\frac{1}{2}} = 3+\sqrt{2}.$$

$\Delta = 50$, 54, not yet solved.

$\Delta = 58$; $\beta = 198$, $\gamma = 26\sqrt{58}$ (Hermite), $\sin \frac{1}{2}\phi = 1$.

$$\kappa = (\sqrt{2}-1)^6 (13\sqrt{58}-99), \quad K'/K = \sqrt{58},$$

$$\lambda = (\sqrt{2}+1)^6 (13\sqrt{58}-99), \quad \Lambda'/\Lambda = \sqrt{(2 \div 29)}$$

$\Delta = 62$; $\gamma = 4\sqrt{2}+60+8\sqrt{(80\sqrt{2}+113)}$,

$$\sqrt{(\gamma+2)} = 4\sqrt{2}+4+(2+\sqrt{2})\sqrt{(4\sqrt{2}+1)}.$$

$\Delta = 66$; not yet solved.

$\Delta = 70$; $\dfrac{\kappa^{-1}-\kappa}{2} = (\sqrt{2}+1)^6 \left(\dfrac{\sqrt{5}+1}{2}\right)^{12}$,

$$\gamma = 31878\sqrt{2}+20160\sqrt{5} \quad \text{(Weber)}.$$

$\Delta = 74$; not yet solved.

$\Delta = 78$; $\gamma = 20(13+9\sqrt{2})$, $\sqrt{(\gamma+2)} = 9\sqrt{2}+10$,

$$\cosh 2\phi = \sqrt{3}(75+52\sqrt{2}), \quad \sinh 2\phi = \sqrt{13}(36+25\sqrt{2}),$$

$$e^{2\phi} = (\sqrt{13}+2\sqrt{3})^2 (3\sqrt{3}+\sqrt{26}),$$

$$\frac{\kappa^{-1}-\kappa}{2} = \left(\frac{\sqrt{13}+3}{2}\right)^6 (\sqrt{26}+5)^2.$$

$\Delta = 94$; $\gamma = 252+180\sqrt{2}+8\sqrt{(1422\sqrt{2}+2011)}$,

$$\sqrt{(\gamma+2)} = 6\sqrt{2}+8+\sqrt{(84\sqrt{2}+118)}.$$

$\Delta = 98$; not yet solved.

$\Delta = 102$; $\quad \kappa^{-1}-\kappa = 2(\sqrt{2}+1)^6(3\sqrt{2}+\sqrt{17})^4$ (Weber).

$\Delta = 106, 110, \ldots$; not yet solved.

$\Delta = 130$; $\quad \kappa^{-1}-\kappa = 2\left(\frac{\sqrt{5}+1}{2}\right)^{18}\left(\frac{\sqrt{13}+3}{2}\right)^6$ (Weber, p. 413).

$\Delta = 134, 138$; not yet solved.

$\Delta = 142$; solved by means of Russell's modular equation of the order 71 (*Proc. Lond. Math. Soc.*, Vol. XXI.). Putting

$$\kappa\lambda = x^8, \quad x^{-1}-x = v = \sqrt[4]{2}\,t;$$

then $\quad t^3-(4\sqrt{2}+3)\,t^2+(7+4\sqrt{2})\,t-9-6\sqrt{2} = 0,$

which has the real root $\quad t = 3\sqrt{2}+3,$

so that $\sqrt{(\gamma+2)} = \sqrt{2}\,t^3+2 = 27\sqrt{2}+38, \;\; \gamma = 2900+2052\sqrt{2}$;

$$\left(\frac{\kappa^{-1}-\kappa}{2}\right)^{\frac{1}{6}}+\left(\frac{\kappa^{-1}-\kappa}{2}\right)^{-\frac{1}{6}} = 9+5\sqrt{2}.$$

The following cases have also been worked out by Weber (*Math. Ann.*, Vol. XXXIII.):—

$\Delta = 82$; $\quad \left(\frac{\kappa^{-1}-\kappa}{2}\right)+\left(\frac{\kappa^{-1}-\kappa}{2}\right)^{-\frac{1}{6}} = \frac{15+\sqrt{41}}{2}.$

