VI.—Theorems relating to a Generalisation of the Bessel-Function. By the Rev. F. H. Jackson, H.M.S. "Irresistible." *Communicated by* Dr W. Peddie.

(MS. received February 17, 1904. Read March 21, 1904. Issued separately May 27, 1904.)

1.

In this paper, theorems which are extensions of the following, are discussed :—

$$J_0(x+y) = J_0(x)J_0(y) - 2J_1(x)J_1(y) + 2J_2(x)J_2(y) - \ldots\ldots \text{ ad inf.} \qquad (\alpha)$$

$$1 = \{J_0(x)\}^2 + 2\{J_1(x)\}^2 + 2\{J_2(x)\}^2 + \ldots\ldots \text{ ad inf.} \qquad (\beta)$$

$$J_n(x) = (-1)^m \left\{ J_{2m+n}(x) - \frac{2m(m+n)}{x} J_{2m+n-1}(x) + \frac{2^2 m(m-1)(m+n)(m+n-1)}{2!\,x^2} J_{2m+n-2}(x) - \ldots\ldots \right\} \qquad (\gamma)$$

$$J_{-n}(x) = (-1)^n J_n(x)$$

$$J_{n+\frac{1}{2}}(x) = (-1)^n \frac{(2x)^{n+\frac{1}{2}}}{\Pi^{\frac{1}{2}}} \frac{d^n}{d(x^2)^n} \left\{ \frac{\sin x}{x} \right\} \qquad (\delta)$$

We define $J_{[n]}(\lambda, x)$ as

$$\sum_{r=0}^{r=\infty} \frac{1}{[r]!\,[n+r]!\,(2)_r (2)_{n+r}} \lambda^{n+2r} x^{[n+2r]}$$

In this expression

$$[n+r]! \text{ is } \Gamma_p([n+r+1]) \quad \text{or} \quad \Pi_p([n+r])$$

$$[n] \text{ is } \frac{p^n - 1}{p-1}$$

$$(2)_{n+r} = [2]^{n+r} \frac{\Gamma_{p^2}([n+r+1])}{\Gamma_p([n+r+1])}$$

The function $\Pi_p([n])$ is defined in the previous paper (*Trans. Roy. Soc. Edin.*, vol. xli. part i.). If λ be changed to $i\lambda$ in $J_{[n]}$, the function will then be more strictly analogous to J_n.

In Weierstrassian form

$$\frac{1}{\Gamma_p([x])} = [x] e^{Px} \prod_{n=1}^{n=\infty} \left\{ \left(1 + p^{-x}\frac{[x]}{[n]}\right) e^{-\frac{x}{[n]}} \right\}$$

$$P = 1 + \frac{1}{[2]} + \frac{1}{[3]} + \quad \ldots - \log \frac{p}{p-1}$$

$$\{2n\}! = (2)_n \Gamma_p([n+1])$$

being in the case of n positive and integral

$$\{2n\}! = [2][4][6] \ldots [2n]$$

analogous to

$$2 \cdot 4 \cdot 6 \ldots\ldots 2n = 2^n . n!$$

This notation enables us to write shortly

$$J_{[n]} = \sum \frac{(-1)^r \lambda^{n+2r}}{\{2n+2r\}!\,\{2r\}!} x^{[n+2r]}$$

2.

If we invert the base element p, we see that $[r]!$ is transformed into $p^{-r(r-1)/2}[r]!$ and that $(2)_r$ becomes transformed into $p^{-r(r-1)/2}(2)_r$. These transformations hold whether r be integral or not. Inverting the base p in the series $J_{[n]}(\lambda)$, we obtain

$$p^{n^2}\sum_{r=0}^{r=\infty}\frac{(-1)^r p^{2r(n+r)}}{[r]!\,[n+r]!\,(2)_r(2)_{n+r}}\left(\frac{\lambda}{p}\right)^{n+2r} \quad . \quad . \quad . \quad . \quad (1)$$

which we denote

$$p^{n^2}\mathfrak{J}_{[n]}\left(\frac{\lambda}{p}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Lommel has shown that

$$J_n(\lambda)J_m(\lambda) = \sum_{r=0}^{r=\infty}(-1)^r\frac{\binom{m+n+2r}{r}}{\Gamma(m+r+1)\Gamma(n+r+1)}\left(\frac{\lambda}{2}\right)^{m+n+2r} \quad . \quad . \quad . \quad (3)$$

The function $\mathfrak{J}_{[n]}$ was formed while seeking to extend the above theorem. The extension was found to be

$$J_{[n]}(\lambda)\mathfrak{J}_{[m]}(\lambda) = J_{[m]}(\lambda)\mathfrak{J}_{[n]}(\lambda) = \sum_{r=0}^{r=\infty}\frac{(-1)^r\Gamma_{p^2}([m+n+2r+1])}{\Gamma_{p^2}([m+n+r+1])\Gamma_{p^2}([m+r+1])\Gamma_{p^2}([n+r+1])\Gamma_{p^2}([r+1])}\left(\frac{\lambda}{[2]}\right)^{m+n+2r} \quad (4)$$

The relation between J and $\mathfrak{J}$ was surmised from the following simple but similar theorem.

$$E_p(\lambda) = 1 + \frac{\lambda}{[1]!} + \frac{\lambda^2}{[2]!} + \ldots\ldots$$

$$E_{\frac{1}{p}}(\lambda) = 1 + \frac{\lambda}{[1]!} + p\frac{\lambda^2}{[2]!} + \ldots\ldots$$

$$E_p(\lambda)E_{\frac{1}{p}}(\lambda) = 1 + \frac{2\lambda}{[1]!} + \frac{2(1+p)\lambda^2}{[2]!} + \ldots\ldots$$

which suggested that $\mathfrak{J}_{[n]}$ might be derived from $J_{[n]}$ by inverting the base p. As I have given the proof of (4) as an example of the use of generalised Gamma-functions in a paper communicated to the Royal Society, London, it will be sufficient to say here, that the theorem may be proved by showing, that the coefficients of the powers of λ on both sides of the equation are identical, being cases of the extension of Vandermonde's theorem (*Proc. Lond. Math. Soc.*, series 2, vol. i. p. 63).

In the notation of Art. (1) we may write the theorem

$$J_{[m]}(\lambda)\mathfrak{J}_{[n]}(\lambda) = \sum_{r=0}^{r=\infty}\frac{(-1)^r\{2m+2n+4r\}!}{\{2m+2n+2r\}!\{2m+2r\}!\{2n+2r\}!\{2r\}!}\lambda^{m+n+2r} \quad . \quad . \quad (5)$$

3.

