



## XXXIX. On the theory and measurement of residual charges

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XXXIX. *On the Theory and Measurement of Residual Charges.* By Prof. A. ANDERSON, M.A., and T. KEANE, B.A.\*

THE following attempt to throw light on the action in a dielectric which results in a residual charge is based on Maxwell's treatment of the subject. He pictured the medium as varying from point to point, both in specific inductive capacity and in specific resistance, and worked out the solution for the case of a dielectric of lamellar structure in which there were discontinuities in the values of these quantities at the bounding surfaces. The case of a single dielectric between two parallel plate electrodes, in which the specific inductive capacity and specific resistance are finite and continuous functions of the distance from one of the plates, admits of easy treatment. We shall only have occasion to require the solution for the steady state acquired by the dielectric after a difference of potential between the plates has been established and kept constant. Let the potential of one plate be  $V$ , and that of the other zero. Then, if  $\tau$  denote the specific resistance and  $K$  the specific inductive capacity at a point whose distance from the first plate is  $x$ , we have

$$\frac{d}{dx} \left( \frac{1}{\tau} \frac{dV}{dx} \right) = 0, \quad \text{and} \quad \frac{d}{dx} \left( K \frac{dV}{dx} \right) = -4\pi\rho,$$

where  $\rho$  is the volume density of electricity. Hence  $-\frac{1}{\tau} \frac{dV}{dx}$ , which is equal to the current  $c$  per unit area, is constant, and therefore

$$c \frac{d}{dx} (K\tau) = 4\pi\rho.$$

Hence, by integration, the whole quantity of electricity stored in the dielectric is

$$\frac{Ac}{4\pi} (K_0\tau_0 - K_1\tau_1),$$

$A$  being the area of one of the plates. Here  $K_1$  and  $K_0$  are the specific inductive capacities, and  $\tau_1$  and  $\tau_0$  the specific resistances, at the surfaces of the dielectric. Hence the amount of electricity stored, or the amount of what is known as the residual charge, or the residual discharge, depends, at any rate in the case under consideration, on the state of the

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surfaces only, and not on the manner of variation of the specific inductive capacity and specific resistance in the interior of the dielectric, except in so far as these affect the value of the current  $c$ . Furthermore, as the residual charge is always of the same sign as the primary charge, it follows that  $K_0\tau_0$  is always greater than  $K_1\tau_1$  when a positive current goes from the first plate to the second, that is, when  $V$  is positive. When  $V$  is negative  $K_0\tau_0$  is always less than  $K_1\tau_1$ . It is clear, therefore, that the surface values of one or both of the quantities  $K$  and  $\tau$  depend on the field of force. The charge  $Q_1$  on the first plate is  $\frac{Ac}{4\pi}K_1\tau_1$ , and the charge  $Q_0$  on the second plate  $-\frac{Ac}{4\pi}K_0\tau_0$ , the sum of the charges on the plates and the charge in the dielectric being, of course, zero. It will be noted that there can be no residual charge if  $K_0\tau_0=K_1\tau_1$ , or if  $c=0$ . If  $q$  denote the charge in the dielectric, we have

$$\frac{q}{Q_1} = \frac{K_0\tau_0 - K_1\tau_1}{K_1\tau_1},$$

or assuming  $K_0=K_1$ , which would seem not to be devoid of probability in an apparently homogeneous dielectric,

$$\frac{q}{Q_1} = \frac{\tau_0 - \tau_1}{\tau_1}.$$

If this be so, we must explain the existence of  $q$  by a difference between  $\tau_0$  and  $\tau_1$ , and  $\tau_0$  must be greater than  $\tau_1$ . That is to say, the resistance of the dielectric at the cathode must be greater than it is at the anode, and this difference must be produced by the field of force. An explanation is supplied by the electron theory according to which the specific resistance of a conductor is given by the expression

$\frac{4\pi\theta}{nule^2}$ . Probably the only quantities in this expression which can be different for the two surfaces of the dielectric are  $n$  and  $l$ ,  $n$  being the number of free electrons per unit volume and  $l$  their mean free path. It is easy to understand that the number of free electrons per unit volume close to the anode will be greater than that close to the cathode, just as the density of the atmosphere decreases with an increase of height. It is possible that  $l$ , the mean free path, may be less at the anode than at the cathode, owing to the increase in the value of  $n$ , but if it be remembered that the motion of a free electron takes place among both fixed atoms and free

