On Partial Differential Equations of the First Order with several dependent variables. By H. W. Lloyd Tanner, M.A.

[*Read January 10th*, 1878.]

1. Let

$$y_1, y_2, \ldots\ldots y_n$$

be n functions of m independent variables

$$x_1, x_2, \ldots\ldots x_m,$$

m being not less than n. The equations proposed for consideration in the present paper are those involving the differential coefficients

$$\frac{dy_i}{dx_j}\,(i=1, \ldots n;\ j=1, \ldots m).$$

It is desirable to show that such equations can always be reduced to a certain standard form, and this will be done in Arts. 2—4. Before attempting the transformation in question, it is desirable to express the given equation as far as possible in terms of Jacobians. For instance, if the expression

$$\frac{dy_1}{dx_1}\cdot\frac{dy_2}{dx_2}-\frac{dy_1}{dx_2}\cdot\frac{dy_2}{dx_1}$$

occurred in the given equation, we should regard it, not as composed of two terms of the second degree, but as a single term

$$\frac{d\,(y_1, y_2)}{d\,(x_1, x_2)}$$

of the first degree. Similarly, the determinant

$$\begin{vmatrix} \frac{dy_1}{dx_1}, & \frac{dy_1}{dx_2}, & \ldots\ldots & \frac{dy_1}{dx_r} \\ \frac{dy_2}{dx_1}, & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \frac{dy_r}{dx_1}, & \ldots & \ldots & \frac{dy_r}{dx_r} \end{vmatrix}$$

would be regarded as a single term of the first degree, and will be indicated as usual by the symbol

$$\frac{d\,(y_1 \ldots y_r)}{d\,(x_1 \ldots x_r)}.$$

It will be assumed that this preliminary reduction has been effected; for, though not essential, it has a very important influence upon the formal simplicity of the transformed equation.

Reduction to the Homogeneous Form.

2. By a homogeneous equation we understand one which has the following properties:—

(1.) All the Jacobians involved are of the n^{th} order; n being the number of dependent variables.

(2.) The equation is algebraically homogeneous with respect to these Jacobians.

(3.) Besides the Jacobians the equation involves only the independent variables.

It may be added that the common degree of the terms of the homogeneous equation is never greater than the degree of any of the non-homogeneous equations to which it is equivalent.

Suppose we are given an equation, or a system of equations, involving n dependent and m independent variables: but it is not assumed that all of these variables are present in each equation. Let $u_1, \dots u_n$ be n independent functions of $x_1 \dots x_m$, $y_1 \dots y_n$; such that

$$u_i = 0 \ (i = 1 \dots\dots n) \qquad (1)$$

are particular solutions of the given equation or system of equations. It will be shown that the transformed system in which $u_1 \dots u_n$ are the dependent variables, and $x_1 \dots x_m$, $y_1 \dots y_n$ the independent variables, will be composed of homogeneous equations.

Differentiate any one of the equations (1) with respect to any x; we get

$$\frac{du_i}{dx_j} + \frac{du_i}{dy_1} \cdot \frac{dy_1}{dx_j} + \dots + \frac{du_i}{dy_n} \cdot \frac{dy_n}{dx_j} = 0,$$

or

$$-\frac{du_i}{dx_j} = \frac{du_i}{dy_1} \cdot \frac{dy_1}{dx_j} + \dots + \frac{du_i}{dy_n} \cdot \frac{dy_n}{dx_j} \qquad (2).$$

Now we know that, if

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj},$$

then

$$\begin{vmatrix} c_{11}, & c_{12}, & \dots & c_{1n} \\ c_{21}, & c_{22}, & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1}, & c_{n2}, & \dots & c_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}, & a_{12}, & \dots & a_{1n} \\ a_{21}, & a_{22}, & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2}, & \dots & a_{nn} \end{vmatrix} \times \begin{vmatrix} b_{11}, & b_{12}, & \dots & b_{1n} \\ b_{21}, & b_{22}, & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1}, & b_{n2}, & \dots & b_{nn} \end{vmatrix}$$

Comparing this with (2), we have

$$(-)^n \frac{d(u_1 \dots u_n)}{d(\xi_1 \dots \xi_n)} = \frac{d(u_1 \dots u_n)}{d(y_1 \dots y_n)} \cdot \frac{d(y_1 \dots y_n)}{d(\xi_1 \dots \xi_n)} \qquad (3),$$

where $\xi_1 \dots \xi_n$ represent any n of the m quantities $x_1 \dots x_m$.

A similar result is obtained when some of the ξ's are replaced by some of the y's. For we have identically

$$\frac{du_i}{dy_j} = \frac{du_i}{dy_1}.0 + \ldots + \frac{du_i}{dy_j}.1 + \ldots + \frac{du_i}{dy_n}.0$$
$$= \frac{du_i}{dy_1}.\frac{dy_1}{dy_j} + \ldots + \frac{du_i}{dy_j}\frac{dy_j}{dy_j} + \ldots + \frac{du_i}{dy_n}.\frac{dy_n}{dy_j} \ldots\ldots\ldots\ldots (4),$$

if we regard the y's as mutually independent, so that $\frac{dy_i}{dy_j}$ vanishes unless $i = j$, when it is unity.

Now take r equations (2), the r values of x_j being called $\xi_1 \ldots \xi_r$; and $n-r$ equations (4), calling the selected y's, $\eta_{r+1}, \ldots \eta_n$. In these n equations let u_i be replaced by $u_1 \ldots u_n$, in succession. Then we have n^2 equations which give

$$(-)^r \frac{d(u_1 \ldots u_r, u_{r+1} \ldots u_n)}{d(\xi_1 \ldots \xi_r, \eta_{r+1} \ldots \eta_n)} = \frac{d(u_1 \ldots u_n)}{d(\eta_1 \ldots \eta_n)} . \frac{d(\eta_1 \ldots \eta_r, \eta_{r+1} \ldots \eta_n)}{d(\xi_1 \ldots \xi_r, \eta_{r+1} \ldots \eta_n)}.$$

Here, $\eta_1 \ldots \eta_n$ and $y_1 \ldots y_n$ are the same quantities arranged in a different order. Let this arrangement be such that one order may be reduced to the other by an even number of transpositions. If this be not the case, it is only necessary to transpose a single pair of $\eta_1 \ldots \eta_r$, *or* of $\eta_{r+1} \ldots \eta_n$ to bring it about. Then we have

$$\frac{d(u_1 \ldots u_n)}{d(\eta_1 \ldots \eta_n)} = \frac{d(u_1 \ldots u_n)}{d(y_1 \ldots y_n)}.$$

Also, in virtue of the hypothesis as to the values of $\frac{dy_i}{dy_j}$, we have

$$\frac{d(\eta_1 \ldots \eta_r, \eta_{r+1} \ldots \eta_n)}{d(\xi_1 \ldots \xi_r, \eta_{r+1} \ldots \eta_n)} = \frac{d(\eta_1 \ldots \eta_r)}{d(\xi_1 \ldots \xi_r)},$$

as may easily be seen by writing it in the form of a determinant. Hence

$$(-)^r . \frac{d(u_1 \ldots u_r, u_{r+1} \ldots u_n)}{d(\xi_1 \ldots \xi_r, \eta_{r+1} \ldots \eta_n)} = \frac{d(u_1 \ldots u_n)}{d(y_1 \ldots y_n)} . \frac{d(\eta_1 \ldots \eta_r)}{d(\xi_1 \ldots \xi_r)} \ldots\ldots (5),$$

an equation which includes (3) as a particular case.

