On a New Connexion of Bessel Functions with Legendre Functions. By E. T. WHITTAKER. Received and Read November 13th, 1902.

1. *Introduction.*

The object of this communication is to establish a new connexion between the Legendre functions $P_n(z)$ and $Q_n(z)$ and the Bessel functions $J_n(z)$ and $Y_n(z)$. Hitherto the functions have been known to be connected in two ways, namely, (1) by Heine's expression of the Bessel functions of the first kind and integral order as limiting cases of the associated Legendre functions of infinite order and finite degree, and (2) by a set of formulæ which express Legendre functions in terms of definite integrals involving Bessel functions. The most general form of these latter integrals was given by Schafheitlin (*Math. Ann.*, Vol. xxx.). The integral

$$P_{2n}(z) = (-1)^n \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty J_{2n+\frac{1}{2}}(x) \cos xz \, . \, x^{-\frac{1}{2}} dx,$$

which is valid for a certain range of values of n, may be given as typical of this set of formulæ.*

The new connexion developed in this paper between the Legendre functions $P_n(z)$, $Q_n(z)$ and the Bessel functions $J_n(z)$, $Y_n(z)$ is not restricted to integer values of the order n of the functions, but can be expressed in various forms applicable to all real or complex values of n. The connexion is expressed by the results numbered (I.) to (V.) below.

It may be noted here that a certain kind of reciprocity exists between the theorems of Schafheitlin and those of this paper; this is in agreement with a remarkable theorem due to Retzval, namely, that, if two functions $u(z)$ and $v(z)$ are connected by a relation of the form

$$u(z) = \int e^{zt} v(t)\, dt,$$

then they are also in general connected by a relation of the form

$$v(z) = \text{constant} \times \int e^{-zt} u(t)\, dt.$$

* An interesting family of integrals of this type is given by Steinthal, *Quart. Jour.*, Vol. XVIII., pp. 330–345.

2. *A Formula which embodies a Simple Case of the Connexion.*

We shall first recall a formula which is due to Lord Rayleigh,* namely,

$$\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} i^n J_{n+\frac{1}{2}}(z) = P_n\left(\frac{d}{i\,dz}\right)\left(\frac{\sin z}{z}\right);$$

this formula may be regarded as embodying the simplest special case of the connexion which is the subject of this paper.

For the sake of completeness a short proof will be given of this formula. It is true when $n=0$ and $n=1$, since it then reduces to the well known results

$$J_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sin z$$

and

$$J_{\frac{3}{2}}(z) = -\left(\frac{2z}{\pi}\right)^{\frac{1}{2}} \frac{d}{dz}\left(\frac{\sin z}{z}\right).$$

We shall now show by induction that it is true in general. For suppose it true for $n = k-2$ and $n = k-1$. Then, from the recurrence formula

$$kP_k(z) - (2k-1)\,zP_{k-1}(z) + (k-1)\,P_{k-2}(z) = 0,$$

we have

$$kP_k\left(\frac{d}{i\,dz}\right)\left(\frac{\sin z}{z}\right)$$

$$= \left\{(2k-1)\frac{d}{i\,dz}P_{k-1}\left(\frac{d}{i\,dz}\right) - (k-1)\,P_{k-2}\left(\frac{d}{i\,dz}\right)\right\}\left(\frac{\sin z}{z}\right),$$

and by hypothesis the right-hand member is equal to

$$(2k-1)\frac{d}{i\,dz}\left\{\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} i^{k-1} J_{k-\frac{1}{2}}(z)\right\} - (k-1)\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} i^{k-2} J_{k-\frac{3}{2}}(z)$$

or $$\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} i^k \left\{\frac{2k-1}{2z} J_{k-\frac{1}{2}}(z) - (2k-1)\frac{dJ_{k-\frac{1}{2}}(z)}{dz} + (k-1)\,J_{k-\frac{3}{2}}(z)\right\}$$

or

$$\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} i^k \left\{-\frac{(k-1)(2k-1)}{z} J_{k-\frac{1}{2}}(z) + (2k-1)\,J_{k+\frac{1}{2}}(z) + (k-1)\,J_{k-\frac{3}{2}}(z)\right\}$$

* *Theory of Sound*, Vol. II., p. 263.

or

$$\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} i^k k J_{k+\frac{1}{2}}(z) + \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} i^k (k-1) \left\{ -\frac{2k-1}{z} J_{k-\frac{1}{2}}(z) + J_{k+\frac{1}{2}}(z) + J_{k-\frac{3}{2}}(z) \right\},$$

and the quantity in the last bracket is zero, in consequence of the recurrence-formulæ for the Bessel functions. We have therefore

$$P_k\left(\frac{d}{i\,dz}\right)\left(\frac{\sin z}{z}\right) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} i^k J_{k+\frac{1}{2}}(z),$$

and consequently the formula is true for $n = k$ if it is true for $n = k-1$ and $n = k-2$. But it was shown to be true for $n = 0$ and $n = 1$. It is therefore true universally.