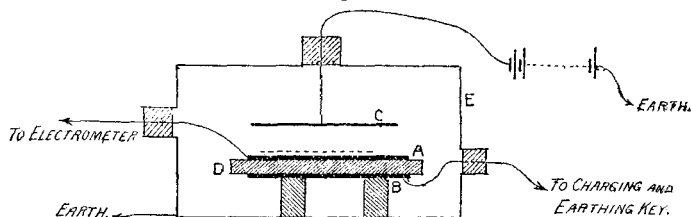
electrons, it will be seen that the change in the value of  $l$  is probably very small. We have, therefore,  $\frac{\tau_0 - \tau_1}{\tau_1} = \frac{n_1 - n_0}{n_0}$ ,

$n_1$  being the number of free electrons per unit volume at the anode, and  $n_0$  the corresponding number at the cathode. According to this view, the residual charge is due to changes of the surface resistances consequent on a displacement of the free electrons under the action of the field of force, and

$\frac{q}{Q_1} = \frac{n_1 - n_0}{n_0}$ . When the field is removed by connecting the two metal plates the residual charge disappears gradually, and, after a time,  $n_1$  becomes equal to  $n_0$ .

It is not necessary, then, for a residual charge, that the dielectric should be heterogeneous, unless the rearrangement of the free electrons caused by electric force be regarded as constituting heterogeneity. In this connexion it may be worth while quoting a remark of Maxwell ('Electricity and Magnetism,' vol. i, 3rd edition, p. 457). "It by no means follows," he says, "that every substance which exhibits the phenomenon is so composed, for it may indicate a new kind of electric polarization of which a homogeneous substance may be capable, and this in some cases may perhaps resemble electrochemical polarization much more than dielectric polarization."

Fig. 1.



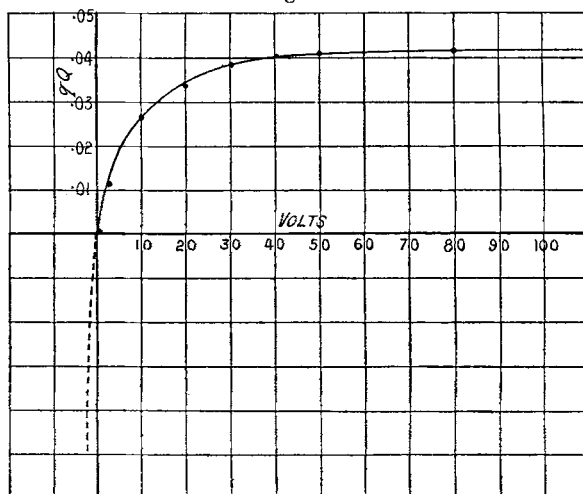
The method of measuring the residual discharge will be understood from fig. 1. The dielectric was a plate D of hard brittle amorphous sulphur of a uniform thickness of .153 cm. placed between two smooth circular plates of zinc A and B, each having a diameter of 10 cm. Both the surfaces of D were like smooth glass. The plate B was insulated by two blocks of paraffin, and the whole condenser was inside an earthed cylindrical box E. B was connected by a wire passing through a plug of paraffin wax, to a key, by means of which it could be charged to any available potential, or earthed, and A was connected by a wire passing through another plug to the electrometer. Another zinc plate C,

connected to a battery of cells, was placed above A, the air between A and C being ionized by means of a small quantity of uranium oxide placed on A. The number of cells in connexion with C was always greater than that required to produce a saturation current in this air space. When a residual charge was to be measured A was at first earthed and B charged up to a definite potential for a definite time. B was then earthed and A insulated immediately afterwards. A special key was constructed to make the interval of time between the earthing of B and the insulating of A as short as possible. The electrometer will now charge up, but will be continuously discharged by the ionized air between C and A, and when the needle comes back to zero, the time which has elapsed from the instant at which A was insulated will be a measure of the residual charge, provided that, when the needle comes back to its resting point, all the charge has disappeared. This can easily be tested by finding whether the rate of motion of the needle after passing the resting point is the same as when no charge had been given to the condenser. It was generally found that a small charge was left when the needle had come back to zero, and for this reason it was always allowed to move through 330 divisions of the scale after having come to zero, and the excess of the time required for these 330 divisions over that required to describe the same interval on the scale when no charge had been given to the condenser, was added to the time to the resting point. The rate of motion of the needle for the last 40 of these 330 divisions was always taken, and if it was found to be less than the rate when the condenser was without charge, a smaller quantity of uranium oxide was used until it was quite certain that all the residual charge had been registered. It will be seen that the method is a very convenient one for measuring either a primary or residual charge, and can be quite readily applied to determine the relation between the amount of residual charge and the time of charging for any given applied electrostatic pressure. Curves showing this relation were obtained for different pressures, and show its known asymptotic character.