In the second factor on the right side of (5), $\eta_1 \ldots \eta_r$ may be any r of the n variables $y_1 \ldots y_n$; and $\xi_1 \ldots \xi_r$ may be any r of the m variables $x_1 \ldots x_m$. Thus this factor is any Jacobian of the r^{th} order: that is, since r may have any value from 1 to n inclusive, it is any Jacobian of any order that can be formed from $y_1 \ldots y_n$ with respect to $x_1 \ldots x_m$. By (5), then, any such Jacobian is expressed as a ratio between two Jacobians of the n^{th} order; the denominator always being

$$\frac{d(u_1 \ldots u_n)}{d(y_1 \ldots y_n)}, = \Delta, \text{ say.}$$

If, then, we substitute in any of the original equations from (5), we get an equation in which all the Jacobians are of the n^{th} order.

Again, each term of the transformed equation is of degree zero with respect to these Jacobians: but if in the original equation any term is of the r^{th} degree, the denominator of the corresponding term in the transformed equation is Δ^r. Suppose the degree of the original equation to be r. Then we can clear transformed equation of fractions by multiplying by Δ^r. But by this the transformed equation is made algebraically homogeneous, of the r^{th} degree.

Lastly, the transformed equation, besides the Jacobians of u, will involve only the variables $x_1 .. x_m$, $y_1 \dots y_n$, which are the independent variables of the transformed equation.

But these characters are just those by which we defined the homogeneous form of equation—so that the required reduction has been effected.

3. In some cases—more especially when the equations are linear—another method is applicable, which involves a change in the dependent variables only, not increasing the number of independent variables. Say, the original equation has the n dependent variables $y_1 \dots y_n$ and the m independent variables $x_1 \dots x_m$. Suppose also that, when $\eta_1 \dots \eta_n$ are taken as dependent variables, the equation becomes homogeneous; $\eta_1 \dots \eta_n$, being functions of $x_1 \dots x_m$, $y_1 \dots y_n$, such that

$$\frac{d(\eta_1 \dots \eta_n)}{d(y_1 \dots y_n)}$$

does not vanish. Then we may write

$$\frac{d\eta_i}{dx_j} = \frac{d\eta_i}{dy_1} \cdot \frac{dy_1}{dx_j} + \dots\dots + \frac{d\eta_i}{dy_n} \cdot \frac{dy_n}{dx_j} + \frac{d\eta_i}{dx_1} \cdot \frac{dx_1}{dx_j} + \dots\dots + \frac{d\eta_i}{dx_m} \cdot \frac{dx_m}{dx_j},$$

since $\frac{dx_i}{dx_j}$ vanishes unless $i = j$, when it becomes unity.

From these equations we infer, by a generalization of the rule for multiplying determinants, that

$$\frac{d(\eta_1 \dots \eta_n)}{d(\xi_1 \dots \xi_n)} = \Sigma \frac{d(\eta_1 \dots \eta_n)}{d(v_1 \dots v_n)} \cdot \frac{d(v_1 \dots v_n)}{d(\xi_1 \dots \xi_n)} \quad \dots\dots\dots\dots\dots (6),$$

where $\xi_1 \dots \xi_n$ are any n of $x_1 \dots x_m$; $v_1 \dots v_n$ are any n of $y_1 \dots y_n$, $\xi_1 \dots \xi_n$;* and the sign of summation includes all the different sets of the v's. The second factor on the right-hand side reduces to a

* If any v were an x not included in $\xi_1 \dots \xi_n$, the second factor on the right-hand side would vanish identically.

Jacobian of the r^{th} order when $v_1 \dots v_n$ include r of $y_1 \dots y_n$ and $n-r$ of $\xi_1 \dots \xi_n$.

Suppose that the y-equation is linear: say it is

$$P\frac{d(y_1 \dots y_n)}{d(\xi_1 \dots \xi_n)} + P'\frac{d(y_1 \dots y_n)}{d(\xi_1' \dots \xi_n')} + \dots\dots = 0 \quad \dots\dots\dots\dots (y),$$

and let the η-equation be

$$Q\frac{d(\eta_1 \dots \eta_n)}{d(\xi_1 \dots \xi_n)} + Q'\frac{d(\eta_1 \dots \eta_n)}{d(\xi_1' \dots \xi_n')} + \dots\dots = 0 \quad \dots\dots\dots\dots (\eta),$$

the second equation differing from the first in not involving any Jacobians of an order lower than the n^{th}. The left-hand side of (η) is, by (6),

$$= \frac{d(\eta_1 \dots \eta_n)}{d(y_1 \dots y_n)}\left\{Q\frac{d(y_1 \dots y_n)}{d(\xi_1 \dots \xi_n)} + Q'\frac{d(y_1 \dots y_n)}{d(\xi_1' \dots \xi_n')} + \dots\dots\right\}$$

+ terms involving y-Jacobians of lower orders.

Comparing with (y), we get

$$\frac{Q}{P} = \frac{Q'}{P'} = \dots\dots$$

Hence, as we are not concerned with the absolute values of $Q, Q', \dots$, but only with their ratios, we may say that in the η-equation the coefficients $Q, Q', \dots$ are equal to the coefficients of the corresponding y-Jacobians in the given equation (y). By comparing the coefficients of the lower Jacobians in (y) with those in (η) after the application of (6), we get a system of equations for $\eta_1 \dots \eta_n$. If we can get n particular solutions of this system, the transformation is effected.

4. As an example, consider the equation

$$\frac{d(y_1, y_2)}{d(x_1, x_2)} + \frac{d(y_1, y_2)}{d(x_2, x_3)} + \frac{y_2}{x_3}\cdot\frac{dy_1}{dx_2} - \frac{y_2}{x_1}\cdot\frac{dy_2}{dx_1}$$
$$+\left(\frac{y_1}{x_1} + \frac{x_2 y_2}{x_1 x_3}\right)\frac{dy_2}{dx_2} + \frac{y_2}{x_1}\cdot\frac{dy_2}{dx_3} + \frac{y_2^2}{x_1 x_3} = 0.$$

We have to find η_1, η_2 so that, if possible, the above may reduce to

$$\frac{d(\eta_1, \eta_2)}{d(x_1, x_2)} + \frac{d(\eta_1, \eta_2)}{d(x_2, x_3)} = 0.$$

A comparison of coefficients gives the following equations for η_1, η_2:

$$\frac{d(\eta_1, \eta_2)}{d(y_1, x_2)} = 0, \qquad \frac{d(\eta_1, \eta_2)}{d(y_2, x_2)} + \frac{y_2}{x_1}\cdot\frac{d(\eta_1, \eta_2)}{d(y_1, y_2)} = 0,$$
$$\frac{d(\eta_1, \eta_2)}{d(x_1, y_1)} + \frac{d(\eta_1, \eta_2)}{d(y_1, x_3)} - \frac{y_2}{x_3}\cdot\frac{d(\eta_1, \eta_2)}{d(y_1, y_2)} = 0,$$

$$\frac{d(\eta_1, \eta_2)}{d(x_1, y_2)} + \frac{d(\eta_1, \eta_2)}{d(y_2, x_3)} - \left(\frac{y_1}{x_1} + \frac{x_2 y_2}{x_1 x_3}\right) \frac{d(\eta_1, \eta_2)}{d(y_1, y_2)} = 0,$$

$$\frac{d(\eta_1, \eta_2)}{d(x_1, x_2)} + \frac{d(\eta_1, \eta_2)}{d(x_2, x_3)} - \frac{y_2^2}{x_1 x_3} \cdot \frac{d(\eta_1, \eta_2)}{d(y_1, y_2)} = 0.$$

