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VII. On a new theorem in analysis

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*s α, β, γ to the axes of elasticity, of the force put in play on m by a displacement $\rho \parallel$ it

$$= \rho \{A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma\}:$$

2. The sum of such resolved forces in the directions of any three rectangular axes is constant

$$= \rho \{A + B + C\}:$$

3. The force produced by a displacement in the direction of one of the axes of elasticity is the greatest, and in that of another the least that can be produced in any one direction by the same displacement in that direction.

I am, Gentlemen, yours, &c.

St. John's College, Cambridge, Aug. 24, 1836. C. J.

VII. *On a new Theorem in Analysis. By the Rev.* Robert Murphy, *M.A.*

LAPLACE first gave a very simple and elegant demonstration of the theorem generally known as Lagrange's, by taking the partial differential coefficients of u relative to x and a, from the equation $y = a + x\,\phi(y)$, where $u = f(y)$; and the simplicity of the process depends on the manner in which x enters this equation, namely, as a multiplier of a function of y.

I have considered a more general equation where x enters the function in any manner, viz. $y = a + \phi(x, y)$ and $u = f(y)$, and have obtained the following theorem:

When x is changed into $x + h$, in this equation y and consequently u are also changed; let the latter become U, and let $\phi(x + h, y) - \phi(x, y) = \Delta\,\phi$ for abridgement; the relation between U and u will then be

$$U = u + \Delta\,\phi \,.\, \frac{du}{da} + \frac{d}{da}\left\{\frac{(\Delta\,\phi)^2}{1.2} \,.\, \frac{du}{da}\right\}$$
$$+ \frac{d}{da^2}\left\{\frac{(\Delta\,\phi)^3}{1.2.3} \,.\, \frac{du}{da}\right\} + \&c.$$

In proving this we shall use ϕ as a contraction of $\phi(x, y)$, and any *accented* symbol will be used to express the *partial* differential coefficient relative to x; thus instead of $\frac{d\,.\,\phi(x, y)}{dx}$ we shall simply write ϕ'.

Now differentiating the proposed equation $y = a + \phi$ relative to x and a, we get

$$\frac{dy}{dx} = \phi' + \frac{d\phi}{dy} \cdot \frac{dy}{dx}$$

$$\frac{dy}{da} = 1 + \frac{d\phi}{dy} \cdot \frac{dy}{da},$$

from which we easily obtain

$$\frac{dy}{dx} = \phi' \cdot \frac{dy}{da};$$

and since $u = f(y)$, therefore

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}, \quad \frac{du}{da} = \frac{du}{dy} \cdot \frac{dy}{da},$$

whence

$$\frac{du}{dx} = \phi' \cdot \frac{du}{da} \quad . \quad . \quad . \quad (1.)$$

Thus far the process is exactly similar to Laplace's, but since ϕ' in the present case is not a function of y only, the remainder of the investigation becomes essentially different from his.

I put

$$\frac{d^n u}{dx^n} = P_{1,n} \frac{du}{da} + \frac{d}{da}\left(P_{2,n} \frac{du}{da}\right) + \frac{d^2}{da^2}\left(P_{3,n} \frac{du}{da}\right) + \&c., (2.)$$

and now proceed to find the value of the general symbol $P_{m,n}$, which is a function both of x and y.

Before differentiating this equation relative to x let it be observed that

$$\frac{d \,.\, P_{m,n}}{dx} = P'_{m,n} + \frac{d \,.\, P_{m,n}}{dy} \cdot \frac{dy}{dx}$$

$$= P'_{m,n} + \frac{d \,.\, P_{m,n}}{dy} \cdot \phi' \frac{dy}{da}$$

$$= P'_{m,n} + \phi' \cdot \frac{d \,.\, P_{m,n}}{da}$$

and also that

$$\frac{d}{dx} \cdot \frac{du}{da} = \frac{d}{da} \cdot \frac{du}{dx} = \frac{d}{da}\left(\phi' \frac{du}{da}\right)$$

by equation (1.); from both of which it follows that

$$\frac{d}{dx}\left(P_{m,n} \frac{du}{da}\right) = \left(P'_{m,n} + \phi' \cdot \frac{dP_{m,n}}{da}\right) \cdot \frac{du}{da}$$

$$+ P_{m,n} \frac{d}{da}\left(\phi' \frac{du}{da}\right)$$

$$= P'_{m,n} \cdot \frac{du}{da} + \frac{d}{da}\left(P_{m,n} \cdot \phi' \frac{du}{da}\right).$$

