Geodesics on Quadrics, not of Revolution. By A. R. FORSYTH.

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Central Quadrics. §§ 1–15.

1. One of the best known properties (Joachimstahl's) of any geodesic drawn upon an ellipsoid (or upon any central quadric) is represented by the equation

$$pD = \text{constant} = k^2,$$

where p is the perpendicular from the centre on the tangent plane at the point, and D is the length of a central semi-diameter parallel to the direction of the geodesic through the point; the quantity k is constant along the geodesic.

But an equation of precisely the same form characterizes lines of curvature upon central quadrics, the difference between the two arising in the value of the constant k for the particular curve. Yet even this difference disappears when the equation is used in a form

$$\frac{d}{ds}(pD) = 0,$$

current along the curve. The property, thus stated, does not distinguish between a geodesic and a line of curvature; it might, indeed, belong to curves of other classes passing through the point. A question is thus suggested as to the curves which are determined by either of the equivalent equations

$$pD = \text{constant}, \quad \frac{d}{ds}(pD) = 0.$$

2. Taking the quadric in the form

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} + \frac{z^2}{\gamma} = 1,$$

and denoting the tangential direction of the curve through x, y, z, by l, m, n, so that

$$l, \; m, \; n = x', \; y', \; z',$$

where dashes imply differentiation with regard to the arc s, we have

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{1}{p^2},$$

$$\frac{l^2}{\alpha}+\frac{m^2}{\beta}+\frac{n^2}{\gamma}=\frac{1}{D^2},$$

$$\frac{lx}{\alpha}+\frac{my}{\beta}+\frac{nz}{\gamma}=0,$$

$$l^2+m^2+n^2=1.$$

From these we have

$$\frac{\dfrac{lx}{\alpha^2}+\dfrac{my}{\beta^2}+\dfrac{nz}{\gamma^2}}{\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}+\dfrac{z^2}{\gamma^2}}=-\frac{1}{p}\,\frac{dp}{ds},$$

$$\frac{\dfrac{ll'}{\alpha}+\dfrac{mm'}{\beta}+\dfrac{nn'}{\gamma}}{\dfrac{l^2}{\alpha}+\dfrac{m^2}{\beta}+\dfrac{n^2}{\gamma}}=-\frac{1}{D}\,\frac{dD}{ds};$$

whence, if we use the characteristic equation in the form

$$\frac{1}{p}\,\frac{dp}{ds}+\frac{1}{D}\,\frac{dD}{ds}=0,$$

we have $$l'\,\frac{D^2l}{\alpha}+m'\,\frac{D^2m}{\beta}+n'\,\frac{D^2n}{\gamma}=-\frac{p^2l}{\alpha^2}x-\frac{p^2m}{\beta^2}y-\frac{p^2n}{\gamma^2}z.$$

Again, we have

$$\frac{lx'}{\alpha}+\frac{my'}{\beta}+\frac{nz'}{\gamma}+\frac{l'x}{\alpha}+\frac{m'y}{\beta}+\frac{n'z}{\gamma}=0,$$

that is, $$l'\,\frac{D^2x}{\alpha}+m'\,\frac{D^2y}{\beta}+n'\,\frac{D^2z}{\gamma}=-1;$$

and $$l'l+m'm+n'n=0.$$

There are thus three equations to determine l', m', n', and they will determine these quantities uniquely unless they are not independent of one another.

When we solve them, we have, as the coefficient of l', the quantity

$$D^4 \begin{vmatrix} \frac{l}{\alpha}, & \frac{m}{\beta}, & \frac{n}{\gamma} \\ \frac{x}{\alpha}, & \frac{y}{\beta}, & \frac{z}{\gamma} \\ l, & m, & n \end{vmatrix},$$

which is equal to

$$\frac{D^4}{\alpha\beta\gamma}\{xmn(\beta-\gamma)+ynl(\gamma-\alpha)+zlm(\alpha-\beta)\} = \frac{D^4}{\alpha\beta\gamma}\Theta, \text{ say.}$$

The value of $\frac{D^4}{\alpha\beta\gamma}\Theta l'$ is

$$= -p^2\left(\frac{lx}{\alpha^2}+\frac{my}{\beta^2}+\frac{nz}{\gamma^2}\right)D^2\left(\frac{ny}{\beta}-\frac{mz}{\gamma}\right)-D^2mn\left(\frac{1}{\gamma}-\frac{1}{\beta}\right).$$