These equations will be solved in the sequel. For the present it suffices to give a particular solution,

$$\eta_1 = x_1 y_1 + x_2 y_2, \quad \eta_2 = x_3 y_2,$$

which is verified without difficulty. The given equation is therefore equivalent to

$$\frac{d(x_1 y_1 + x_2 y_2, x_3 y_2)}{d(x_1, x_2)} + \frac{d(x_1 y_1 + x_2 y_2, x_3 y_2)}{d(x_2, x_3)} = 0.$$

The above example will illustrate an important point, namely, that even when the complete reduction sought is not attainable, yet the number of terms may be considerably reduced, and the equations proportionally simplified. For suppose the second member of the original equation to be Y instead of zero; then the η-equation would be of the form

$$\frac{d(\eta_1, \eta_2)}{d(x_1, x_2)} + \frac{d(\eta_1, \eta_2)}{d(x_2, x_3)} = H.$$

Thus the seven terms of the original equation would be reduced to three.

Another point of importance arises. If the y-equation of the last example be transformed into a homogeneous equation by the general method, the new equation will consist of seven terms. Suppose now that the change of variables above indicated be made in the new equation. This will reduce to two terms, thus showing that in a homogeneous equation a change of the independent variables may cause a considerable simplification. To this point I hope to return hereafter.

---

5. The results of the preceding Articles enable us to confine our attention to homogeneous equations. We shall further restrict ourselves to linear equations—viz., equations which are included in the form

$$\Sigma P \cdot \frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = 0 \quad \text{.........................} (7),$$

where $x_1 \dots x_n$ are any n of the m independent variables; P is any function of $x_1 \dots x_m$; and the sign of summation extends to the $\dfrac{\lfloor m}{\lfloor n \, \lfloor m-n}$ terms which can be formed by taking different sets of $x_1 \dots x_n$.

Of this equation (7) we shall discuss three cases:

1) When it consists of a single term, or is reducible to this form.
2) When it consists of two terms, or is reducible to this form.
3) When only certain specified terms occur.

Under each case will be discussed the systems of simultaneous equations corresponding thereto.

Single Term Equations.

6. The simplest form of the general equation (7), in which, namely, only one term occurs, is

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = 0 \quad \dots\dots\dots\dots (8).$$

The general solution is $\phi(y_1 \dots y_n) = 0 \quad \dots\dots\dots\dots (9),$

where ϕ is an arbitrary function, involving, besides its expressed arguments, any quantity not explicitly involved in (8). If, for instance, $y_1 \dots y_n$ are functions of $x_1 \dots x_n$, $x_{n+1} \dots x_m$, the solution of (8) may be written $\phi(y_1 \dots y_n, x_{n+1} \dots x_m) \quad \dots\dots\dots\dots (10).$

Let us now seek the conditions under which the general equation (7) is reducible to this well-known form—viz., when it may be written

$$\frac{d(y_1 \dots y_n, u_1 \dots u_p)}{d(x_1 \dots\dots\dots\dots x_m)} = 0 \quad (m = n+p) \quad \dots\dots\dots\dots (11).$$

This implies $\dfrac{|\underline{n+p}}{|\underline{n}\;|\underline{p}}$ equations of the form

$$P = \pm\lambda \,.\, \frac{d(u_1 \dots\dots u_p)}{d(x_{n+1} \dots x_{n+p})} \quad \dots\dots\dots\dots (12),$$

by means of which the $\dfrac{|\underline{n+p}}{|\underline{n}\;|\underline{p}}$ coefficients P are expressed in terms of $p+1$ independent quantities λ, $u_1 \dots u_p$. The former number is greater than the latter, so that the P's are subject to certain conditions when the transformation is possible, save when either n or p is zero or when $n = 1$. In the first case there is no dependent variable; in the second case the equation is (8) itself; in the third case we have only one dependent variable, or the equation is a common partial differential equation of the first order. Except in these cases the P's must satisfy certain relations which it is our object to discover. Supposing these relations to be satisfied, we then require the forms of $u_1 \dots u_p$. To attain the former we might eliminate λ, $u_1 \dots u_p$ from the equations (12), and find $u_1 \dots u_9$ by solving the system formed by eliminating λ; but we prefer to avail ourselves of the peculiar form of (12), and to make use of an artifice by means of which both ends may be attained without difficulty.

It is convenient to assume that $\xi_1 \dots \xi_m$ are the same as $x_1 \dots x_m$, but arranged in a different order. Let $\xi_1 \dots \xi_m$ be so chosen that in (7) the coefficient of

$$\frac{d(y_1 \dots\dots y_n)}{d(\xi_{1+p} \dots \xi_{n+p})} \quad\dots\dots\dots\dots\dots\dots\dots\dots\dots (13)$$

is not zero, and let the equation be divided throughout by this coefficient.

Consider next the p Jacobians formed from (13) by replacing ξ_{p+1} by $\xi_1, \xi_2 \,..\, \xi_p$ respectively. Let the coefficients of these Jacobians in (7) be denoted by $P^1_{p+1}, P^2_{p+1}, \dots P^p_{p+1}$ respectively, the suffix denoting the ξ of (13) which is removed, the index denoting the ξ which replaces it. Then a comparison of (7) and (11) gives

$$1 = \lambda \frac{d(u_1 \,..\, u_p)}{d(\xi_1 \dots \xi_p)},$$

$$P^p_{p+1} = -\lambda \frac{d(u_1 \quad u_p)}{d(\xi_1 \,..\, \xi_{p-1}, \xi_{p+1})},$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$P^1_{p+1} = \pm\lambda \frac{d(u_1 \dots u_p)}{d(\xi_2 \,..\, \xi_{p+1})},$$

equations which may be conveniently expressed in the form

$$\pm P^1_{p+1}, \mp P^2_{p+1}, \dots -P^p_{p+1}, 1 = \lambda \frac{d(u_1 \dots u_p)}{d(\xi_1 \dots \xi_{p+1})} \quad\dots\dots\dots (14),$$

which is meant to imply that the i^{th} member on the left is equal to the determinant formed on the right by erasing the i^{th} constituent of the denominator.