It will be noticed that this result connects the case in which the Legendre function $P_n(z)$ can be expressed in a finite form in terms of elementary functions (namely, the case of $n =$ an integer) with the case in which the Bessel function $J_m(z)$ can be expressed in a finite form in terms of elementary functions (namely, m equal to half an odd integer).

3. *Transition to a Definite-Integral form of the Connexion.*

It is obvious that Rayleigh's equation has a meaning only when n is an integer. In order to extend it to the case in which n is not an integer, we must therefore express the result in a form which is capable of this generalization.

For this purpose we observe that, if r be a postive integer, we have

$$\int_{-1}^{1} e^{izt} t^r dt = (-i)^r \frac{d^r}{dz^r} \int_{-1}^{1} e^{izt} dt = 2\,\frac{d^r}{i^r dz^r}\left(\frac{\sin z}{z}\right).$$

Since, when n is an integer, $P_n(z)$ is a sum of positive integral powers of z, we have therefore

$$\int_{-1}^{1} e^{izt} P_n(t)\,dt = 2P_n\left(\frac{d}{i\,dz}\right)\left(\frac{\sin z}{z}\right);$$

and therefore, by (1.), we have

$$J_{n+\frac{1}{2}}(z) = (-i)^n \left(\frac{z}{2\pi}\right)^{\frac{1}{2}} \int_{-1}^{1} e^{izt} P_n(t)\,dt,$$

and on combining the two parts of the range of integration, namely, from -1 to 0 and from 0 to 1, this becomes

$$J_{n+\frac{1}{2}}(z) = \left(\frac{2z}{\pi}\right)^{\frac{1}{2}} \int_0^1 \cos\left(zt - \frac{n\pi}{2}\right) P_n(t)\,dt.$$

This equation has so far been proved for the case in which n is an integer. But each member of it has a meaning, even when n is not an integer, and we may therefore now inquire how far the result is still true when the restriction to integral values of n is removed.

4. *General Case in which n is unrestricted: a Family of Integrals which satisfy Bessel's Differential Equation.*

It will now appear that the integral last obtained is merely one member of a large family of definite integrals by which the Bessel functions can be represented for all values of z and of n, real or complex. To show this, let u denote the integral

$$z^{\frac{1}{2}} \int_\gamma e^{zti} P_{n-\frac{1}{2}}(t)\,dt,$$

where $P_{n-\frac{1}{2}}(t)$ is the Legendre function of order $(n-\frac{1}{2})$, and where the path of integration, γ, is left for the present unspecified.

Then, on performing the differentiations, we have

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{n^2}{z^2}\right)u$$

$$= (\tfrac{1}{4} - n^2)\,z^{-\frac{3}{2}} \int_\gamma e^{zti} P_{n-\frac{1}{2}}(t)\,dt + 2z^{-\frac{1}{2}} i \int_\gamma e^{zti}\, t P_{n-\frac{1}{2}}(t)\,dt$$

$$+ z^{\frac{1}{2}} \int_\gamma e^{zti} (1-t^2) P_{n-\frac{1}{2}}(t)\,dt,$$

and, since $P_{n-\frac{1}{2}}(t)$ satisfies Legendre's equation, namely,

$$\frac{d}{dt}\left\{(1-t^2)\frac{dP_{n-\frac{1}{2}}(t)}{dt}\right\} + (n^2 - \tfrac{1}{4})\,P_{n-\frac{1}{2}}(t) = 0,$$

it follows that the right-hand member can be written in the form

$$z^{-\frac{3}{2}} \int_\gamma e^{zti} d\left\{(1-t^2)\frac{dP_{n-\frac{1}{2}}(t)}{dt}\right\} + 2z^{-\frac{1}{2}} i \int_\gamma e^{zti}\, t P_{n-\frac{1}{2}}(t)\,dt$$

$$+ z^{\frac{1}{2}} \int_\gamma e^{zti} (1-t^2) P_{n-\frac{1}{2}}(t)\,dt,$$

or $$\int_\gamma d\left\{z^{-\frac{3}{2}} e^{zti} (1-t^2)\frac{dP_{n-\frac{1}{2}}(t)}{dt} - iz^{-\frac{1}{2}}(1-t^2)\, e^{zti} P_{n-\frac{1}{2}}(t)\right\}.$$