The curve shown in fig. 2 gives the ratio  $q/Q$ , that is the ratio of the residual charge of the condenser to the primary charge for different values of the applied pressure, the time of charging being always greater than that required to make the residual charge closely approach its asymptotic value. It was thought that, inasmuch as the amount of the residual charge depends on the surface resistance, there might be a difference for plates of different metals, but no such difference

could be detected having a greater value than the possible experimental error, which was about one in a hundred. It was also thought that there might be a lower limit to the electrostatic pressure capable of producing a residual charge,

Fig. 2.



but no such limit could be found, and it seems, therefore, that the smallest electric force if applied long enough to the dielectric will produce some displacement of the free electrons. Though the curve in fig. 2 is, no doubt, an exponential curve, it approximates very closely to an equilateral hyperbola, the various readings, of which a few are indicated in the figure, falling into position with surprising accuracy. This fact, in itself, might be an indication that the residual charge is not mainly due to any haphazard space variation of  $\bar{K}$  and  $\tau$ , but to something quite simple in its nature, and wholly distinct from the ordinary mechanical properties of the dielectric, and to something, too, produced by the impressed electric force.

The curve approximates closely to that whose equation is

$$y(x+10)=\cdot 05 x,$$

$y$  being the ratio  $q/Q$ , and  $x$  the pressure in volts applied to the condenser. Thus we have, roughly,

$$\frac{q}{Q} = \frac{\cdot 05 V}{V+10},$$

$V$  being the difference of potential of the two plates in volts.

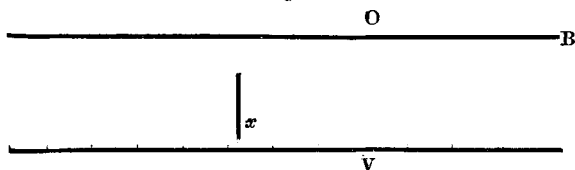
When  $V$  is indefinitely increased  $q/Q$  tends to the value  $\cdot 05$  or one twentieth. Since

$$\frac{n_1 - n_0}{n_1} = \frac{\cdot 05V}{V + 10},$$

the value of  $\frac{n_0}{n_1}$  for high pressures is  $\frac{19}{20}$ .

The following is a mathematical discussion of the problem which arises in this experiment.

Fig. 3.



Consider a dielectric between two parallel plates A and B at potentials  $V$  and  $0$  respectively, and let  $d$  be the density of the free electrons in the natural state, that is the quantity of negative electricity possessed by unit volume owing to the free electrons which it contains. Then when the field is put on, the density in the neighbourhood of A will be greater than the density in the neighbourhood of B. Let  $\rho$  denote the density at any point. In the natural state the force due to the free electrons will be

$$-4\pi d\left(x - \frac{a}{2}\right),$$

where  $a$  is the distance between the plates, the specific inductive capacity of the dielectric being taken as unity. But, since the whole force is zero in the natural state, it is clear that the force due to the positive atoms and bound electrons will be

$$4\pi d\left(x - \frac{a}{2}\right).$$

Hence, if  $X$  be the impressed force, the force at any point will be

$$X + 2\pi d(2x - a) + \text{Force due to the free electrons,} \\ \text{as rearranged by the total field.}$$

Denoting the latter force by  $P$ , we have the force at any point per unit quantity of positive electricity equal to

$$X + 2\pi d(2x - a) + P.$$

The force  $P$  is positive at  $A$ , vanishes for a certain value of  $x$ , and is negative at  $B$ . If

$$X=0, \quad P=-2\pi d(2x-a).$$

Let  $p$  be the pressure due to the free electrons; then, since  $p=K\rho$ , we have

$$\frac{K}{p} \cdot \frac{dp}{dx} = -X - 2\pi d(2x-a) - P$$

$$\text{and} \quad \frac{dP}{dx} = -4\pi\rho,$$

$$\therefore K \frac{d}{dx} \left( \frac{1}{p} \frac{dp}{dx} \right) = 4\pi(\rho - d).$$