Now the equation $\qquad \dfrac{d(u, u_1, \dots u_p)}{d(\xi_1, \dots \xi_{p+1})} = 0$

is identically satisfied by $u = u_1, u_2 \dots u_n$. This equation may be written

$$\frac{du}{d\xi_1} \cdot \frac{d(u_1 \dots u_p)}{d(\xi_2 \dots \xi_{p+1})} - \frac{du}{d\xi_2} \cdot \frac{d(u_1 \dots u_p)}{d(\xi_1, \xi_3 \dots \xi_{p+1})} + \dots\dots$$

$$\pm \frac{du}{d\xi_{p+1}} \cdot \frac{d(u_1 \dots u_p)}{d(\xi_1 \dots \xi_p)} = 0,$$

or, because of (14),

$$P^1_{p+1} \frac{du}{d\xi_1} + P^2_{p+1} \frac{du}{d\xi_2} + \dots\dots + P^p_{p+1} \frac{du}{d\xi_p} + \frac{du}{d\xi_{p+1}} = 0.$$

This, then, is an equation which must be satisfied by $u_1 \dots u_p$; but evidently it is insufficient of itself to determine them, since it gives no information as to how they depend upon $\xi_{p+2} \dots \xi_m$. To supply this deficiency we form other equations, starting in each case with the

Jacobian (13), but replacing, for one equation ξ_{p+2}, for another ξ_{p+3}, and so on. Thus we form a system of n equations, viz.:

$$\left.\begin{aligned} \Delta_{p+1}u &= P^1_{p+1}\frac{du}{d\xi_1}+\ldots\ldots+P^p_{p+1}\frac{du}{d\xi_p}+\frac{du}{d\xi_{p+1}}=0 \\ \Delta_{p+2}u &= P^1_{p+2}\frac{du}{d\xi_1}+\ldots\ldots+P^p_{p+2}\frac{du}{d\xi_p}+\frac{du}{d\xi_{p+2}}=0 \\ \ldots \quad & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \Delta_{p+n}u &= P^1_{p+n}\frac{du}{d\xi_1}+\ldots\ldots+P^p_{p+n}\frac{du}{d\xi_p}+\frac{du}{d\xi_{p+n}}=0 \end{aligned}\right\} \quad \ldots\ldots\ldots(15),$$

which must be satisfied by $u_1 \ldots u_p$.

But the system (15) comprises n equations, which are mutually independent, since each contains a differential coefficient not occurring in any of the others: there are $n+p$ independent variables. Hence (15) cannot have more than p common solutions, nor can it have so many, unless, identically,

$$\Delta_r P^i_s = \Delta_s P^i_r \quad (i=1\ldots p;\ r, s=p+1, \ldots p+n)\ \ldots\ldots\ldots(16),$$

the Δ's being as defined in (15).

When these conditions are fulfilled, the common integrals of (15) are given by integrating

$$\left.\begin{aligned} d\xi_1 - P^1_{p+1}\,d\xi_{p+1} - P^1_{p+2}\,d\xi_{p+2} - \ldots\ldots - P^1_{p+n}\,d\xi_{p+n} &= 0 \\ d\xi_2 - P^2_{p+1}\,d\xi_{p+1} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots - P^2_{p+n}\,d\xi_{p+n} &= 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad & \ldots \\ d\xi_p - P^p_{p+1}\,d\xi_{p+1} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots - P^p_{p+n}\,d\xi_{p+n} &= 0 \end{aligned}\right\} \ldots\ldots(17),$$

which then are reducible to p exact differentials. It does not follow, however, that the p integrals of (17) may be put for $u_1 \ldots u_p$. For these integrals are determined from a knowledge of $np+1$ of the P's, and without any reference to the remaining $\frac{\lfloor n+p}{\lfloor n \lfloor p} - np - 1$; and if $u_1 \ldots u_p$ are the p integrals of (17), then the remaining P's are determined without ambiguity from the P's used in (15) or (17). These other P's must therefore satisfy certain conditions which may be determined either by forming other equations similar to (15), and expressing that all such equations must be consequences of (15), or by making use of the known relations between the determinants of the matrix

$$\frac{d\,(u_1 \ldots u_p)}{d\,(x_1 \ldots x_{n+p})}.$$

But practically it is most convenient in any given case to remember that the integrals of (17) include all the values of u which can be used in (11); and by comparison with the given equation (7) to determine how many, and which, of these integrals are suitable for our purpose. If there be p integrals of (17), and all of these be suitable, the general solution of (7) will be

$$\phi\,(y_1 \ldots y_n, u_1 \ldots u_p) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (18),$$

$u_1 \ldots u_p$ being in this case the p integrals of (17). If, on the other hand, there are less than p integrals of (17), or all the integrals are not available, then we get a solution,

$$\phi\,(y_1 \ldots y_n \cdot u_1 \ldots u_{p'}) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (19),$$

where $p' < p$. This is not a general solution, for, as will appear in the sequel, it will satisfy not only (7), but $p-p'$ other differential equations.

The knowledge of this solution is however useful, as it enables us to reduce the number of independent variables by p'. This is effected by taking a new set of independent variables, say $\xi_1 \ldots \xi_m$, of which p' are to be $u_1 \ldots u_{p'}$, say $\xi_1 = u_1, \ldots \xi_{p'} = u_{p'}$; for clearly

$$\frac{d\,(u_1 \ldots u_{p'},\ y_1 \ldots y_n)}{d\,(u_1 \ldots u_{p'},\ \xi_{p'+1} \ldots \xi_{p'+n})} = \frac{d\,(y_1 \ldots y_n)}{d\,(\xi_{p'+1} \ldots \xi_{p'+n})},$$

and so on; so that of the new independent variables only $\xi_{p'+1} \ldots \xi_{p'+n}$ will remain.

7. A few simple examples may be useful.

$$\frac{d\,(y_1, y_2)}{d\,(x_1, x_3)} - \frac{d\,(y_1, y_2)}{d\,(x_1, x_4)} - \frac{d\,(y_1, y_2)}{d\,(x_2, x_3)} + \frac{d\,(y_1, y_2)}{d\,(x_2, x_4)} = 0.$$

Let us put $\quad \xi_1 = x_1, \quad \xi_2 = x_3, \quad \xi_3 = x_2, \quad \xi_4 = x_4.$

Then
$$P_3^2 = 0, \qquad P_3^1 = -1,$$
$$P_4^2 = -1, \qquad P_4^1 = 0.$$

Thus (17) become

$$d\xi_1 + d\xi_3 = 0, \quad \xi_1 + \xi_3 \text{ or } x_1 + x_2 = u_1;$$
$$d\xi_2 + d\xi_4 = 0, \quad \xi_2 + \xi_4 \text{ or } x_3 + x_4 = u_2.$$

Hence the required solution is

$$\phi\,(x_1 + x_2,\ x_3 + x_4,\ y_1,\ y_2) = 0.$$

[It seems worth while to show *à posteriori* how such an integral equation satisfies the proposed differential equation; viz., the integral equation is $\quad y_2 = \phi\,(y_1, x_1 + x_2, x_3 + x_4),$

so that $\quad dy_2 = A\,dy_1 + B\,(dx_1 + dx_2) + C\,(dx_3 + dx_4);$

i. e.,

$$\frac{dy_2}{dx_1} = A\frac{dy_1}{dx_1} + B, \qquad \frac{dy_2}{dx_3} = A\frac{dy_1}{dx_3} + C,$$
$$\frac{dy_2}{dx_2} = A\frac{dy_1}{dx_2} + B, \qquad \frac{dy_2}{dx_4} = A\frac{dy_1}{dx_4} + C;$$

or, eliminating A, B, C,

$$\frac{\frac{dy_2}{dx_1}-\frac{dy_2}{dx_2}}{\frac{dy_1}{dx_1}-\frac{dy_1}{dx_2}}=\frac{\frac{dy_2}{dx_3}-\frac{dy_2}{dx_4}}{\frac{dy_1}{dx_3}-\frac{dy_1}{dx_4}}.]$$

