



# V. Some remarks on the finite integration of linear partial differential equations with constant coefficients

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duct for the images of the corner pole will be found to be

$$P_2 = \frac{2\sqrt{2}}{\pi} \Pi \left( \coth \frac{\pi x}{2} \coth \frac{\pi x'}{4} \right) \Pi \left( \frac{\coth \frac{\pi x'}{2}}{\coth \frac{\pi x}{4}} \right),$$

where  $x$  is to take the successive values 4, 8, 12, ..., and  $x'$  the values 2, 6, 10, ...

Writing this product  $P_2$  in the form

$$\frac{2\sqrt{2}}{\pi} \cdot \frac{\tanh \frac{\pi}{2} \cdot \tanh \frac{3\pi}{2} \dots}{\tanh \frac{2\pi}{2} \cdot \tanh \frac{4\pi}{2} \dots} \cdot \coth \pi \cdot \coth 2\pi \cdot \coth 3\pi \dots,$$

it is brought into connexion with  $(\delta)$  and  $(\epsilon)$  (§ 23); and its value is thus found to be

$$P_2 = \frac{2\sqrt{2}}{\pi} \sqrt{\frac{\pi}{2K_0}} \sqrt{\frac{\pi}{2^{\frac{5}{2}}K_0}} = \frac{2^{\frac{5}{2}}}{K_0}.$$

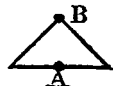
Hence

$$Q^5 = \frac{2^{\frac{5}{2}}}{K_0^5}, \text{ or } Q = \frac{\sqrt{2}}{K_0}.$$

In other words, the resistance of a square plate with the poles arranged as in fig. 11 is given by (21), if for  $m+n$  we read its value  $\frac{1}{2}$ , and consider  $\alpha$  equal to  $\frac{1}{2}$ .

One more distinct case may be mentioned, namely the right-angled triangle shown in fig. 12. This also requires two factors raised to different powers; and the result is the expression (21) with  $\alpha = \frac{1}{6}$ .

Fig. 12.



[To be continued.]

V. *Some Remarks on the Finite Integration of Linear Partial Differential Equations with constant Coefficients.* By the Rev. S. EARNSHAW, M.A.

*To the Editors of the Philosophical Magazine and Journal.*

GENTLEMEN,

IN your Magazine for June 1849 you kindly printed a short paper of mine "On the Transformation of Linear Partial Differential Equations with constant Coefficients to Fundamental Forms," in which I promised to make "in a future communication a few remarks on the finite integration" of equations of the second order with two or three independent

variables. With your permission I will now fulfil that promise.

In the communication referred to it was shown that these two classes of equations can by a change of variables be reduced to the following fundamental forms :—

$$(1) \quad \frac{d^2u}{dx dy} = au; \quad (2) \quad \frac{d^2u}{dx^2} = \frac{du}{dy}; \quad (3) \quad \frac{d^2u}{dx dy} = \frac{d^2u}{dz^2};$$

$$(4) \quad \frac{d^2u}{dx dy} = \frac{du}{dz}; \quad (5) \quad \frac{d^2u}{dx dy} = \frac{d^2u}{dz^2} + a \frac{du}{dz};$$

so that if we can succeed in integrating these forms, then we may consider the general linear partial differential equations of the second order, with two or three variables and with constant coefficients, to have been integrated. When certain relations among the coefficients exist the finite integration is easy; but in other cases the difficulty of finite integration was never overcome. I propose to point out the cause of the difficulty which occurs.

1. If  $U$  be an integral in finite terms, then are also  $\frac{dU}{dx}$ ,  $\frac{dU}{dy}$ ; and if  $U$  be differentiated any number of times with regard to the independent variables, the results will be integrals also. Consequently from any one integral we can deduce an unlimited number of other integrals; and all these must be included with  $U$  in the general integral; and each must stand therein multiplied by a separate and independent arbitrary constant. And if so, how is their sum to be gathered into a finite form? Surely the arbitrary constants will be an insuperable barrier to finiteness of expression.

2. Again, if  $U$  contain in the body of it an arbitrary constant  $c$ , then will  $\frac{dU}{dc}$ ,  $\frac{d^2U}{dc^2}$ ,  $\frac{d^3U}{dc^3}$ , ... be cointegrals with  $U$ , and their sum, each multiplied with an arbitrary constant, must enter into the general integral; so that, as before, we have what appears to be an insuperable barrier to finiteness of expression.

3. If  $U$  be expanded in a series in powers of  $c$ , each coefficient of its powers will be an integral, and we have again the same difficulty repeated. Thus there are three impediments, any one of which seems to be an insuperable barrier to finiteness of expression.

An example will make this clear. Let us take the equation  $\frac{d^2u}{dx^2} = \frac{du}{dy}$ . A known integral of this equation is

$$F(x, y) = A e^{ax + c^2 y}. \quad \dots \quad (1)$$

If this be differentiated any number of times with regard to

the independent variables no new integral results from it, and consequently the integral before us evades by its form the difficulty to finiteness stated in (1). But if it be differentiated with regard to  $c$ , we obtain an infinite series of dissimilar integrals; and when each has been multiplied by an arbitrary constant the sum of the series will not be expressible in a finite form; so that the difficulty stated in (2) holds good here.

Again, if the above integral be expanded in a series according to the powers of  $c$ , the coefficients of these powers will be unlimited in number, and each of them will be an integral and finite in form, as follows:—

$$1; \frac{x}{1}; \frac{x^2}{1.2} + \frac{y}{1}; \frac{x^3}{1.2.3} + \frac{xy}{1.1}; \frac{x^4}{1.2.3.4} + \frac{x^2y}{1.2.1} + \frac{y^2}{1.2}; \&c. \quad (2)$$

Hence the general integral of the equation must include the following infinite series of independent integrals,

$$A + B \frac{x}{1} + C \left( \frac{x^2}{1.2} + \frac{y}{1} \right) + D \left( \frac{x^3}{1.2.3} + \frac{xy}{1.1} \right) + \dots;$$

and it is at once evident that this series cannot be summed, by reason of the arbitrary multipliers of its terms.

Thus it is clearly seen to be a hopeless task to seek for a finite integral of any of the five equations mentioned above; and it is evident that the difficulty arises directly out of their property of linearity.

Some few years ago I discovered that the equation, above quoted as an example, admits of an integral of the form

$$F(x, y) = Ay^{-\frac{1}{2}} e^{-\frac{x^2}{4y}}; \quad (3)$$

and thus we have two integrals, (1) and (3), of one equation which are as distinct in form as can well be conceived; besides which we have an unlimited number of other distinct integrals of the same equation in (2). It has, in consequence of this curious abundance of distinct integrals of the same equation, appeared to me a worthy object to ascertain the nature of the connexion existing among them, and to what their abundance and distinctness are due. I believe I have completely succeeded in this object, and that I have been enabled thereby to find the various integrals which each of the five fundamental differential equations can have in finite forms. The investigations are too long for your Magazine; and I propose therefore to commit them to the press for private circulation among such mathematicians as may desire to possess them.

Sheffield, May 31, 1876.

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