$\Delta = 190$; $\quad \frac{\kappa^{-1}-\kappa}{2} = \left(\frac{\sqrt{5}+1}{2}\right)^{18}(\sqrt{10}+3)^6.$

$\Delta = 210$;

$$\frac{\kappa^{-1}-\kappa}{2} = \left(\frac{\sqrt{5}+1}{2}\right)^6(\sqrt{3}+\sqrt{2})^6\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^6(3\sqrt{14}+5\sqrt{5}).$$

$\Delta = 330$;

$$\frac{\kappa^{-1}-\kappa}{2} = \left(\frac{\sqrt{5}+1}{2}\right)^{12}(\sqrt{6}+\sqrt{5})^6\left(\frac{\sqrt{15}+\sqrt{11}}{2}\right)^6(\sqrt{11}+\sqrt{10})^2.$$

$\Delta = 462$;

$$\frac{\kappa^{-1}-\kappa}{2} = \left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^6(\sqrt{7}+\sqrt{6})^6\left(\frac{\sqrt{11}+\sqrt{7}}{2}\right)^6(22\sqrt{22}+39\sqrt{7})^2.$$

A few cases for $\Delta \equiv 0$, mod. 4, taken from Weber's paper in the *Math. Ann.*, Vol. XXXIII., p. 406, may be added here; Weber's f_1 being connected with κ by the relation

$$\tfrac{1}{4} f_1^{12} = \kappa^{-1} - \kappa.$$

$$\Delta = 4; \quad \kappa^{-1} - \kappa = 4\sqrt{2}, \quad \kappa^{-1} + \kappa = 6, \quad \kappa = (\sqrt{2}-1)^2.$$

$$\Delta = 8; \quad \kappa^{-1} - \kappa = 4\sqrt{2}\,(\sqrt{2}+1)^{\frac{3}{2}}, \quad \kappa^{-1} + \kappa = 8\sqrt{2}+10,$$

$$\kappa = (\sqrt{2}+1)^2 \{1 - \sqrt{(2\sqrt{2}-2)}\}^2.$$

$$\Delta = 12; \quad \kappa^{-1} - \kappa = 2\sqrt{2}\,(\sqrt{3}+1)^3, \quad \kappa^{-1} + \kappa = 30+16\sqrt{3};$$

$$\lambda = (\sqrt{2}-1)^2 (\sqrt{3}-\sqrt{2})^2, \quad \Lambda'/\Lambda = \sqrt{12},$$

$$\gamma' = (\sqrt{2}+1)^2 (\sqrt{3}-\sqrt{2})^2, \quad \Gamma'/\Gamma = \sqrt{(3 \div 4)}.$$

$$\Delta = 16; \quad \kappa^{-1} - \kappa = 4\sqrt[4]{2}\,(\sqrt{2}+1)^3, \quad \kappa^{-1} + \kappa = 6\,(16+11\sqrt{2}).$$

$$\Delta = 24; \quad \kappa^{-1} - \kappa = 4\sqrt{2}\,(\sqrt{2}+1)\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^3 (\sqrt{3}+\sqrt{2})^{\frac{3}{2}};$$

and so on, for $\Delta = 28, 40, 48, \ldots$ up to 1848, in Weber's paper.

But all the solutions for $\Delta = 4n$ can be deduced immediately from the solutions for $\Delta = n$, by means of the quadric transformation and the relations

$$\gamma' = \frac{1-\kappa}{1+\kappa}, \quad \lambda = \frac{1-\kappa'}{1+\kappa'},$$

or

$$\sqrt{\gamma'} = (1-\kappa)/\kappa', \qquad \sqrt{\lambda} = (1-\kappa')/\kappa,$$

when

$$\tfrac{1}{2}\Lambda'/\Lambda = K'/K = 2\Gamma'/\Gamma = \sqrt{n}.$$

[A very complete list of similar numerical results in Complex Multiplication will be found at the end of Prof. H. Weber's recently published *Elliptische Functionen*, 1891.

Prof. Weber points out (p. 424) that $\Delta = 235, 267, 403$ can be solved in the same manner as $\Delta = 35, 51, \ldots$

Mr. Russell has also found simple solutions for $\Delta = 59$, 107, and 139.]