Consider the series

$$J_{[0]}(\lambda)\mathfrak{J}_{[0]}(\lambda) + \frac{[4]}{[2]}J_{[1]}(\lambda)\mathfrak{J}_{[1]}(\lambda) + \ldots \quad \ldots + p^{s.s-1}\frac{[4s]}{[2s]}J_{[s]}(\lambda)\mathfrak{J}_{[s]}(\lambda) + \ldots\ldots \quad (6)$$

By means of (5) we write this—

$$\left\{1-\frac{[4][2]}{[2][2][2][2]}\lambda^2+\ldots\ldots\ldots(-1)^r\frac{\{4r\}!}{\{2r\}!\{2r\}!\{2r\}!\{2r\}!}\lambda^{2r}+\ldots\ldots\right\}$$

$$+\frac{[4]}{[2]}\left\{\frac{[4][2]}{[4][2][2][2]}\lambda^2-\ldots\ldots+\frac{\{4r\}!}{\{2r+2\}!\{2r\}!\{2r\}!\{2r-2\}}\lambda^{2r}+\ldots\ldots\right\} \quad (7)$$

$$\cdots\cdots\cdots\cdots$$

$$p^{r(r-1)}\frac{[4r]}{[2r]}\left\{\frac{\{4r\}!}{\{4r\}!\{2r\}!\{2r\}!\{0\}!}\lambda^{2r}+\ldots\ldots\right\}$$

$$\cdots\cdots\cdots\cdots$$

We see by inspection that the coefficient of λ^2 vanishes: the coefficient of λ^{2r} is the expression

$$(1-)^r\frac{\{4r\}!}{\{2r\}!\{2r\}!\{2r\}!\{2r\}!}\left\{1-\frac{[4]}{[2]}\frac{[2r]}{[2r+2]}+p^2\frac{[8]}{[4]}\frac{[2r][2r-2]}{[2r+2][2r+4]}-\ldots\right.$$

$$\left.+(-1)^r p^{r(r-1)}\frac{[4r]}{[2r]}\frac{[2r][2r-2]\ldots\ldots[4][2]}{[2r+2][2r+4]\ldots\ldots[4r]}\right\} \quad (8)$$

The series within the large brackets is easily summed term by term. The sum of the first two terms is $-p^2\dfrac{[2r-2]}{[2r+2]}$, which is a factor of the third term. The sum of $-p^2\dfrac{[2r-2]}{[2r+2]}$ and the third term is

$$p^6\frac{[2r-4][2r-2]}{[2r+2][2r+4]}$$

Continuing in this way, we obtain that the sum of the first r terms is

$$(-1)^{r-1}p^{r(r-1)}\frac{[2r-2][2r-4][2r-6]\ldots\ldots[4][2]}{[2r+2][2r+4][2r+6]\ldots\ldots[4r-2]}$$

which is equal to the last term, but is of opposite sign. The coefficient of λ^{2r} is zero, and only the constant unity is left. Therefore

$$1=J_{[0]}(\lambda)\mathfrak{J}_{[0]}(\lambda)+\frac{[4]}{[2]}J_{[1]}(\lambda)\mathfrak{J}_{[1]}(\lambda)+\ldots\ldots+p^{s,s-1}\frac{[4s]}{[2s]}J_{[s]}(\lambda)\mathfrak{J}_{[s]}(\lambda)+\ldots\ldots \quad (9)$$

which, if $p=1$, reduces to

$$1=\{J_0\}^2+2\{J_1\}^2+2\{J_2\}^2+\ldots\ldots \quad (10)$$

4.

Consider now the series

$$J_{[0]}(\lambda)\mathfrak{J}_{[0]}(\lambda)-\frac{[4]}{[2]}J_{[1]}(\lambda)\mathfrak{J}_{[1]}(\lambda)+\ldots\ldots+p^{r,r-1}\frac{[4r]}{[2r]}J_{[r]}(\lambda)\mathfrak{J}_{[r]}(\lambda)-\ldots \quad (11)$$

Referring to expression (8), we see that the coefficient of λ^{2r} is

$$(-1)^r\frac{\{4r\}!}{\{2r\}!\{2r\}!\{2r\}!\{2r\}!}\left\{1+\frac{[4]}{[2]}\frac{[2r]}{[2r+2]}+p^2\frac{[8]}{[4]}\frac{[2r][2r-2]}{[2r+2][2r+4]}+\ldots\ldots\right\} \quad (12)$$

The series within the large brackets, although simple in form, offers considerable difficulty in summation. The sum is

$$2(p^2+1)(p^4+1)\ \ldots\ldots\ (p^{2r-2}+1)\cdot(p^2+1)(p^4+1)\ \ldots\ (p^{2r}+1)\frac{\{2r\}!\{2r\}!}{\{4r\}!}. \qquad (13)$$

The sum in general for all values of r is

$$2\frac{[4]^{2r}}{[2]^{2r}}\cdot\frac{[2r]}{[4r]}\cdot\frac{\Gamma_{p^4}([r+1])\Gamma_{p^4}([r+1])}{\Gamma_{p^2}([2r+2])} \qquad (14)$$

The reason that the simple series ($p=1$)

$$1+\frac{4}{2}\cdot\frac{2r}{2r+2}+\frac{8}{4}\cdot\frac{2r(2r-2)}{(2r+2)(2r+4)}+ \quad .$$

is easily summed as

$$2^{2r}\frac{r!\,r!}{2r!}$$

while the general series offers difficulty, is that the functions Γ_p are present, both to the base p^4 and the base p^2 in the series (12) and in expression (14).

HEINE has shown in his "*Kugelfunctionen*" that

$$\phi[a,b,c,p,x]=1+\frac{(1-a)(1-b)}{(1-p)(1-c)}x+\frac{(1-a)(1-ap)(1-b)(1-bp)}{(1-p)(1-p^2)(1-c)(1-cp)}x^2+\ \ldots\ldots$$

$$=\prod_{n=0}^{n=\infty}\frac{(1-bxp^n)(1-\frac{c}{b}p^n)}{(1-xp^n)(1-cp^n)}\ \phi\left[b,\frac{abx}{c},bx,p,\frac{c}{b}\right] \qquad (15)$$

Consider now the series

$$1+\frac{[2]}{[1]}\frac{[r]}{[r+1]}+p\frac{[4]}{[2]}\frac{[r][r-1]}{[r+1][r+2]}+\ \ldots\ldots\ \ldots+p^{s(s-1)/2}\frac{[2s]}{[s]}\frac{[r][r-1]\ldots.[r-s+1]}{[r+1][r+2]\ldots[r+s]}+\ldots. \qquad (16)$$

Since

$$\frac{[2]}{[1]}=\frac{p^2-1}{p-1}=p+1$$

$$\frac{[4]}{[2]}=\frac{p^4-1}{p^2-1}=p^2+1$$

. . . .