Applying this formula to differentiate equation (2.) relative to x, we get

$$\left.\begin{aligned} \frac{d^{n+1} u}{d x^{n+1}} = P'_{1,n} \frac{du}{da} &+ \frac{d}{da}\left(P'_{2,n} \frac{du}{da}\right) \\ &+ \frac{d^2}{da^2}\left(P'_{3,n} \frac{du}{da}\right) + \&c. \\ &+ \frac{d}{da}\left(P_{1,n} \phi' \frac{du}{da}\right) \\ &+ \frac{d^2}{da^2}\left(P_{2,n} \phi' \frac{du}{da}\right) + \&c. \end{aligned}\right\} \quad . \quad . \quad . \quad (3.)$$

But if we write $n + 1$ for n in the formula (2.), we also have

$$\frac{d^{n+1} u}{d x^{n+1}} = P_{1,n+1} \cdot \frac{du}{da} + \frac{d}{da}\left(P_{2,n+1} \frac{du}{da}\right) + \frac{d^2}{da^2}\left(P_{3,n+1} \frac{du}{da}\right)$$
$$+ \&c.,$$

which compared with the preceding expression shows the following law for the formation of the functions $P_{m,n}$, viz.

$$P_{m,n+1} = P'_{m,n} + \phi' \cdot P_{m-1,n} \quad . \quad . \quad . \quad . \quad (4.)$$

And since $P_{m,1}$ is known, (for $P_{1,1} = \phi'$, $P_{2,1} = 0$, $P_{3,1} = 0$, &c.,) we can thus form successively the quantities $P_{m,2}$, $P_{m,3}$, &c., and the general law of these quantities may be thus found.

Put

$$1.2.3 \ldots m \, P_{m,n} = (\phi^m)'''^{(n)} - A_m . \phi \, (\phi^{m-1})'''^{(n)} + B_m \phi^2 \, (\phi^{m-2})'''^{(n)}$$
$$- C_m \phi^3 \, (\phi^{m-3})'''^{(n)} + \&c.$$

Hence

$$\left.\begin{aligned} 1.2.3 \ldots m \, P'_{m,n} = (\phi^m)''^{(n+1)} &- A_m \phi \, (\phi^{m-1})''^{(n+1)} \\ &+ B_m \phi^2 \, (\phi^{m-2})''^{(n+1)} - C_m \phi^3 \, (\phi^{m-3})''^{(n+1)} + \&c. \\ &- A_m \phi' \, (\phi^{m-1})'''^{(n)} + 2 B_m \phi \phi' \, (\phi^{m-2})'''^{(n)} \\ &- 3 C_m \phi^2 \phi' \, (\phi^{m-3})'''^{(n)} + \&c. \end{aligned}\right\}$$

Also

$$1.2.3 \ldots m \, \phi' P_{m-1,n} = m \phi' \, (\phi^{m-1})'''^{(n)} - m A_{m-1} \phi \phi' \, (\phi^{m-2})'''^{(n)}$$
$$+ m B_{m-1} \phi^2 \phi' \, (\phi^{m-3})'''^{(n)} + \&c.$$

And by equation (4.) the sum of these expressions must be equal to $1.2.3 \ldots m\, P_{m,n+1}$, that is,

$$= (\phi^m)^{\prime\prime(n+1)} - A_m \phi\, (\phi^{m-1})^{\prime\prime(n+1)} + B_m \phi^2 (\phi^{m-2})^{\prime\prime(n+1)}$$

$$- C_m \phi^3 (\phi^{m-3})^{\prime\prime(n+1)} \text{ \&c.}$$

From whence we have

$$A_m = m,\ 2\, B_m = m\, A_{m-1} \text{ or } B_m = \frac{m\,(m-1)}{2}$$

$$3\, C_m = m\, B_{m-1} \text{ or } C_m = \frac{m\,(m-1)\,(m-2)}{2.3} \text{ \&c.};$$

and therefore

$$1.2.3 \ldots m\, P_{m,n} = (\phi^m)^{\prime\prime\prime(n)} - m\, \phi\, (\phi^{m-1})^{\prime\prime\prime(n)}$$

$$+ \frac{m\,(m-1)}{2} \phi^2 (\phi^{m-2})^{\prime\prime\prime(n)} + \text{\&c.}, \quad . \quad . \quad . \quad (5.)$$

which series terminates at the m^{th} term, since $(\phi^{m-m})^{\prime\prime(n)} = 0$.