The equation of Art. 4, viz.,

$$\frac{d(\eta_1, \eta_2)}{d(x_1, x_2)}+\frac{d(\eta_1, \eta_2)}{d(x_2, x_3)}=0$$

is found in the same way to have for its solution

$$\phi(x_1+x_3, \eta_1, \eta_2)=0,$$

so that the general solution of the y-equation of that Art. is

$$\phi(x_1+x_3, x_1y_1+x_3y_3, x_3y_2)=0.$$

As a third example, take

$$\frac{d(y_1, y_2, y_3)}{d(x_1, x_2, x_3)}+\frac{d(y_1, y_2, y_3)}{d(x_3, x_4, x_5)}+\frac{d(y_1, y_2, y_3)}{d(x_2, x_4, x_0)}=0.$$

Here let
$$\xi_4=x_2, \quad \xi_5=x_4, \quad \xi_6=x_0,$$
$$\xi_1=x_1, \quad \xi_2=x_3, \quad \xi_3=x_5.$$

On trying, we find that every P_j^i vanishes; so that (17) would become

$$d\xi_1=0, \quad d\xi_2=0, \quad d\xi_3=0;$$

or
$$u_1=x_1, \quad u_2=x_3, \quad u_3=x_5.$$

But without trial we know at once that these cannot be available; for, if they were, x_1, x_3, x_5 would not occur amongst the independent variables of the proposed equation. [*Cf.* (10).]

This, then, is an equation not soluble in the form (18) or (19). By another method we shall get its solution hereafter (Art. 14).

8. In the preceding paragraphs it has been assumed that the P's do not involve the dependent variables. It may happen, however, that we have to deal with an equation which is 'homogeneous' except in this respect, and the trouble of reducing to the homogeneous form may be avoided in many cases. Suppose, then, that some or all of the P's involve $y_1 \dots y_n$, so that the same will be true of $u_1 \dots u_p$. Now, in forming the determinant

$$\frac{d(y_1 \dots y_n, u_1 \dots u_p)}{d(x_1 \dots\dots\dots\dots x_{n+p})},$$

we differentiate $u_1 \dots u_p$ as if the y's were constant. For let any constituent of the determinant be

$$\frac{du_i}{dx_j}+\frac{du_i}{dy_1}\cdot\frac{dy_1}{dx_j}+\dots\dots+\frac{du_i}{dy_n}\cdot\frac{dy_n}{dx} \quad\dots\dots\dots\dots\dots\dots(19).$$

In the same column we shall have the constituents

$$\frac{dy_1}{dx_j}, \frac{dy_2}{dx_j}, \ldots\ldots \frac{dy_n}{dx_j}.$$

Multiply the rows in which these constituents occur by

$$-\frac{du_i}{dy_1}, -\frac{du_i}{dy_2}, \ldots\ldots, -\frac{du_i}{dy_n} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

respectively, and add to the row to which (19) belongs. This, of course, will not affect the value of the determinant, and it will reduce (19) to

$$\frac{du_i}{dx_j}.$$

But observing that the multipliers (20) are the same for all values of j, we find that all the constituents in the row of (19) are similarly reduced. Applying the same in turn to all the rows involving u, we get the result announced—viz., that in forming

$$\frac{d(y_1 \ldots y_n, u_1 \ldots u_p)}{d(x_1 \ldots\ldots\ldots\ldots x_{n+p})},$$

we may regard $y_1 \ldots y_n$ as constants when occurring in $u_1 \ldots u_p$.

Hence, in seeking the integrals of (15) or (17), we may regard any y which occurs in the coefficients as a constant.

Simultaneous Mononomial Equations.

9. These equations have for their general solution

$$\phi(y_1 \ldots y_n, u_1 \ldots u_p) = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20);$$

but $n+p$ is now less than the number of independent variables. The system of differential equations which has (20) for its general solution may be written

$$\frac{d(y_1 \ldots y_n, u_1 \ldots u_p)}{d(x_1 \ldots\ldots\ldots\ldots x_m)} = 0 \quad (m > n+p) \quad\ldots\ldots\ldots\ldots (21).$$

Let q be the number of independent equations that can be formed from (21). Then we have

$$\left.\begin{aligned} q &= m-(n+p)+1 \\ \text{or} \quad m+1 &= n+p+q \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22).$$

For if we take the equations formed by retaining

$$\begin{array}{l} x_1 \ldots\ldots x_{n+p-1}, x_{n+p}, \\ x_1 \ldots\ldots x_{n+p-1}, x_{n+p+1}, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ x_1 \ldots\ldots x_{n+p-1}, x_{m-1}, \\ x_1 \ldots\ldots x_{n+p-1}, x_m, \end{array}$$

supposing $x_1 \dots x_{n+p}$ so chosen that in the first equation the coefficient of

$$\frac{d(y_1 \dots\dots y_n)}{d(x_{p+1} \dots x_{p+n})},$$

that is, the determinant $$\frac{d(u_1 \dots u_p)}{d(x_1 \dots x_p)}$$

does not vanish. Then in the $\overline{i+1}^{\text{th}}$ equation we shall have a Jacobian

$$\frac{d(y_1 \dots y_{n-1}, y_n)}{d(x_{p+1} \dots x_{p+n-1}, x_{p+n+i})},$$

which does not occur in any of the others. Therefore the $m-(n+p)+1$ equations thus formed are independent.

Also in a matrix

$$\left\| \begin{matrix} a_{11}, & a_{12} & \dots\dots & a_{1,m+n} \\ \dots & \dots & \dots & \dots \\ a_{m+1,1}, & \dots\dots & & a_{m+1,m+n} \end{matrix} \right\|$$

we know that, if the n determinants, formed by combining the first m columns with each of the last n columns in turn, all vanish, then all the determinants vanish, provided

$$\left| \begin{matrix} a_{21} & \dots\dots & a_{2,m} \\ \dots & \dots & \dots \\ a_{m+1,1} & \dots\dots & a_{m+1,m} \end{matrix} \right|$$

is not zero. Applying this to (21), we learn that, if the $m-(n+p)+1$ equations formed as above all vanish, then every equation of (21) is satisfied—viz., there are *only* $m-(n+p)+1$ independent equations in the system. So that (22) is proved.

10. Suppose now we have a system of q equations each of the form (7). We might take any of the equations involving all the independent variables, and find the most general solution it admits of the form (20); then, by substitution in the other equations, determine the limitations imposed upon ϕ, $u_1 \dots u_p$. But it is more convenient generally to adopt another course. We reduce the given system as nearly as possible to the form of (21) when expanded as explained above. Each equation will contain a Jacobian not occurring in any of the other equations. In these Jacobians we shall have $n-1$ independent variables (say $x_{p+1} \dots x_{p+n-1}$) common to all; and q (say $x_{p+n} \dots x_m$) each of which occurs only in one. The other Jacobians must not involve these q independent variables, but may contain any of the $m-q$ others. This arrangement is always possible if there be a general solution of the form (20).

Having made this arrangement, let the q independent variables, each of which is characterised by occurring in one and only one equation of the system, be indicated by $x_{m-q+1}, x_{m-q+2}, \dots x_m$; or, what is the same thing, by $x_{n+p}, x_{n+p+1}, \dots x_{n+p+q-1}$. To distinguish the equations of the given system conveniently, let that which contains x_{m-q+i} or $x_{n+p+i-1}$ be called the i^{th} equation of the system.