. . .

we write series (16) as the sum of two HEINE's series

$$S_1+p\frac{p^r-1}{p^{r+1}-1}S_2=\left\{1+\frac{p^r-1}{p^{r+1}-1}+p\frac{(p^r-1)(p^{r-1}-1)}{(p^{r+1}-1)(p^{r+2}-1)}+\ldots\ .\right\}$$

$$+p\frac{p^r-1}{p^{r+1}-1}\left\{1+\frac{p^{r-1}-1}{p^{r+2}-1}p^2+\ \ldots.\ +p^{s(s+3)/2}\frac{(p^{r-1}-1)\ldots(p^{r-s}-1)}{(p^{r+2}-1)\ldots(p^{r+s+1}-1)}+\ldots\right\} \qquad (17)$$

These series we transform by means of (15).

First, for the series S_1 we put

$$a=p$$
$$b=p^{-r}$$
$$c=p^{r+1}$$
$$x=-p^r$$

and obtain, after obvious reductions,

$$S_1 = 1 + \frac{p^r - 1}{p^{r+1} - 1} + p\frac{(p^r - 1)(p^{r-1} - 1)}{(p^{r+1} - 1)(p^{r+2} - 1)} + \ldots .$$

$$= \frac{2(1+p)(1+p^2)\ldots\ldots(1+p^{r-1})}{(1-p^{r+1})(1-p^{r+2})\ldots.(1-p^{2r})}\left\{1 - \frac{p^{2r}-1}{p-1}\cdot\frac{p}{2} + \frac{(p^{2r}-1)(p^{2r-2}-1)}{(p^2-1)(p^2-1)}\frac{p^4}{2}\right.$$

$$\left. - \frac{(p^{2r}-1)(p^{2r-2}-1)(p^{2r-4}-1)}{(p^2-1)(p^3-1)(p^4-1)}\cdot\frac{p^9}{2} + \ldots.\right\} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In the same way, if we put

$$a = p$$
$$b = p^{1-r}$$
$$c = p^{r+2}$$
$$x = -p^{r+1}$$

we obtain, after reductions

$$S_2 = 1 + \frac{p^{r-1} - 1}{p^{r+2} - 1}p^2 + \ldots\ldots$$

$$= \frac{(1+p^2)(1+p^3)\ldots\ldots(1+p^r)}{(1-p^{r+2})(1-p^{r+3})\ldots\ldots(1-p^{2r})}\left\{1 - \frac{p^{2r-2}-1}{(p-1)(p^2+1)}p^3 + \frac{(p^{2r-2}-1)(p^{2r-4}-1)}{(p-1)(p^2-1)(p^2+1)(p^3+1)}p^8 - \ldots\right\} \quad (19)$$

So that $S_1 + p\frac{p^r - 1}{p^{r+1} - 1}S_2$ may be written

$$\frac{(1+p)(1+p^2)\ldots\ldots(1+p^{r-1})}{(1-p^{r+1})(1-p^{r+2})\ldots.(1-p^{2r})}\left\{2 - \frac{p^{2r}-1}{p-1}p + \frac{(p^{2r}-1)(p^{2r-2}-1)}{(p^2-1)(p^2-1)}p^4 - \ldots\ldots\ldots\right.$$

$$\left. - \frac{p^{2r}-1}{p+1}p + \frac{(p^{2r}-1)(p^{2r-2}-1)}{(p^2-1)(p^2+1)}p^4 - \ldots\ldots\ldots\right\} \quad . \quad (20)$$

Adding the terms with like numerators together, we obtain

$$\frac{(1+p)(1+p^2)\ldots\ldots(1+p^{r-1})}{(1-p^{r+1})(1-p^{r+2})\ldots\ldots(1-p^{2r})}\left\{2 - 2p^2\frac{p^{2r}-1}{p^2-1} + 2p^6\frac{(p^{2r}-1)(p^{2r-2}-1)}{(p^2-1)(p^4-1)} - \ldots\ldots\right\} \quad . \quad (21)$$

The series within the large brackets is the simplest type of series, and its sum is well known to be

$$2(1-p^2)(1-p^4) \ldots\ldots (1-p^{2r}) \quad . \quad . \quad . \quad (22)$$

We have therefore

$$1 + \frac{[2]}{[1]}\frac{[r]}{[r+1]} + p\frac{[4]}{[2]}\frac{[r][r-1]}{[r+1][r+2]} + \ldots\ldots$$

$$= \frac{2(1+p)(1+p^2)\ \ldots\ (1+p^{r-1})\cdot(1-p^2)(1-p^4)\ \ldots\ (1-p^{2r})}{(1-p^{r+1})(1-p^{r+2})\ \ldots\ (1-p^{2r})} \quad . \quad . \quad (23)$$

Changing the base p to p^2 we obtain the series whose sum was sought

$$1 + \frac{[4]}{[2]}\frac{[2r]}{[2r+2]} + p^2\frac{[8]}{[4]}\frac{[2r][2r-2]}{[2r+2][2r+4]} + \ldots\ldots$$

$$= \frac{2(1+p^2)(1+p^4)\ldots(1+p^{2r-2})\cdot(1-p^4)(1-p^8)\ldots(1-p^{4r})}{(1-p^{2r+2})(1-p^{2r+4})\ldots\ldots\ldots(1-p^{4r})} \quad . \quad . \quad (24)$$

The coefficient of λ^{2r} is obtained by multiplying this sum by

$$\frac{\{4r\}!}{\{2r\}!\{2r\}!\{2r\}!\{2r\}!}$$

which gives us

$$(-1)^r \frac{2(1+p^2)(1+p^4)(1+p^6) \ . \ . \ . \ (1+p^{2r-2}) \cdot (1+p^2)(1+p^4) \ . \ . \ . \ . \ (1+p^{2r})}{\{2r\}! \, \{2r\}!} \quad . \qquad . \quad (25)$$

as the coefficient of λ^{2r}. If r be not integral, the infinite products in HEINE's transformation do not reduce to finite products but to expressions in terms of the Γ_p functions, ultimately giving the sum of the series in the form (14).

Having obtained the coefficient of λ^{2r}, we have established, subject to convergence, the theorem

$$J_{[0]}(\lambda)\mathfrak{J}_{[0]}(\lambda) - \frac{[4]}{[2]}J_{[1]}(\lambda)\mathfrak{J}_{[1]}(\lambda) + p^2\frac{[8]}{[4]}J_{[2]}(\lambda)\mathfrak{J}_{[2]}(\lambda) + \ . \ . \ . \ . \ .$$

$$= 1 - \frac{2(1+p^2)\lambda^2}{\{2\}! \, \{2\}!} + \ . \ . \ . \ . \ . \ . \ (-1)^r \frac{2(1+p^2) \ . \ . \ . \ (1+p^{2r-2}) \cdot (1+p^2) \ . \ . \ . \ (1+p^{2r})\lambda^{2r}}{\{2r\}! \, \{2r\}!} - \ . \ . \ . \ . \ . \quad (26)$$

which is the extension of

$$J_0(2\lambda) = \{J_0(\lambda)\}^2 - 2\{J_1(\lambda)\}^2 + 2\{J_2(\lambda)\}^2 - \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad . \qquad . \quad (27)$$

This is a particular case of the addition theorem for J_0.

$$J_0(\lambda + \lambda_1) = J_0(\lambda)J_0(\lambda_1) - 2J_1(\lambda)J_1(\lambda_1) + \ . \ . \ .$$

5.