This result is clearly equally true if the Legendre function of the second kind, $Q_{n-\frac{1}{2}}(t)$, is used instead of $P_{n-\frac{1}{2}}(t)$, since no property of $P_{n-\frac{1}{2}}(t)$ has been required beyond the fact that it satisfies Legendre's equation. We have therefore the result

The integrals
$$z^{\frac{1}{2}} \int_\gamma e^{zti} P_{n-\frac{1}{2}}(t)\, dt$$
and
$$z^{\frac{1}{2}} \int_\gamma e^{zti} Q_{n-\frac{1}{2}}(t)\, dt \qquad \text{(I.)}$$

satisfy Bessel's differential equation of order n, namely,

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{n^2}{z^2}\right) u = 0,$$

provided γ is any path of integration (it may be either an open path or a closed circuit) which fulfils the condition that the quantity

$$e^{zti}(1-t^2)\left(\frac{dP_{n-\frac{1}{2}}(t)}{dt} - izP_{n-\frac{1}{2}}(t)\right)$$

[*or, in the second case, the corresponding expression in* $Q_{n-\frac{1}{2}}(t)$] *resumes its initial value after describing* γ.

Suitable contours γ can be found in plenty to satisfy this condition. Thus it is evident that an arc joining the points $t = -1$ and $t = +1$ is a possible path γ for the integrals (II.), p. 203; that is, the expression

$$z^{\frac{1}{2}} \int_{-1}^{1} e^{zti} P_{n-\frac{1}{2}}(t)\, dt$$

is a solution of Bessel's differential equation of order n.

5. *Determination of some Notable Members of the Family of Integrals found in* § 4.

The theorem of § 4 enables us to express the connexion between the Legendre and Bessel functions in a variety of forms; for, if we write

$$I = z^{\frac{1}{2}} \int_{-1}^{1} e^{zti} P_{n-\frac{1}{2}}(t)\, dt,$$

it has just been shown that the quantity I is a solution of Bessel's differential equation of order n, for all values of n, real or complex. To find its precise relation to the known solutions $J_n(z)$, we need only find the first term of the asymptotic expansion of I for large positive values of z. This is obtained in the following way:—We have, integrating by parts,

$$I = z^{\frac{1}{2}} \left[\frac{e^{zti}}{zi} P_{n-\frac{1}{2}}(t) \right]_{-1}^{1} - \frac{1}{iz^{\frac{1}{2}}} \int_{-1}^{1} e^{zti} \frac{dP_{n-\frac{1}{2}}(t)}{dt} dt.$$

The first term of the asymptotic expansion of I is therefore

$$\frac{1}{iz^{\frac{1}{2}}} \{ e^{zi} P_{n-\frac{1}{2}}(1) - e^{-zi} P_{n-\frac{1}{2}}(-1) \}$$

or
$$\frac{1}{iz^{\frac{1}{2}}} \{ e^{zi} - e^{-zi+(n-\frac{1}{2})i\pi} \},$$

or
$$2z^{-\frac{1}{2}} e^{(n-\frac{1}{2})\frac{1}{2}i\pi} \cos \left\{ z - (n+\tfrac{1}{2}) \frac{\pi}{2} \right\}.$$

Since the first term of the asymptotic expansion of $J_n(z)$ is

$$\left(\frac{2}{\pi z} \right)^{\frac{1}{2}} \cos \left\{ z - (n+\tfrac{1}{2}) \frac{\pi}{2} \right\},$$

it follows that
$$I = (2\pi)^{\frac{1}{2}} e^{(n-\frac{1}{2})\frac{1}{2}i\pi} J_n(z),$$

so
$$J_n(z) = \left(\frac{z}{2\pi} \right)^{\frac{1}{2}} e^{-(n-\frac{1}{2})\frac{1}{2}i\pi} \int_{-1}^{1} e^{zti} P_{n-\frac{1}{2}}(t)\, dt.$$

Combining the parts of the integral which arise from the ranges -1 to 0 and 0 to 1 respectively, *we have a connexion between the Bessel and Legendre functions expressed by the formula*

$$J_n(z) = \left(\frac{2z}{\pi} \right)^{\frac{1}{2}} \int_0^1 \cos \left\{ zt - (n-\tfrac{1}{2}) \frac{\pi}{2} \right\} P_{n-\frac{1}{2}}(t)\, dt \qquad \text{(II.)}$$

which is valid for all values of z and of n, real or complex. This result was previously (in § 3) shown to be true for the particular case in which n is half an odd integer.