Now $\frac{lx}{\alpha^2}+\frac{my}{\beta^2}+\frac{nz}{\gamma^2} = -\frac{1}{\alpha}\left(\frac{my}{\beta}+\frac{nz}{\gamma}\right)+\frac{my}{\beta^2}+\frac{nz}{\gamma^2}$

$$= -\frac{my}{\beta}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)-\frac{nz}{\gamma}\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right);$$

and therefore the coefficient of $-p^2D^2$ is

$$-\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\frac{mn}{\beta^2}y^2-\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)\frac{n^2}{\beta\gamma}yz+\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\frac{m^2}{\beta\gamma}yz+\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)\frac{mn}{\gamma^2}z^2.$$

Also the quantity $-D^2mn\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)$ is equal to $-p^2D^2$ multiplied by

$$\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{mn}{\beta^2}y^2+\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{mn}{\alpha^2}x^2+\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{mn}{\gamma^2}z^2.$$

Hence the whole expression for $\frac{D^4}{\alpha\beta\gamma}\Theta l'$ is

$$p^2D^2\left\{-\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{mn}{\alpha^2}x^2 +\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)\left(\frac{mn}{\beta^2}y^2+\frac{n^2}{\beta\gamma}yz\right) +\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\left(\frac{m^2}{\beta\gamma}yz+\frac{mn}{\gamma^2}z^2\right)\right\}.$$

But $$\frac{mn}{\beta^2}y^2+\frac{n^2}{\beta\gamma}yz=\frac{ny}{\beta}\left(\frac{my}{\beta}+\frac{nz}{\gamma}\right)=-\frac{lnxy}{\alpha\beta},$$

and $$\frac{m^2}{\beta\gamma}yz+\frac{mn}{\gamma^2}z^2=\frac{mz}{\gamma}\left(\frac{my}{\beta}+\frac{nz}{\gamma}\right)=-\frac{lmxz}{\alpha\gamma};$$

and therefore

$$\frac{D^4}{\alpha\beta\gamma}\Theta l'=p^2D^2\frac{x}{\alpha}\left\{-\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{mnx}{\alpha}-\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)\frac{lny}{\beta}-\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\frac{lmz}{\gamma}\right\}$$
$$=-p^2D^2\frac{x}{\alpha^2\beta\gamma}\Theta,$$

so that, as D does not vanish, we have*

and similarly $$\left.\begin{aligned}\Theta l'&=-\Theta\frac{p^2}{D^2}\frac{x}{\alpha}\\ \Theta m'&=-\Theta\frac{p^2}{D^2}\frac{y}{\beta}\\ \Theta n'&=-\Theta\frac{p^2}{D^2}\frac{z}{\gamma}\end{aligned}\right\}$$

3. If Θ does not vanish, we have

$$\frac{l'}{\frac{x}{\alpha}}=\frac{m'}{\frac{y}{\beta}}=\frac{n'}{\frac{z}{\gamma}}=-\frac{p^2}{D^2},$$

which are the equations† of a geodesic through x, y, z.

But, if Θ vanishes, the equations do not determine l', m', n'. In that case, we have

$$xmn(\beta-\gamma)+ynl(\gamma-\alpha)+zlm(\alpha-\beta)=0,$$

or, what is the equivalent,

$$\frac{x}{l}(\beta-\gamma)+\frac{y}{m}(\gamma-\alpha)+\frac{z}{n}(\alpha-\beta)=0.$$

This, together with $$\left.\begin{aligned}\frac{lx}{\alpha}+\frac{my}{\beta}+\frac{nz}{\gamma}&=0\\ l^2+m^2+n^2&=1\end{aligned}\right\},$$

* See Salmon's *Solid Geometry*, 3rd edition, p. 353, note.
† Frost's *Solid Geometry*, 3rd ed., p. 314.

suffices to determine the (two) sets of values at x, y, z for l, m, n. That these two sets correspond to the lines of curvature can be seen easily as follows. The direction of either of the lines of curvature is normal to a confocal; so that, if ϕ be a root (other than zero) of the equation

$$\frac{x^2}{a-\theta}+\frac{y^2}{\beta-\theta}+\frac{z^2}{\gamma-\theta}=1,$$

the direction cosines λ, μ, ν of the line of curvature, that is normal to the ϕ confocal, are proportional to

$$\frac{x}{a-\phi},\ \frac{y}{\beta-\phi},\ \frac{z}{\gamma-\phi}.$$

Hence
$$\frac{x}{\lambda}(\beta-\gamma)+\frac{y}{\mu}(\gamma-a)+\frac{z}{\nu}(a-\beta)$$