Defining * $\mathrm{Sin}_p\,(\lambda)$ and $\mathrm{Cos}_p\,(\lambda)$ as

$$\mathrm{Sin}_p\,(\lambda) = \lambda - \frac{\lambda^3}{[3]!} + \frac{\lambda^5}{[5]!} -$$

$$\mathrm{Cos}_p\,(\lambda) = 1 - \frac{\lambda^2}{[2]!} + \frac{\lambda^4}{[4]!} - \ . \ . \quad . \ . \ .$$

we obtain

$$\mathrm{Sin}_p\,(\lambda)\,\mathrm{Cos}_{\frac{1}{p}}\,(\lambda_1) + \mathrm{Cos}_p\,(\lambda)\,\mathrm{Sin}_{\frac{1}{p}}\,(\lambda_1) = (\lambda+\lambda_1) - \frac{(\lambda+\lambda_1)(\lambda+\lambda_1 p^2)(\lambda+\lambda_1 p^4)}{[3]!} + \ . \ . \ . \ .$$

$$\mathrm{Sin}_p\,(\lambda)\,\mathrm{Cos}_{\frac{1}{p}}\,(\lambda) + \mathrm{Cos}_p\,(\lambda)\,\mathrm{Sin}_{\frac{1}{p}}\,(\lambda) = 2\lambda - \frac{2(1+p^2)(1+p^4)\lambda^3}{[3]!} + \ . \ . \ . \ . \ .$$

This suggests that the extension of the addition theorem of $J_0(\lambda + \lambda_1)$ will be on similar lines.

Consider now the series

$$J_{[0]}(\lambda)\mathfrak{J}_{[0]}(\lambda_1) - \frac{[4]}{[2]}J_{[1]}(\lambda)\mathfrak{J}_{[1]}(\lambda_1) + \ . \ . \ . \ . \ . \ . \ . \ (-1)^r p^{r(r-1)}\frac{[4r]}{[2r]}J_{[1]}(\lambda)\mathfrak{J}_{[1]}(\lambda_1) + \ . \ . \ . \ . \ . \quad . \quad (28)$$

* *Proc. Edin. Math. Soc.*, vol. xxii., 1904.

and the product

$$J_{[n]}(\lambda)\mathfrak{J}_{[n]}(\lambda_1)$$

$$= \left\{ \frac{\lambda_n}{\{2n\}!} - \frac{\lambda^{n+2}}{\{2n+2\}!\,\{2\}!} + \dots + \frac{\lambda^{n+2r}}{\{2n+2r\}!\,\{2r\}!} - \dots \right\} \left\{ \frac{\lambda_1^{\,n}}{\{2n\}!} - \dots\dots + \frac{\lambda_1^{\,n+2r} p^{2r(n+r)}}{\{2n+2r\}!\,\{2r\}!} - \dots \right\} \quad (29)$$

From series (28) we are to form a new series, of which the successive terms will be homogeneous in $\lambda\lambda_1$ and of degrees 0, 2, 4, 6, ... $2r$, ... respectively.
The first term of (28) gives rise to the constant, unity.
The terms of the second degree arising from $J_{[0]}\mathfrak{J}_{[0]}$ are

$$-\frac{\lambda^2}{\{2\}!\,\{2\}!} \text{ and } -\frac{\lambda_1^{\,2}p^2}{\{2\}!\,\{2\}!}$$

The term of the second degree arising from $J_{[1]}\mathfrak{J}_{[1]}$ is

$$-\frac{[4]}{[2]}\frac{\lambda\lambda_1}{\{2\}!\,\{2\}!}$$

There are no other terms of the second degree; the sum of these terms is

$$\frac{1}{\{2\}!\,\{2\}!}\left\{ \lambda^2 + \frac{[4]}{[2]}\lambda\lambda_1 + p^2\lambda_1^{\,2} \right\}$$
$$= \frac{1}{\{2\}!\,\{2\}!}\left\{ \lambda^2 + (p^2+1)\lambda\lambda_1 + p^2\lambda_1^{\,2} \right\}$$
$$= \frac{(\lambda+\lambda_1 p^2)(\lambda+\lambda_1)}{\{2\}!\,\{2\}!}$$

Terms of the fourth degree arise only from the first, second, and third terms of (28), being respectively

$$\frac{\lambda^4}{\{4\}!\,\{4\}!} + \frac{\lambda^2\lambda_1^{\,2}p^2}{\{2\}!\,\{2\}!\,\{2\}!\,\{2\}!} + \frac{\lambda_1^{\,4}p^8}{\{4\}!\,\{4\}!}$$
$$\frac{[4]}{[2]}\left\{ \frac{\lambda^3\lambda_1}{\{4\}!\,\{2\}!\,\{2\}!} + \frac{\lambda\lambda_1^{\,3}p^4}{\{4\}!\,\{2\}!\,\{2\}!} \right\} \quad . \quad . \quad . \quad . \quad (30)$$
$$p^2\frac{[8]}{[4]}\frac{\lambda^2\lambda_1^{\,2}}{\{4\}!\,\{4\}!}$$

Remembering that

$$\{4\}! = [4]\,[2] \quad \text{and} \quad \{2\}! = [2]$$

we write the sum of (30)

$$\frac{1}{[4]\,[2]\,[4]\,[2]}\left\{ \lambda^4 + \frac{[4]\,[4]}{[2]\,[2]}\lambda^3\lambda_1 + \frac{[4]\,[4]}{[2]\,[2]}\lambda^2\lambda_1^{\,2}p^2 + \frac{[8]}{[4]}\lambda^2\lambda_1^{\,2}p^2 + \frac{[4]\,[4]}{[2]\,[2]}\lambda\lambda_1^{\,3} + p^8\lambda_1^{\,4} \right\}$$

Replacing

$$\frac{[4]}{[2]} \text{ by } (p^2+1) \quad \text{and} \quad \frac{[8]}{[4]} \text{ by } p^4+1$$

the expression within the large brackets reduces to

$$(\lambda+\lambda_1)(\lambda+p^2\lambda_1)(\lambda+p^2\lambda_1)(\lambda+p^4\lambda_1)$$

The term of the sixth degree in λ, λ_1 I have verified as

$$-\frac{1}{[6]\,[4]\,[2]\,[6]\,[4]\,[2]}\left\{ (\lambda+\lambda_1)(\lambda+\lambda_1p^2)(\lambda+\lambda_1p^4)\cdot(\lambda+\lambda_1p^2)(\lambda+\lambda_1p^4)(\lambda+\lambda_1p^6) \right\} \quad . \quad (31)$$