Suppose now that the first equation of the system is reducible to

$$\frac{d(y_1 \dots y_n, u_1 \dots u_p)}{d(x_1 \dots\dots\dots x_{n+p})} = 0,$$

and let the x's be so arranged that the coefficient of

$$\frac{d(y_1 \dots y_n)}{d(x_{1+p} \dots x_{n+p})}$$

is not zero. Divide the equation throughout by this coefficient, and indicate the new coefficient of the Jacobian formed by writing x_j for x_{i+p} in the above by $\qquad {}_1P^j_{i+p}$,

where the "1" to the left indicates the number of the equation.

We have then

$$\pm {}_1P^1_{i+p}, \mp {}_1P^2_{i+p}, \dots\dots, -{}_1P^p_{i+p}, 1 = \lambda_1 \frac{d(u_1 \dots u_n)}{d(x_1 \dots x_p, x_{i+p})}.$$

But $u_1 \dots u_p$ satisfy the equation

$$\frac{d(u, u_1 \dots u_p)}{d(x_1 \dots x_p, x_{i+p})} = 0$$

identically. That is, they satisfy the equations

$${}_1P^1_{i+p}\frac{du}{dx_1} + {}_1P^2_{i+p}\frac{du}{dx_2} + \dots\dots + {}_1P^p_{i+p}\frac{du}{dx_p} + \frac{du}{dx_{i+p}} = 0 \dots\dots (15')$$
$$(i = 1, 2, \dots n).$$

The n equations (15′) are not sufficient to determine u, for they do not say how u depends upon $x_{n+p+1} \dots x_m$. To supply this knowledge we form from each of the remaining $q-1$ equations of the given system one other equation for u. From the i^{th} equation, for instance, we get

$${}_iP^1_{m-q+i}\frac{du}{dx_1} + \dots\dots + {}_iP^p_{m-q+i}\frac{du}{dx_p} + \frac{du}{dx_{m-q+i}} = 0 \dots\dots\dots (15'')$$
$$(i = 2, 3, \dots q).$$

It is here tacitly assumed that the coefficient of

$$\frac{d(y_1 \dots y_n)}{d(x_{1+p} \dots x_{n-1+p}, x_{n-1+p+i})}$$

does not vanish; but the coefficient in question being

$$\lambda_i \frac{d(u_1 \dots u_p)}{d(x_1 \dots x_p)},$$

it appears that this is no new assumption.

If in (15′) i take all values from 1 to n, and in (15″) all values from 2 to q [the value n in (15′) and 1 in (15″) would give the same equation], we get a system of $n+q-1$ independent equations involving m independent variables. These can at most have $m-(n+q)+1$, or p solutions. Thus, if $u_1 \dots u_p$ exist, they are determined by (15′), (15″), or by integrating the p expressions

$$dx_i - {}_1P^i_{1+p}\,dx_{1+p} - \dots - {}_1P^i_{n+p}\,dx_{n+p} - {}_2P^i_{n+p+1}\,dx_{n+p+1} - \dots - {}_qP^i_m\,dx_m = 0$$
$$(i = 1, 2, \dots p). \qquad \dots\dots(17').$$

In order that (17′) should have p integrals, certain conditions involving the differential coefficients of the P's must be satisfied, as (16). In order that the P's not included in the above system (17′), or (15′), (15″), may be consistent, they must satisfy certain algebraical conditions. Of these latter, one is obviously that ${}_jP^i_{k+p}$ is the same for all values of j, k having one of the values $1, 2, \dots n-1, n-1+j$.

If any of these conditions be not satisfied, then there are less than p values of u; and there is, consequently, no general solution of the given system of the form (20). But the knowledge of p' $(p' < p)$ values of u enables us to simplify the system, reducing the number of independent variables by p'.

Binomial Equations.

11. The most general form of this equation is

$$\frac{d(u_1 \dots u_m)}{d(x_1 \dots x_m)} + P\frac{d(u_1 \dots u_n,\ u_{n+1} \dots u_m)}{d(y_1 \dots y_n,\ x_{n+1} \dots x_m)},$$

which, by (5), reduces to

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = P \quad \dots\dots\dots\dots\dots\dots\dots\dots (24),$$

P being a function of x, y. I have not been able to obtain a general solution of this equation; but it is easy to get, in two distinct ways, as many solutions as we please.

12. In (24) let us replace $x_1 \dots x_n$ by a new set of independent variables $\xi_1 \dots \xi_n$. Then we have

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = \frac{d(y_1 \dots y_n)}{d(\xi_1 \dots \xi_n)} \cdot \frac{d(\xi_1 \dots \xi_n)}{d(x_1 \dots x_n)} = P.$$

Say now we attribute to $y_1 \dots y_{n-1}$ any values whatever in terms of $x_1 \dots x_n$, and replace $\xi_1 \dots \xi_{n-1}$ by these values. Then the last-written equation becomes

$$\frac{d(\xi_1 \dots \xi_n)}{d(x_1 \dots x_n)} \cdot \frac{dy_n}{d\xi_n} = P \quad \dots\dots\dots\dots\dots\dots\dots(25).$$

Lastly, take any convenient value for ξ_n such that the left-hand member does not vanish. By means of these suppositions

$$\frac{d(\xi_1 \ldots \xi_n)}{d(x_1 \ldots x_n)}$$

becomes a known function of $x_1 \ldots x_n$; while P is a given function of $y_1 \ldots y_n$, $x_1 \ldots x_n$. Both these quantities may therefore be expressed in terms of y_n, $\xi_1 \ldots \xi_n$; so that (25) is an ordinary differential equation which gives the value of y_n under the supposed conditions.

As an example, take $$\frac{d(y_1, y_2)}{d(x_1, x_2)} = 1 \quad \text{.............................(26).}$$

Let us find the form of y_2 when $y_1 = x_1^2 + x_2^2$.

Transform to new variables

$$\xi_1 = x_1^2 + x_2^2, \quad \xi_2 = \frac{x_2}{x_1}.$$

Then (24) becomes $$2\left(1 + \frac{x_2^2}{x_1^2}\right)\frac{dy_2}{d\xi_2} = 1;$$

$$\therefore \quad 2\frac{dy_2}{d\xi_2} = \frac{1}{1+\xi_2^2},$$

$$\begin{aligned} 2y_2 &= \phi(\xi_1) + \tan^{-1}\xi_2 \\ &= \phi(x_1^2 + x_2^2) + \tan^{-1}\frac{x_2}{x_1}. \end{aligned}$$