Let  $n$  = number of free electrons per unit volume at any point, and  $N$  the number of free electrons per unit volume in the natural state. Then  $d=Ne$ , and  $\rho=ne$ ,  $e$  being the charge of an electron. We have then

$$K \frac{d}{dx} \left( \frac{1}{n} \frac{dn}{dx} \right) = 4\pi e(n - N),$$

$$\text{or} \quad \frac{d}{dx} \left( \frac{1}{n} \frac{dn}{dx} \right) = \lambda(n - N),$$

$$\text{where} \quad \lambda \text{ is written for } \frac{4\pi e}{K}.$$

This differential equation being independent of  $X$  will be true for any uniform field applied to the dielectric, and in the particular case we must have

$$\frac{K}{n} \frac{dn}{dx} = -X - 2\pi Ne(2x-a) - P,$$

$$\text{where} \quad \frac{dP}{dx} = -4\pi ne.$$

$$\text{Writing} \quad 2v = \left( \frac{dn}{dx} \right)^2,$$

the above differential equation becomes

$$\frac{d}{dn} \left( \frac{v}{n^2} \right) = \lambda \left( 1 - \frac{N}{n} \right),$$



$$\therefore v = \lambda n^2 (n - N \log n + A),$$

$$\text{or} \quad \left(\frac{dn}{dx}\right)^2 = 2\lambda n^2 (n - N \log n + A),$$

where A is a constant. Since this equation is true for every uniform field of force, we must have

$$A = N \log N - N,$$

and, therefore,

$$\left(\frac{dn}{dx}\right)^2 = 2\lambda n^2 \left(n - N - N \log \frac{n}{N}\right),$$

$$\therefore \frac{dn}{dx} = -n \left[2\lambda \left(n - N - N \log \frac{n}{N}\right)\right]^{\frac{1}{2}},$$

it being clearly negative at every point. P is therefore determined, and is equal to

$$-X - 2\pi N e (2x - a) + 4\pi e \left[\frac{2}{\lambda} \left(n - N - N \log \frac{n}{N}\right)\right]^{\frac{1}{2}}.$$

The experiment on the sulphur plate shows that, very probably,  $n$  never differs very much from  $N$ .

$$\text{Let} \quad \frac{n}{N} = 1 + \beta, \text{ where } \beta \text{ is small.}$$

$$\text{Then} \quad \frac{dn}{dx} = -N(1 + \beta) \left[2\lambda \cdot \frac{\beta^2}{2}\right]^{\frac{1}{2}},$$

$$\text{or} \quad \frac{d\beta}{dx} = -\sqrt{\lambda N} \cdot \beta(1 + \beta).$$

$$\therefore \frac{\beta}{1 + \beta} = C e^{-\sqrt{\lambda N} \cdot x}$$

where C is a constant.

$$\text{Hence} \quad n = \frac{N}{1 - C e^{-\sqrt{\lambda N} \cdot x}}.$$

The constant C must be determined by experiment. In the experiment above described, when  $q/Q$  was .05, or when the electric force applied was 80 volts per .153 cm.,

$$\frac{n_1 - n_0}{n_1} = \frac{1}{20}.$$

from which it follows that

$$C = \frac{1}{20 - 19 e^{-\sqrt{\lambda N} \cdot a}}.$$

*Note.*—In the above the existence of the variation of conductivity which is necessary in Maxwell's theory of the residual charge is made to depend on the displacement of the free electrons in the dielectric. But it must be pointed out that this displacement, without any bodily charge in the dielectric, would give rise to a residual charge. When the primary discharge takes place, bound charges are left on both plates which are gradually set free, as the displaced electrons return to the state of uniform distribution. The residual charge is therefore, probably, a more complicated phenomenon than that contemplated in Maxwell's theory.

# *XL. The Motion of the Needle of a Quadrant Electrometer.*

*By W. F. G. SWANN, D.Sc., A.R.C.S., Assistant Lecturer in Physics at the University of Sheffield\*.*

VERY small currents are frequently measured by observing the rate of movement of the needle of a quadrant electrometer as the electricity enters one of the quadrants. Such currents are of course often measured by connecting the quadrant to earth through a very high resistance, and noting the steady deflexion which is produced when the electricity passes into the quadrant at the same rate as it leaves through the high resistance. The former method is more sensitive however, and is very convenient in practice, but even though the electricity passes into the quadrant at a uniform rate, the needle does not move with uniform velocity, owing to its inertia. This effect must doubtless have been noticed by many observers, but though, unless suitable precautions are taken, considerable errors may result I have seen no accounts of any systematic precautions of this nature. In view of the present extensive employment of electrometers, the following discussion of the theory and suggestion of a simple method of avoiding the error may be of interest.

If  $\theta$  is the angle of deflexion of the needle, the equation of motion is

$$K \frac{d^2\theta}{dt^2} + b \frac{d\theta}{dt} + a\theta = F(t), \quad . \quad . \quad . \quad (1)$$

where the constants have the usual significance, and  $F(t)$  is the couple on the needle due to the electricity which has

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