The term of degree $2r$ is the following expression—

$$\left\{\frac{\lambda^{2r}}{\{2r\}!\{2r\}!}+\frac{\lambda^{2r-2}\lambda_1^2p^2}{\{2r-2\}!\{2r-2\}!\{2\}!\{2\}!}+\ldots\ldots+\frac{\lambda_1^{2r}p^{2r^2}}{\{2r\}!\{2r\}!}\right\}$$

$$+\frac{[4]}{[2]}\left\{\frac{\lambda^{2r-1}\lambda_1}{\{2r\}!\{2r-2\}!\{2\}!}+\frac{\lambda^{2r-3}\lambda_1^3p^4}{\{2r-2\}!\{2r-4\}!\{4\}!\{2\}!}+\ldots\ldots+\frac{\lambda_1^{2r-1}\lambda p^{2r(r-1)}}{\{2\}!\{2r-2\}!\{2r\}!}\right\} \quad (32)$$

$$+p^2\frac{[8]}{[4]}\left\{\frac{\lambda^{2r-2}\lambda_1^2}{\{2r\}!\{2r-4\}!\{4\}!}+\frac{\lambda^{2r-4}\lambda_1^4p^6}{\{2r-2\}!\{2r-6\}!\{6\}!\{2\}!}+\ldots\ldots+\frac{\lambda_1^{2r-2}\lambda^2p^{2r(r-2)}}{\{4\}!\{2r-4\}!\{2r\}!}\right\}$$

. . . .

$$+p^{r(r-1)}\frac{[4r]}{[2r]}\left\{\frac{\lambda^r\lambda_1^r}{\{2r\}!\{2r\}!}\right\}$$

We have shown in Art. (4) that in case $\lambda=\lambda_1$ this expression is

$$\frac{(\lambda+\lambda)(\lambda+\lambda p^2)(\lambda+\lambda p^4)\ldots(\lambda+\lambda p^{2r-2})\cdot(\lambda+\lambda p^2)(\lambda+\lambda p^4)\ldots\ldots(\lambda+\lambda p^{2r})}{\{2r\}!\{2r\}!} \quad (33)$$

It has been directly verified that for particular values of r (1, 2, 3) the forms, in case λ be not equal to λ_1, are

$$-\frac{(\lambda+\lambda_1)(\lambda+\lambda_1p^2)}{\{2\}!\{2\}!}$$

$$\frac{(\lambda+\lambda_1)(\lambda+\lambda_1p^2)(\lambda+\lambda_1p^2)(\lambda+\lambda_1p^4)}{\{4\}!\{4\}!}$$

$$-\frac{(\lambda+\lambda_1)(\lambda+\lambda_1p^2)(\lambda+\lambda_1p^4)\cdot(\lambda+\lambda_1p^2)(\lambda+\lambda_1p^4)(\lambda+\lambda_1p^6)}{\{6\}!\{6\}!}$$

respectively. This indirectly establishes the form of the coefficient of degree $2r$ in λ and λ_1. A direct proof of the algebraic identity would, however, be preferable. Writing now

$$\sum_{n=0}^{n=\infty}(-1)^np^{n(n-1)}\frac{[4n]}{[2n]}J_{[n]}(\lambda)J_{[n]}(\lambda_1)=1+\sum_{n=1}^{n=\infty}(-1)^n\frac{(\lambda+\lambda_1)(\lambda+\lambda_1p^2)\ldots(\lambda+\lambda_1p^{2n-2})\cdot(\lambda+\lambda_1p^2)\ldots\ldots(\lambda+\lambda_1p^{2n})}{\{2n\}!\{2n\}!} \quad (34)$$

If $p=1$, we obtain the addition theorem of J_0

$$J_0(\lambda+\lambda_1)=J_0(\lambda)J_0(\lambda_1)-2J_1(\lambda)J_1(\lambda_1)+\ldots\ldots \quad (35)$$

6.

The analogue of LOMMEL's theorem

$$J_n(\lambda)=(-1)^m\left\{J_{2m+n}(\lambda)-\frac{2m(m+n)}{\lambda}J_{2m+n-1}(\lambda)+\ldots\ldots\right\}$$

I have shown by two distinct methods * that

$$\mathop{\mathrm{L}}_{\kappa=\infty}\frac{(p^{\gamma-\alpha}-1)(p^{\gamma-\alpha+1}-1)\ldots(p^{\gamma-\beta+\kappa-1}-1)\cdot(p^{\gamma-\beta}-1)(p^{\gamma-\beta+1}-1)\ldots\ldots(p^{\gamma-\beta+\kappa-1}-1)}{(p^{\gamma-\alpha-\beta}-1)(p^{\gamma-\alpha-\beta+1}-1)\ldots(p^{\gamma-\alpha-\beta+\kappa-1}-1)\cdot(p^{\gamma}-1)(p^{\gamma+1}-1)\ldots(p^{\gamma+\kappa-1}-1)}$$

$$=1+p\frac{[\alpha][\beta]}{[1][\gamma]}+p^2\frac{[\alpha][\alpha+1][\beta][\beta+1]}{[1][2][\gamma][\gamma+1]}+\ldots \quad (36)$$

* *Proc. Lond. Math. Soc.*, series 2, vol. i. pp. 71, 72, 1903, and *Amer. Jour. Math.*, vol. XXVI., 1904.

In terms of the function Γ_p of this paper, this theorem is

$$\frac{1}{p^{\alpha\beta}}\frac{\Gamma_p([\gamma-\alpha-\beta])\Gamma_p([\gamma])}{\Gamma_p([\gamma-\alpha])\Gamma_p([\gamma-\beta])} = 1 + p\frac{[\alpha][\beta]}{[1][\gamma]} + \dots \qquad (37)$$

Change the base p to p^2 and put

$$\alpha = -m$$
$$\beta = -m-n$$
$$\gamma = r-m+1$$

we obtain

$$\frac{\Gamma_{p^2}([r+m+n+1])\Gamma_{p^2}([r-m+1])}{\Gamma_{p^2}([r+1])\Gamma_{p^2}([r+n+1])} = 1 + \frac{[2m][2m+2n]}{[2][2r-2m+2]}p^{2-4m-2n} + \dots\dots$$

$$+\frac{[2m]\dots\dots[2m-2s+2]\cdot[2m+2n]\dots\dots[2m+2n-2s+2]}{[2][4]\dots[2s]\cdot[2r-2m+2]\dots[2r-2m+2s]}p^{2s^2-2s(2m+n)} + \dots\dots \qquad (38)$$

Now consider

$$J_{[2m+n]}(\lambda) - \frac{p^2}{p^{4m+2n}}\frac{[2m][2m+2n]}{[2]\lambda}J_{[2m+n-2]}(\lambda) + \frac{p^8}{p^{8m+4n}}\frac{[2m][2m-2][2m+2n][2m+2n-2]}{[2][4]}J_{[2m+n-4]}(\lambda) + \dots \qquad (39)$$