The general theorem of § 4 enables us not only to express $J_n(z)$ in terms of Legendre functions of the first kind, but also in terms of Legendre functions of the second kind—namely, the Q-functions which are defined by the equation

$$Q_n(z) = \frac{1}{2^{n+1}} \int_{-1}^{+1} (1-t^2)^n (z-t)^{-n-1}\, dt.$$

For, in the plane of a complex variable s, let S denote a contour which encircles the part of the real axis between the points $s=1$ and $s=-1$, and which begins and ends at infinity. The direction in which S tends to infinity will be supposed to be such as to make the real part of szi negative. Then, if t be a real quantity lying between 1 and -1, we have

$$\int_S (s-t)^{-n-\frac{1}{2}} e^{zsi}\, ds = z^{n-\frac{1}{2}} e^{zti} \int_V v^{-n-\frac{1}{2}} e^{iv}\, dv,$$

where $v = z(s-t)$, and where V denotes a contour in the v-plane which encircles the origin and begins and ends at infinity in the v-plane.

But, from Heine's expression for the Gamma-function, we have

$$\frac{1}{\Gamma(n+\frac{1}{2})} = \frac{1}{2\pi} e^{-(n+\frac{1}{2})\frac{1}{2}i\pi} \int_V v^{-n-\frac{1}{2}} e^{iv}\, dv,$$

and therefore

$$\int_S (s-t)^{-n-\frac{1}{2}} e^{zsi}\, ds = \frac{2\pi}{\Gamma(n+\frac{1}{2})} z^{n-\frac{1}{2}} e^{zti+(n+\frac{1}{2})\frac{1}{2}i\pi}.$$

If we substitute this result in Bessel's integral

$$J_n(z) = \frac{z^n}{2^n \pi^{\frac{1}{2}} \Gamma(n+\frac{1}{2})} \int_{-1}^{1} e^{zti} (1-t^2)^{n-\frac{1}{2}}\, dt,$$

which is valid when the real part of $(n+\frac{1}{2})$ is positive, we have

$$J_n(z) = \frac{z^{\frac{1}{2}} e^{-(n+\frac{1}{2})\frac{1}{2}i\pi}}{2^{n+1}\pi^{\frac{3}{2}}} \int_S \int_{-1}^{1} (1-t^2)^{n-\frac{1}{2}} (s-t)^{-n-\frac{1}{2}} e^{zsi}\, ds\, dt,$$

and, on substituting the expression given above for the Legendre function of the second kind, *we obtain the formula*

$$J_n(z) = \frac{z^{\frac{1}{2}} e^{-(n+\frac{1}{2})\frac{1}{2}i\pi}}{2^{\frac{1}{2}}\pi^{\frac{3}{2}}} \int_S e^{zsi} Q_{n-\frac{1}{2}}(s)\, ds. \qquad \text{(III.)}$$

This integral is of the type found in § 4*; it connects the Bessel function* $J_n(z)$ *with the Legendre functions of the second kind.* It is valid for all values of z, and for all values of n whose real part is greater than $-\frac{1}{2}$.

6. *The Second Solution* $Y_n(z)$ *of Bessel's Differential Equation; and the General Solution.*

We now proceed to consider the second solution of Bessel's differential equation. When the order n of the equation is not an integer, it is known that $J_n(z)$ and $J_{-n}(z)$ are two independent solutions; and therefore the second solution when n is not an integer is

$$J_{-n}(z),$$

or, by (II.), $\left(\frac{2z}{\pi}\right)^{\frac{1}{2}}\int_0^1 \cos\left(zt+\frac{n\pi}{2}+\frac{\pi}{4}\right)P_{-n-\frac{1}{2}}(t)\,dt.$

But the Legendre function of the first kind satisfies the relation

$$P_k(z) = P_{-k-1}(z)$$

for all values of k, real or complex; and therefore we have

$$J_{-n}(z) = \left(\frac{2z}{\pi}\right)^{\frac{1}{2}}\int_0^1 \cos\left\{zt+\frac{2\pi}{2}+\frac{\pi}{4}\right\}P_{n-\frac{1}{2}}(t)\,dt.$$