If we put 1, 2, 3, &c. successively for m in this formula, and substitute in the expression (2), stopping at $P_{n,n}$, we should have $\frac{d^n u}{d x^n}$ explicitly obtained; but there is no necessity for this.

Again, since $\Delta \phi = \phi\,(x + h, y) - \phi\,(x, y)$, therefore

$$(\Delta \phi)^m = \{\phi\,(x + h, y)\}^m - m\, \phi\,(x, y)\, \{\phi\,(x + h, y)\}^{m-1}$$

$$+ \frac{m\,(m-1)}{2} \{\phi\,(x, y)\}^2 \{\phi\,(x + h, y)\}^{m-2} - \text{\&c.}$$

Each term in this series may be expanded according to powers of h by Taylor's theorem; and if we take the coefficient of $\frac{h^n}{1.2.3 \ldots n}$ in each term, the coefficient of $\frac{h^n}{1.2.3 \ldots n}$ in $(\Delta \phi)^m$ is exactly the same as the series (5.).

By this comparison we find that $P_{m,n}$ is the coefficient of $\frac{h^n}{1.2.3 \ldots n}$ in the expansion of $\frac{(\Delta \phi)^m}{1.2.3 \ldots m}$.

Recur now to the series (2.), and we get

$\frac{d^n u}{d x^n}$ = the coefficient of $\frac{h^n}{1.2.3 \ldots n}$ in the series following, viz.

$$\Delta \phi \cdot \frac{du}{da} + \frac{d}{da} \left\{ \frac{(\Delta \phi)^2}{1.2} \cdot \frac{du}{da} \right\} + \frac{d^2}{d a^2} \left\{ \frac{(\Delta \phi)^3}{1.2.3} \cdot \frac{du}{da} \right\} + \text{\&c.}$$

But by Taylor's theorem the same quantity is the coefficient of $\frac{h^n}{1.2.3\ldots n}$; in the expansion for $U - u$, we therefore find the theorem announced at the commencement, viz.

$$U = u + \Delta\phi \cdot \frac{du}{da} + \frac{d}{da}\left\{\frac{(\Delta\phi)^2}{1.2} \cdot \frac{du}{da}\right\}$$
$$+ \frac{d^2}{da^2}\left\{\frac{(\Delta\phi)^3}{1.2.3} \cdot \frac{du}{da}\right\} + \&c.$$

We have not space here to point out the applications of this general theorem, and shall therefore close this paper with two remarks.

First, if $\phi(x, y)$ be of the form $x\phi(y)$, then $\Delta\phi = h\phi y$; we have then

$$U = u + h\phi(y) \cdot \frac{du}{da} + \frac{h^2}{1.2}\frac{d}{da}\left\{(\phi y)^2 \cdot \frac{du}{da}\right\}$$
$$+ \frac{h^3}{1.2.3}\frac{d^2}{da^2}\left\{(\phi y)^3 \cdot \frac{du}{da}\right\} \&c.;$$

and if we suppose $x = 0$, then $y = a$, and U is then the value of $f(y)$ determined from the equation $y = a + h\phi(y)$: we thus fall on Lagrange's theorem.

Secondly, that if the proposed equation were

$$y = F\{a + \phi(x, y)\} \text{ and } u = f(y),$$

the fundamental equation (1.) would remain the same, and therefore this theorem admits of the same extension that Laplace gave to Lagrange's. R. M.

VIII. *On the Property of the Parabola demonstrated by* Mr. Lubbock *in the Phil. Mag. for August. By* A CORRESPONDENT.

To the Editors of the Philosophical Magazine and Journal.

GENTLEMEN,

IN the last [August] Number of your Journal there is a demonstration by Mr. Lubbock of a very beautiful property of the parabola. Mr. L. does not seem to be aware that this problem, which he ascribes to the French, is in fact due to Prof. Wallace of Edinburgh, who before the end of the last century communicated it to Mr. Leybourne, by whom it was published as a prize question. Not having the book at present, I cannot tell in which volume of the Repository it is to