The coefficient of λ^{n+2r} is the infinite series

$$\frac{(-1)^{r-m}}{\{2n+2m+2r\}!\{2r-2m\}!}\left[1+\frac{p^2}{p^{4m+2n}}\frac{[2m][2m+2n]}{[2][2r-2m+2]}+\dots\dots\right] \qquad (40)$$

which by (38) reduces to

$$\frac{(-1)^{r-m}}{\{2n+2m+2r\}!\{2r-2m\}!}\times\frac{\Gamma_{p^2}([m+n+r+1])\Gamma_{p^2}([r-m+1])}{\Gamma_{p^2}([r+1])\Gamma_{p^2}([r+n+1])} \qquad (41)$$

Now remembering

$$\{2s\}! = [2]^s\Gamma_{p^2}([s+1]) = (2)_s\Gamma_p([s+1])$$

the expression (40) reduces to

$$(-1)^{r-m}\frac{p^{2m(m+n)}}{\Gamma_p([r+1])\Gamma_p([n+r+1])(2)_r(2)_{n+r}} \qquad (42)$$

which is $(-1)^{-m}p^{2m(m+n)}$ × coefficient of λ^{n+2r} in the series $J_{[n]}$.

This establishes

$$(-1)^m p^{2m(m+n)}J_{[n]}(\lambda) = J_{[2m+n]}(\lambda) - \frac{p^2}{p^{4m+2n}}\frac{[2m][2m+2n]}{[2]\lambda}J_{[2m+n-1]}(\lambda) + \dots \text{ ad inf.} \qquad (43)$$

an extension of

$$J_n = (-1)^m\left\{J_{2m+n} - \frac{2m(m+n)}{\lambda}J_{2m+n-1} + \dots\dots\dots\right\} \qquad (44)$$

LOMMEL defined J_n for negative integral values of n, so as to make this theorem always hold: for example, suppose n a negative integer, and put it equal to $-m$, then we have by this theorem

$$J_{[-n]} = (-1)^n J_{[n]} \qquad (45)$$

extending

$$J_{-n} = (-1)^n J_n$$

also

$$\mathfrak{J}_{[-n]} = (-1)^n\mathfrak{J}_{[n]}$$

as may be shown by inverting the base p in expression (43).

7.

$$J_{n+\frac{1}{2}}(x) = (-1)^n \frac{(2x)^{n+\frac{1}{2}}}{\Pi^{\frac{1}{2}}} \frac{d^n}{d(x^2)^n} \left\{ \frac{\sin x}{x} \right\}$$

If we now define

$$\text{Sin}_p(\lambda,x) \text{ as } \lambda x - \frac{\lambda^3 x^{[3]}}{[3]!} + \frac{\lambda^5 x^{[5]}}{[5]!} - \dots \qquad (46)$$

then

$$\frac{\text{Sin}_p(\lambda,x^{\frac{1}{p}})}{\lambda x^{\frac{1}{p}}} = 1 - \frac{\lambda^2 x^{[2]}}{[3]!} + \dots + (-1)^{n+r}\frac{\lambda^{2n+2r} x^{[2n+2r]}}{[2n+2r+1]!} + \dots$$

Operating on this with $\mathfrak{D}^{[n]}$ which is

$$\left\{ \frac{d}{d(x^{p^{2n-2}[2]})} \dots \dots \left\{ \frac{d}{d(x^{p^2[2]})} \cdot \left\{ \frac{d}{d(x^{[2]})} \right\} \right\} \dots \right\} \qquad (47)$$

the first n terms of the series are destroyed, while the term involving $x^{[2n+2r]}$ is reduced to

$$(-1)^{n+r}\lambda^{2n+2r}\frac{1}{[2]^n}\frac{[2n+2r][2n+2r-2]\dots\dots[2r+2]}{[2n+2r+1]!}x^{p^{2n}[2r]} \qquad (48)$$

Taking

$$J_{[n+\frac{1}{2}]}(\lambda,x) = \sum(-1)^r \frac{\lambda^{n+\frac{1}{2}+2r}}{[n+r+\frac{1}{2}]!\,[r]!\,(2)_r(2)_{n+r+\frac{1}{2}}} x^{[n+\frac{1}{2}+2r]}$$

$$[n+r+\tfrac{1}{2}]! = [n+r+\tfrac{1}{2}][n+r-\tfrac{1}{2}]\dots\dots[\tfrac{3}{2}]\Gamma_p([\tfrac{3}{2}])$$

$$(2)_{n+r+\frac{1}{2}} = (p^{n+r+\frac{1}{2}}+1)(p^{n+r-\frac{1}{2}}+1)\dots\dots(p^{\frac{3}{2}}+1)\cdot(2)_{\frac{1}{2}}$$

and

$$(2)_{\frac{1}{2}}\Gamma_p([1+\tfrac{1}{2}]) = [2]^{\frac{1}{2}}\Gamma_{p^2}([1+\tfrac{1}{2}])$$

therefore

$$[n+r+\tfrac{1}{2}]!\,(2)_{n+r+\frac{1}{2}} = [2n+2r+1]\dots\dots[3] \times [2]^{\frac{1}{2}}\Pi_{p^2}([\tfrac{1}{2}]) \qquad (49)$$

So we obtain

$$J_{[n+\frac{1}{2}]}(\lambda,x) = \frac{\lambda^{n+\frac{1}{2}}x^{[n+\frac{1}{2}]}}{[2]^{\frac{1}{2}}\Gamma_{p^2}([\frac{3}{2}])}\sum(-1)^r\frac{\lambda^{2r}x^{p^{n+\frac{1}{2}}[2r]}}{[2n+2r+1]\dots[5][3]\cdot[2][4]\dots[2r]} \qquad (50)$$

and by a change of the variable x

$$\lambda^{2n}J_{[n+\frac{1}{2}]}(\lambda,x^{p^{n+\frac{1}{2}}}) = (-1)^n\frac{\{[2]\lambda\}^{n+\frac{1}{2}}x^{p^{n+\frac{1}{2}}[n+\frac{1}{2}]}}{[2]\Gamma_{p^2}([\frac{3}{2}])}\cdot\Delta^{[n]}\left\{\frac{\text{Sin}_p(\lambda,x)}{\lambda x}\right\}. \qquad (51)$$

$\Delta^{[n]}$ denoting the operator

$$\left\{ \frac{d}{d(x^{p^{2n-1}[2]})} \dots \dots \left\{ \frac{d}{d(x^{p^3[2]})} \cdot \left\{ \frac{d}{d(x^{p[2]})} \right\} \right\} \dots \right\} \qquad (52)$$

For further properties of Sin_p, Cos_p, and their connection with symbolical solutions of certain differential equations, reference may be made to a paper on "Basic Sines and Cosines" (*Proc. Edin. Math. Soc.*, 1904).