Since the general solution of Bessel's equation (n being still supposed to be not an integer) consists of any linear combination of $J_n(z)$ and $J_{-n}(z)$, it follows that this general solution is represented by any linear combination of

$$z^{\frac{1}{2}}\int_0^1 \cos zt\,P_{n-\frac{1}{2}}(t)\,dt$$

and

$$z^{\frac{1}{2}}\int_0^1 \sin zt\,P_{n-\frac{1}{2}}(t)\,dt.$$

Consider now the case in which n is an integer. In this case we have

$$J_{-n}(z) = (-1)^n J_n(z);$$

and therefore the function $J_{-n}(z)$ no longer provides an independent second solution of the differential equation. We take therefore the second solution $Y_n(z)$ introduced by Hankel, which is defined to be

$$\text{limit}_{\epsilon=0}\frac{(-1)^n J_{-(n-\epsilon)}(z)-J_{(n-\epsilon)}(z)}{\epsilon}.$$

Substituting the values found above for $J_{-(n-\epsilon)}(z)$ and $J_{(n-\epsilon)}(z)$, we have

$$Y_n(z) = \text{limit}_{\epsilon=0}\left(\frac{2z}{\pi}\right)^{\frac{1}{2}}\int_0^1\frac{1}{\epsilon}\left\{(-1)^n\cos\left(zt+\frac{\pi}{4}+\frac{n\pi}{2}-\frac{\epsilon\pi}{2}\right)\right.$$
$$\left.-\cos\left(zt+\frac{\pi}{4}-\frac{n\pi}{2}+\frac{\epsilon\pi}{2}\right)\right\}P_{n-\frac{1}{2}}(t)\,dt$$

$$= \text{limit}_{\epsilon=0}\left(\frac{2z}{\pi}\right)^{\frac{1}{2}}\int_0^1\frac{2\sin\frac{\epsilon\pi}{2}}{\epsilon}\sin\left(zt+\frac{\pi}{4}-\frac{n\pi}{2}\right)P_{n-\frac{1}{2}}(t)\,dt;$$

and therefore *we have the formula*

$$Y_n(z) = (2\pi z)^{\frac{1}{2}}\int_0^1 \sin\left\{zt-(n-\tfrac{1}{2})\frac{\pi}{2}\right\}P_{n-\frac{1}{2}}(t)\,dt, \qquad \text{(IV.)}$$

which expresses the Bessel function of the second kind in terms of the Legendre function.

Since the most general solution of Bessel's differential equation of order n consists of a linear combination of $J_n(z)$ and $Y_n(z)$, it follows that this can be expressed as a linear combination of

$$z^{\frac{1}{2}} \int_0^1 \cos zt \, P_{n-\frac{1}{2}}(t) \, dt$$

and

$$z^{\frac{1}{2}} \int_0^1 \sin zt \, P_{n-\frac{1}{2}}(t) \, dt.$$

But this result, which is thus obtained for the case in which n is integral, is the same as the result already obtained for the case in which n is not integral; and therefore we see that *in all cases the general solution of Bessel's equation is a linear combination of*

$$\left.\begin{array}{l} z^{\frac{1}{2}} \int_0^1 \cos zt \, P_{n-\frac{1}{2}}(t) \, dt \\ \textit{and} \quad z^{\frac{1}{2}} \int_0^1 \sin zt \, P_{n-\frac{1}{2}}(t) \, dt \end{array}\right\}. \qquad \text{(V.)}$$

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*On Groups which are Linear and Homogeneous in both Variables and Parameters.** *By* W. Burnside. Received and Read November 13th, 1902.

In this paper I propose to discuss the nature of the characteristic determinant, first of any simply transitive linear homogeneous group, and secondly of any transitive linear homogeneous group. The result of this discussion, leading to a quite general form for the

* Prof. L. E. Dickson has called my attention to two errors in my paper "On the Continuous Group defined by any given Group of Finite Order" (*Proc. Lond. Math. Soc.*, Vol. xxix., pp. 546-565). These occur in §§ 5, 6, dealing with particular cases of the characteristic determinant of a simply transitive linear homogeneous group. The induction in § 5 is faulty; and, as Prof. Dickson has shown, it is *not* the case that, if the characteristic determinant is the power of a single linear factor, the group is necessarily Abelian. The error in § 6 occurs in the third line from the bottom of p. 553. The correct inference from the previous reasoning, there, should obviously be not $\nu = \mu$, but that ν is a multiple of μ. The results of §§ 5, 6 are used in § 10 in establishing one of the main results of the paper. In order that this should rest on a sound basis it is essential that these errors should be corrected.