Or, again, take $$\frac{d(y_1, y_2)}{d(x_1, x_2)} = x_1 y_2 - x_2 y_1.$$

Suppose now that $y_1 = \frac{x_1}{x_2}$. Take for new variables

$$\xi_1 = \frac{x_1}{x_2}, \quad \xi_2 = x_1 x_2.$$

Then (24) becomes $$2\frac{x_1}{x_2} \cdot \frac{dy_2}{d\xi_2} = x_1(y_2 - 1);$$

$$2\frac{dy_2}{d\xi_2} = x_2(y_2 - 1) = \sqrt{\frac{\xi_2}{\xi_1}}(y_2 - 1);$$

$$\therefore \quad \frac{2dy_2}{y_2 - 1} = \frac{\xi_2^{\frac{1}{2}}}{\xi_1^{\frac{1}{2}}}\,d\xi_2,$$

whence $$y_2 = 1 + \phi\left(\frac{x_1}{x_2}\right) . e^{\frac{x_1 x_2^2}{8}}.$$

13. It is evident that by attributing thus various forms to $y_1 \ldots y_{n-1}$ we can get as many solutions as we please, and that these solutions include all possible solutions of (24) in their number. But the process, though convenient in some cases, is not of much use when $y_1 \ldots y_n$ are

restricted to certain forms, and we do not know what the forms may be. In such cases we may get solutions of (24); but $y_1 \dots y_{n-1}$ may have been chosen so as to violate some of the other conditions, and the solutions are useless. It becomes desirable, therefore, to employ a method which does not imply a knowledge of the form of $y_1 \dots y_{n-1}$. This will be now indicated by means of the same particular example,

$$\frac{d(y_1, y_2)}{d(x_1, x_2)} = 1 \quad\dots\dots\dots\dots \quad (26).$$

Now we have

$$\frac{d(y_1, y_2)}{d(x_1, x_2)} = \frac{d(y_1, y_2)}{d(y_1', y_2')} \cdot \frac{d(y_1', y_2')}{d(y_1'', y_2'')} \cdots\cdots \frac{d(y_1^{(n)}, y_2^{(n)})}{d(x_1, x_2)} = 1;$$

so that (26) will be satisfied if each of the determinants in the middle term of the last equation is unity. And, moreover, $y_1^{(r)}$, $y_2^{(r)}$ will be particular values of y_1, y_2. Now assume

$$\frac{dy_1^{(n)}}{dx_1} = 0, \quad \frac{dy_1^{(n)}}{dx_2} \cdot \frac{dy_2^{(n)}}{dx_1} = -1 \quad\dots\dots\dots\dots \quad (27),$$

so that
$$\frac{d(y_1^{(n)} . y_2^{(n)})}{d(x_1 . x_2)} = 1.$$

Then
$$y_1^{(n)} = \phi_1(x_2),$$
$$y_2^{(n)} = \psi_1(x_2) - \frac{x_1}{\phi_1'(x_2)},$$

ϕ_1, ψ_1 being arbitrary functions.

In just the same way we should get

$$y_1^{(n-1)} = \phi_2\{y_2^{(n)}\} = \phi_2\left\{\psi_1(x_2) - \frac{x_1}{\phi_1'(x_2)}\right\},$$

$$y_2^{(n-1)} = \psi_2\{y_2^{(n)}\} - \frac{y_1^{(n)}}{\phi_2'(y_2^{(n)})}$$
$$= \psi_2\left\{\psi_1(x_2) - \frac{x_1}{\phi_1'(x_2)}\right\} - \phi_1(x_2) : \phi_2'\left\{\psi_1(x_2) - \frac{x_1}{\phi_1'(x_2)}\right\}.$$

Continuing in this way, we shall get expressions of very great generality for y_1, y_2. But no solution thus obtained will be a general solution; nor can we even say that these solutions will include every possible solution, since we do not know that the forms attributed to y_1, say, include every possible form of function of x_1, x_2.

The manner of dealing with the more general equation

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = 1$$

is precisely similar. The results may be in certain cases extended to the equation (24), when P is a product of a function of $y_1 \dots y_n$ by a function of $x_1 \dots x_n$. For we have

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)}, = Y . X \quad\dots\dots\dots\dots (28),$$

$$= \frac{d(y_1 \dots y_n)}{d(y'_1 \dots y'_n)} \cdot \frac{d(y'_1 \dots y'_n)}{d(y''_1 \dots y''_n)} \cdot \frac{d(y''_1 \dots y''_n)}{d(x_1 \dots x_n)}.$$

Now let $y''_1 \dots y''_n$ be particular solutions of

$$\frac{d(y''_1 \dots y''_n)}{d(x_1 \,..\, x_n)} = X;$$

and $y'_1 \dots y'_n$ be particular solutions of

$$\frac{d(y'_1 \dots y'_n)}{d(y_1 \dots y_n)} = \frac{1}{Y}.$$

Lastly, let us find as general a solution as possible of

$$\frac{d(y'_1 \dots y'_n)}{d(y''_1 \dots y''_n)} = 1.$$

By eliminating y', y'' from these results we shall get a solution of (28) of considerable generality.

We have already met with examples of systems of simultaneous binomial equations; for instance, (14) is such. These, however, in all cases in which their general solution has been attained, are included in the next class of equations. It will be convenient to defer the consideration of them.

Equations with Selected Terms.

14. The last class of equations with which we propose to deal only include certain selected terms of the general equation

$$\Sigma P \frac{d(y_1 \dots y_n)}{d(\xi_1 \dots \xi_n)} = 0 \quad\dots\dots\dots\dots (7).$$

These selected terms are distinguished by the circumstance that by replacing one of the independent variables in each Jacobian by one of the independent variables excluded from it, we arrive at one and the same Jacobian, which may or may not actually occur in the equation.

For convenience we will call the dependent variables $u_1 \dots u_n$; and will suppose the defining Jacobian of the equation to be

$$\frac{d(u_1 \dots u_n)}{d(y_1 \dots y_n)}.$$

The remaining independent variables we shall indicate by $x_1 \dots x_p$, so that there are in all $n+p$ independent variables. The general form of the Jacobian which can occur in an equation of the class we now consider is

$$\frac{d(u_1 \dots u_i \dots u_n)}{d(y_1 \dots x_j \dots y_n)};$$

viz., *one* of the y's is replaced by *one* of the x's. The most general form of the equation itself is

$$P\frac{d(u_1 \dots u_n)}{d(y_1 \dots y_n)} + \Sigma P_j^i \frac{d(u_1 \dots u_i \dots u_n)}{d(y_1 \dots x_j \dots y_n)} = 0 \quad \dots\dots\dots\dots (29).$$

A reference to (5) will show that

$$\frac{d(u_1 \dots u_i \dots u_n)}{d(y_1 \dots x_j \dots y_n)} : \frac{d(u_1 \dots u_n)}{d(y_1 \dots y_n)} = -\frac{dy_i}{dx_j}.$$

Thus (29) reduces to

$$\Sigma P_j^i \frac{dy_i}{dx_j} = P \; (i = 1 \dots n,\; j = 1 \dots m) \quad \dots\dots\dots\dots (30),$$

which is a partial differential equation of the first order—involving only Jacobians of the first order.

If it should happen that all the i's in (29) or (30) were the same, then (30) would be a partial differential equation with *one* dependent variable. It is to this circumstance that the form of (15), for instance, is due.