CONTINUATION OF PAPER—

"THEOREMS RELATING TO A GENERALISATION OF THE BESSEL-FUNCTION."

(MS. received April 19, 1904.)

8.

The theorem

$$\mathrm{J}_{[0]}(a)\mathfrak{J}_{[0]}(b)-\frac{[4]}{[2]}\mathrm{J}_{[1]}(a)\mathfrak{J}_{[1]}(b)+\ \dots\dots\ +(-1)^s p^{s^2-s}\frac{[4s]}{[2s]}\mathrm{J}_{[s]}(a)\mathfrak{J}_{[s]}(b)-\ \dots\dots$$

$$=1-\frac{(a+b)(a+bp^2)}{[2]^2}+\frac{(a+b)(a+bp^2)(a+bp^2)(a+bp^4)}{[2]^2[4]^2}-\ \dots\dots \qquad (\alpha)$$

discussed in the first part of this paper may be obtained very naturally from the properties of a certain function analogous to the exponential function. Elsewhere,* by means of the function E_p I have obtained

$$\mathrm{J}_{[0]}(a)\mathfrak{J}_{[0]}\left(\frac{b}{p}\right)-2p\mathrm{J}_{[1]}(a)\mathfrak{J}_{[1]}\left(\frac{b}{p}\right)+\ \dots\dots\ +(-1)^s 2p^{s^2}\mathrm{J}_{[s]}(a)\mathfrak{J}_{[s]}\left(\frac{b}{p}\right)-\ \dots\dots$$

$$=1-\frac{(a+b)^2}{[2]^2}+\frac{(a+b)^2(a+p^2b)^2}{[2]^2[4]^2}-\ \dots\dots \qquad (\beta)$$

We naturally expect to find some general form to which both (α) and (β) will belong, as particular cases. The following is the general theorem which will be obtained from the function E_p, just as the addition theorem for Bessel coefficients is obtained by means of the exponential function. Exp. $\left(\frac{x}{2}\left(t-\frac{1}{t}\right)\right)$

$$\mathrm{J}^{\nu}_0(a\,,b)=\mathrm{J}_{[0]}(a)\mathfrak{J}_{[0]}(bp^{\nu-1})-p^{1-\nu}\frac{[4\nu]}{[2\nu]}\mathrm{J}_{[1]}(a)\mathfrak{J}_{[1]}(bp^{\nu-1})+p^{2(2-\nu)}\frac{[8\nu]}{[4\nu]}\mathrm{J}_{[2]}(a)\mathfrak{J}_{[2]}(bp^{\nu-1})-\ \dots\dots \qquad (\gamma)$$

$$\mathrm{J}^{\nu}_0(a\,,b)=1-\frac{(a+b)(a+bp^{2\nu})}{[2]^2}+\frac{(a+b)(a+bp^2)(a+bp^{2\nu})(a+bp^{2\nu+2})}{[2]^2[4]^2}-\ \dots\dots$$

In case $\nu=0$ we have the quasi-addition theorem (β). If, however, $\nu=1$ we have the quasi-addition theorem (α). The corresponding theorems for the function $\mathrm{J}^{\nu}_n(a\,,b)$ will be briefly noticed.

$$\mathrm{J}^{\nu}_n(a\,,b)=\frac{(a+b)(a+bp^2)\ \dots\ (a+bp^{2n-2})}{[2][4]\ \dots\ [2n]}\left\{1-\frac{(a+bp^{2n})(a+bp^{2\nu})}{[2n+2][2]}\right.$$

$$\left.+\frac{(a+bp^{2n})(a+bp^{2n+2})(a+p^{2\nu}b)(a+p^{2\nu+2}b)}{[2n+2][2n+4][2][4]}-\ \dots\right\} \qquad (\delta)$$

The expression for $\mathrm{J}^{\nu}_n(a\,,b)$ will be given also in the case when n is not a positive integer.

* *Proc. Lond. Math. Soc.*, shortly to be published.

9.

In this article certain results will be obtained which will be required in subsequent work. We define the function $E_p(a)$ as

$$E_p(a)^* = 1 + \frac{a}{[1]!} + \frac{a^2}{[2]!} + \dots\dots$$

If we invert the base p

$$E_{\frac{1}{p}}(a) = 1 + \frac{a}{[1]!} + p\frac{a^2}{[2]!} + \dots\dots + p^{s(s-1)/2}\frac{a^s}{[s]!} + \dots\dots$$

without difficulty we have

$$E_p(a)E_{\frac{1}{p}}(b) = 1 + \frac{(a+b)}{[1]!} + \frac{(a+b)(a+bp)}{[2]!} + \dots\dots \qquad (\epsilon)$$

Changing p to p^2

$$E_{p^2}(a)E_{p^{-2}}(b) = 1 + \frac{(a+b)}{\frac{p^2-1}{p^2-1}} + \frac{(a+b)(a+bp^2)}{\frac{p^2-1}{p^2-1}\cdot\frac{p^4-1}{p^2-1}} + \dots\dots$$

$$= 1 + \frac{[2](a+b)}{[2]} + \frac{[2]^2(a+b)(a+bp^2)}{[2][4]} + \dots\dots$$

$$E_{p^2}\left(\frac{a}{[2]}\right)E_{p^{-2}}\left(\frac{b}{[2]}\right) = 1 + \frac{(a+b)}{[2]} + \frac{(a+b)(a+bp^2)}{[2][4]} + \dots\dots \qquad (\eta)$$

In part (1) we have established

$$J_{[-n]}(a) = (-1)^n J_{[n]}(a) \qquad (\theta)$$

Inverting the base p we obtain also from this

$$\mathfrak{J}_{[-n]}(a) = (-1)^n \mathfrak{J}_{[n]}(a) \qquad (\kappa)$$

10.

A consideration of the product of the two absolutely convergent series

$$E_{p^2}\left(\frac{at}{[2]}\right) = 1 + \frac{at}{[2]} + \frac{a^2t^2}{[2][4]} + \dots\dots + \frac{a^nt^n}{\{2n\}!} + \dots\dots$$

$$E_{p^2}\left(-\frac{at^{-1}}{[2]}\right) = 1 - \frac{at^{-1}}{[2]} + \frac{a^2t^{-2}}{[2][4]} - \dots\dots (-1)^n\frac{a^nt^{-n}}{\{2n\}!} + \dots\dots$$

shows us that

$$E_{p^2}\left(\frac{at}{[2]}\right)E_{p^2}\left(-\frac{at^{-1}}{[2]}\right) = \sum_0^{\infty} J_{[n]}(a)t^n + \sum_1^{\infty}(-1)^n J_{[n]}(a)t^{-n}$$

$$= \sum_{-\infty}^{+\infty} J_{[n]}(a)t^n \qquad (\lambda)$$

In precisely the same manner, if we consider the product of

$$E_{\frac{1}{p^2}}\left(\frac{bt}{[2]}\right) = 1 + \frac{bt}{[2]} + p^2\frac{b^2t^2}{[2][4]} + \dots\dots + p^{n(n-1)}\frac{b^nt^n}{\{2n\}!} + \dots$$

$$E_{\frac{1}{p^2}}\left(-\frac{p^{2\nu}bt^{-1}}{[2]}\right) = 1 - \frac{p^{2\nu}bt^{-1}}{[2]} + p^2\frac{p^{4\nu}b^2t^{-2}}{[2][4]} - \dots\dots (-1)^s p^{n(n-1)}\frac{p^{2n\nu}b^nt^n}{\{2n\}!} + \dots\dots$$

* Proc. Edin. Math. Soc., vol. xxii.

we obtain

$$\mathrm{E}_{\frac{1}{p^2}}\Big(\frac{bt}{[2]}\Big)\mathrm{E}_{\frac{1}{p^2}}\Big(-\frac{p^{2\nu}bt^{-1}}{[2]}\Big) = \sum_{n=0}^{n=\infty} p^{n(n-\nu)}\mathfrak{J}_{[n]}(bp^{\nu-1})t^n + \sum_{n=1}^{n=\infty}(-1)^n p^{n(n+\nu)}\mathfrak{J}_{[n]}(bp^{\nu-1})t^{-n}$$

$$= \sum_{-\infty}^{+\infty} p^{n(n-\nu)}\mathfrak{J}_{[n]}(bp^{\nu-1})t^n \qquad (\mu)$$

We have now, on taking the product of (λ) and (μ),

$$\sum_{-\infty}^{+\infty}\mathrm{J}_{[n]}(a)t^n \times \sum_{-\infty}^{+\infty} p^{n(n-\nu)}\mathfrak{J}_{[n]}(bp^{\nu-1})t^n = \mathrm{E}_{p^2}\Big(\frac{at}{[2]}\Big)\mathrm{E}_{p^2}\Big(-\frac{at^{-1}}{[2]}\Big)\mathrm{E}_{\frac{1}{p^2}}\Big(\frac{bt}{[2]}\Big)\mathrm{E}_{\frac{1}{p^2}}\Big(-\frac{p^{2\nu}bt^{-1}}{[2]}\Big)$$

The product of the four basic-exponential functions on the right of this expression is the product of two convergent series

$$\Big\{1+\frac{(a+b)t}{[2]}+\frac{(a+b)(a+p^2b)t^2}{[2][4]}+\dots\Big\} \times \Big\{1-\frac{(a+p^{2\nu}b)t^{-1}}{[2]}+\frac{(a+p^{2\nu}b)(a+p^{2\nu+2}b)}{[2][4]}t^{-1}\Big\} \qquad (B)$$

This result follows from result (η) of article (9).

If now we equate coefficients of the various powers of t in (B) with the corresponding coefficients in

$$\sum_{-\infty}^{+\infty}\mathrm{J}_{[n]}(a)t^n \times \sum_{-\infty}^{+\infty} p^{n(n-\nu)}\mathfrak{J}_{[n]}(bp^{\nu-1})t^n$$

remembering that

$$\mathrm{J}_{[n]} = (-1)^n\mathrm{J}_{[-n]}$$
$$\mathfrak{J}_{[n]} = (-1)^n\mathfrak{J}_{[-n]}$$

we obtain from the terms which are independent of t

$$\mathrm{J}_{[0]}(a)\mathfrak{J}_{[0]}(bp^{\nu-1}) - (p^{1-\nu}+p^{1-\nu})\mathrm{J}_{[1]}(a)\mathfrak{J}_{[1]}(bp^{\nu-1}) + (p^{2(2-\nu)}+p^{2(2+\nu)})\mathrm{J}_{[2]}(a)\mathfrak{J}_{[2]}(bp^{\nu-1}) - \dots \qquad (\rho)$$

$$= 1 - \frac{(a+b)(a+p^{2\nu}b)}{[2]^2} + \dots\dots$$

which by an obvious reduction becomes

$$\mathrm{J}_{[0]}(a)\mathfrak{J}_{[0]}(bp^{\nu-1}) - p^{1-\nu}\frac{[4\nu]}{[2\nu]}\mathrm{J}_{[1]}(a)\mathfrak{J}_{[1]}(bp^{\nu-1}) + \dots + (-1)^r p^{r(r-\nu)}\frac{[4r\nu]}{[2r\nu]}\mathrm{J}_{[r]}(a)\mathfrak{J}_{[r]}(bp^{\nu-1}) + \dots \qquad (\sigma)$$

$$= \mathrm{J}^{\nu}_{0}(a, b)$$

Equating the coefficients of t^n we obtain

$$\sum_{m=-\infty}^{m=+\infty}\mathrm{J}_{[m]}(a)\mathfrak{J}_{[n-m]}(bp^{\nu-1})p^{(n-m)(n-m-\nu)} = \mathrm{J}^{\nu}_{n}(a, b) \qquad (\tau)$$

the expression for J^{ν}_{n} being that given in article (8) expression (δ).

11.

When n is not a positive integer the expression

$$(a+b)(a+p^2b)\dots(a+p^{2n-2}b) \text{ in } \mathrm{J}^{\nu}_{n}(a, b)$$

must be replaced by

$$\mathop{\mathrm{L}}_{\kappa=\infty}\frac{(a+b)(a+p^2b)(a+p^4b)\dots(a+p^{2\kappa-2}b)}{(a+p^{2n}b)(a+p^{2n+2}b)\dots(a+p^{2n+2\kappa-2}b)}a^n$$

$$p < 1$$

If, however, $p > 1$,

$$\underset{\kappa=\infty}{\mathrm{L}} \frac{(a+bp^{2n-2})(a+bp^{2n-4}) \ldots (a+bp^{2n-2\kappa})}{(a+p^{-2}b)(a+p^{-4}b) \ldots (a+p^{-2\kappa}b)} a^n$$

$$p > 1$$

is the effective representative of the product $(a+b)(a+p^2b) \ldots$ to n factors. This corresponds to the change of $n!$ in the Bessel coefficients into $\Gamma(n+1)$ in the case of Bessel-Functions. The series expansions of the products given above may be found in *Proc. L.M.S.*, series 2, vol. i. pp. 63–88. The theorem analogous to NEUMANN's theorem

$$\mathrm{J}_0(a^2+b^2+2ab\cos\theta) = \mathrm{J}_0(a)\mathrm{J}_0(b) + 2\sum(-1)^s \mathrm{J}_s(a)\mathrm{J}_s(b)\cos s\theta \quad . \quad (\xi)$$

I have investigated in a paper (*Proc. L.M.S.*). The function E_p being used in a manner similar to the use of the exponential (pp. 25, 26, 27, Gray and Matthew's *Treatise on Bessel-Functions*), gives us a rather complicated extension of (ξ).