Suppose, however, that the i's are different. Say we have r different y's and s different x's in (30): let these be $y_1 \dots y_r$, $x_1 \dots x_s$. Now choose r new dependent variables $\eta_1 \dots \eta_r$ functions of $y_1 \dots y_n$, $x_1 \dots x_m$, and m new independent variables $\xi_1 \dots \xi_m$ functions of $x_1 \dots x_m$, so that, if possible, (30) may reduce to the form

$$\frac{d\eta_1}{d\xi_1} + \frac{d\eta_2}{d\xi_2} + \dots + \frac{d\eta_r}{d\xi_r} = 0 \quad \dots\dots\dots\dots (31).$$

The general solution of this equation is

$$\eta_1, -\eta_2, \dots, \pm\eta_r = \frac{d(\phi_1, \phi_2, \dots \phi_{r-1})}{d(\xi_1, \xi_2, \dots \xi_r)},$$

where $\phi_1, \dots \phi_{r-1}$ are $r-1$ arbitrary functions of $(\xi_1 \dots \xi_m)$ (see "Messenger of Mathematics," vol. vii., p. 107). This is also the general solution of (30), if after the differentiations $\eta_1 \dots \eta_r$, $\xi_1 \dots \xi_m$ be replaced by their values in terms of $x_1 \dots x_m$, $y_1 \dots y_n$. Also the solution of (29) may be written

$$\psi_k \{u_1, \dots, u_n, y_{r+1}, \dots, y_n, \eta_k (-)^k \Delta_k\} = 0 \quad \dots\dots\dots\dots (32),$$
$$k = 1, \dots r,$$

where $$\Delta_1, \Delta_2, \dots \Delta_r = \frac{d(\phi_1 \dots \phi_{r-1})}{d(\xi_1 \dots \xi_r)}.$$

For $u_1, u_2, \dots u_n = 0$ are solutions of (30).

A simple example of this class of equation is afforded by

$$\frac{d(u_1, u_2, u_3)}{d(x_1, x_2, x_3)} + \frac{d(u_1, u_2, u_3)}{d(x_3, x_4, x_5)} + \frac{d(u_1, u_2, u_3)}{d(x_2, x_4, x_6)}.$$

If we write
$$x_2 = \eta_1, \quad x_3 = \eta_2, \quad x_4 = \eta_3,$$
$$x_1 = \xi_1, \quad x_5 = \xi_2, \quad x_6 = \xi_3.$$

The above becomes

$$\frac{d(u_1, u_2, u_3)}{d(\eta_1, \eta_2, \xi_1)} + \frac{d(u_1, u_2, u_3)}{d(\xi_2, \eta_2, \eta_3)} - \frac{d(u_1, u_2, u_3)}{d(\eta_1, \xi_3, \eta_3)} = 0,$$

whence
$$\frac{d\eta_3}{d\xi_1} + \frac{d\eta_1}{d\xi_2} - \frac{d\eta_2}{d\xi_3} = 0,$$

or
$$\eta_3, \ -\eta_1, \ -\eta_2 = \frac{d(\phi_1, \phi_2)}{d(\xi_1, \xi_2, \xi_3)},$$

i.e.,
$$x_4, \ -x_2, \ -x_3 = \frac{d(\phi_1, \phi_2)}{d(x_1, x_5, x_6)} = \Delta_1, \Delta_5, \Delta_6 \text{ say},$$

where ϕ_1, ϕ_2 are arbitrary functions of x_1, x_5, x_6. The solution of the given equation is, therefore,

$$\psi_1 \{u_1, u_2, u_3, x_4 - \Delta_1\} = 0,$$
$$\psi_2 \{u_1, u_2, u_3, x_2 + \Delta_5\} = 0,$$
$$\psi_3 \{u_1, u_2, u_3, x_3 + \Delta_6\} = 0.$$

15. There is no special difficulty as to simultaneous equations. As an example we may take the equations left in a previous article, viz.,

$$\frac{d(\eta_1, \eta_2)}{d(y_1, x_2)} = 0, \quad \frac{d(\eta_1, \eta_2)}{d(y_2, x_2)} + \frac{y_2}{x_1}\frac{d(\eta_1, \eta_2)}{d(y_1, y_2)} = 0,$$

$$\frac{d(\eta_1, \eta_2)}{d(x_1, y_1)} + \frac{d(\eta_1, \eta_2)}{d(y_1, x_3)} - \frac{y_2}{x_3} \cdot \frac{d(\eta_1, \eta_2)}{d(y_1, y_2)} = 0,$$

$$\frac{d(\eta_1, \eta_2)}{d(x_1, y_2)} + \frac{d(\eta_1, \eta_2)}{d(y_2, x_3)} - \left(\frac{y_1}{x_1} + \frac{x_2 y_2}{x_1 x_3}\right) \frac{d(\eta_1, \eta_2)}{d(y_1, y_2)} = 0,$$

$$\frac{d(\eta_1, \eta_2)}{d(x_1, x_2)} + \frac{d(\eta_1, \eta_2)}{d(x_3, x_1)} - \frac{y_2^2}{x_1 x_3} \cdot \frac{d(\eta_1, \eta_2)}{d(y_1, y_2)} = 0.$$

From these, taking y_1, y_2 as new dependent variables, and writing only independent equations,

$$\frac{dy_2}{dx_2} = 0, \qquad \frac{dy_2}{dx_1} - \frac{dy_2}{dx_3} - \frac{y_2}{x_3} = 0,$$

$$\frac{dy_1}{dx_2} + \frac{y_2}{x_1} = 0, \qquad \frac{dy_1}{dx_1} - \frac{dy_1}{dx_3} + \frac{y_1}{x_1} + \frac{x_2 y_2}{x_1 x_3} = 0.$$

From the first pair we get

$$y_2 = \frac{1}{x_3}\,\phi\,(x_1+x_3),$$

and then, from the second pair,

$$y_1 = -\frac{x_2}{x_1 x_3}\,.\,\phi\,(x_1+x_3) + \frac{1}{x_1}\,\psi\,(x_1+x_3),$$

$$= -\frac{x_2}{x_1}\,y_2 + \frac{1}{x_1}\,\psi\,(x_1+x_3).$$

These lead to the general solution found in Art. 7.

On the Relation between the Functions of Laplace and Bessel.

By Lord RAYLEIGH, M.A., F.R.S.

[*Read January* 10, 1878.]

In § 783 of Thomson and Tait's "Natural Philosophy" a suggestion is made to examine the transition from formulæ dealing with Laplace's spherical functions to the corresponding formulæ proper to a plane. It is evident at once from this point of view that Bessel's functions are merely particular cases of Laplace's more general functions; but the fact seems to be very little known. Of two valuable works recently published on this subject,* one makes no mention of Bessel's functions, and the other states expressly that they are not connected with the main subject of the book; other mathematicians also, to whom I have mentioned the matter, have been unaware of the relation. Under these circumstances it may not be superfluous to point out briefly the correspondence of some of the formulæ.

The Bessel's function of zero order J_0 is the limiting form of Legendre's function $P_n(\mu)$, when n is indefinitely great, and μ $(= \cos\theta)$ such that $n \sin\theta$ is finite, equal say to z. The simplest proof of this assertion is perhaps that obtained from Murphy's series for P_n. Thus (Todhunter, § 23),

$$P_n(\cos\theta) = \cos^{2n}\frac{\theta}{2}\left\{1-\left(\frac{n}{1}\tan\frac{\theta}{2}\right)^2+\left(\frac{n\,.\,n-1}{1\,.\,2}\tan^2\frac{\theta}{2}\right)^2-\ldots\ldots\right\} \qquad \ldots\ldots\ldots\ldots(1).$$

---

* "An Elementary Treatise on Laplace's Functions, Lamé's Functions, and Bessel's Functions." By I. Todhunter. 1875.

"An Elementary Treatise on Spherical Harmonics and subjects connected with them." By the Rev. N. M. Ferrers. 1877.