is proportional to

$$(\beta-\gamma)(a-\phi)+(\gamma-a)(\beta-\phi)+(a-\beta)(\gamma-\phi),$$

that is, it vanishes; and

$$\frac{\lambda x}{a}+\frac{\mu y}{\beta}+\frac{\nu z}{\gamma}$$

is proportional to $\dfrac{x^2}{a(a-\phi)}+\dfrac{y^2}{\beta(\beta-\phi)}+\dfrac{z^2}{\gamma(\gamma-\phi)}$,

that is, to

$$\frac{1}{\phi}\left\{\frac{x^2}{a-\phi}+\frac{y^2}{\beta-\phi}+\frac{z^2}{\gamma-\phi}-\left(\frac{x^2}{a}+\frac{y^2}{\beta}+\frac{z^2}{\gamma}\right)\right\},$$

so that it vanishes. Hence the two sets of values, determined for l, m, n, correspond to the lines of curvature.

Consequently, *the equation*

$$\frac{d}{ds}(pD)=0$$

determines either a geodesic or a line of curvature. When taken in the form

$$pD=k^2,$$

it determines either a geodesic or one of the lines of curvature according to the value of k.

The only exception is when both sets of equations, viz.,

$$\frac{l'}{\frac{x}{a}} = \frac{m'}{\frac{y}{\beta}} = \frac{n'}{\frac{z}{\gamma}} = -\frac{p^2}{D^2},$$

and $$\Theta = 0,$$

are satisfied. This circumstance occurs when the geodesic, determined by the former, touches a line of curvature, determined by the latter; at the point, l, m, n have the same values. And, in fact, the quantity k, which is the parameter of a geodesic, can be equal to the parameter of some line of curvature, which accordingly is touched by the geodesic.

4. But, though the discrimination between the geodesic and the line of curvature cannot be made by the explicit form

$$\frac{d}{ds}(pD) = 0,$$

it can be secured by introducing into the differential equation the ellipsoidal surface-parameters. Denoting these by λ_1 and λ_2, the roots (other than zero) of the equation

$$\frac{x^2}{a-\theta} + \frac{y^2}{\beta-\theta} + \frac{z^2}{\gamma-\theta} = 1,$$

we have, as usual, $$\frac{1}{p^2} = \frac{\lambda_1\lambda_2}{a\beta\gamma},$$

$$x^2 = \text{A}\,(a-\lambda_1)(a-\lambda_2),$$

$$y^2 = \text{B}\,(\beta-\lambda_1)(\beta-\lambda_2),$$

$$z^2 = \Gamma\,(\gamma-\lambda_1)(\gamma-\lambda_2),$$

where, if $\square$ denote $$(a-\beta)(a-\gamma)(\beta-\gamma),$$

then $$\square\text{A} = a\,(\beta-\gamma),$$

$$\square\text{B} = \beta\,(\gamma-a),$$

$$\square\Gamma = \gamma\,(a-\beta)$$

they satisfy the equations

$$\mathrm{A}+\mathrm{B}+\Gamma=0, \qquad \frac{\mathrm{A}}{\alpha}+\frac{\mathrm{B}}{\beta}+\frac{\Gamma}{\gamma}=0.$$

Now
$$-2\frac{dx}{ds}=\mathrm{A}^{\frac{1}{2}}\left\{\left(\frac{\alpha-\lambda_2}{\alpha-\lambda_1}\right)^{\frac{1}{2}}\frac{d\lambda_1}{ds}+\left(\frac{\alpha-\lambda_1}{\alpha-\lambda_2}\right)^{\frac{1}{2}}\frac{d\lambda_2}{ds}\right\},$$
$$-2\frac{dy}{ds}=\mathrm{B}^{\frac{1}{2}}\left\{\left(\frac{\beta-\lambda_2}{\beta-\lambda_1}\right)^{\frac{1}{2}}\frac{d\lambda_1}{ds}+\left(\frac{\beta-\lambda_1}{\beta-\lambda_2}\right)^{\frac{1}{2}}\frac{d\lambda_2}{ds}\right\},$$
$$-2\frac{dz}{ds}=\Gamma^{\frac{1}{2}}\left\{\left(\frac{\gamma-\lambda_2}{\gamma-\lambda_1}\right)^{\frac{1}{2}}\frac{d\lambda_1}{ds}+\left(\frac{\gamma-\lambda_1}{\gamma-\lambda_2}\right)^{\frac{1}{2}}\frac{d\lambda_2}{ds}\right\}$$

Hence
$$4=\left(2\frac{dx}{ds}\right)^2+\left(2\frac{dy}{ds}\right)^2+\left(2\frac{dz}{ds}\right)^2$$
$$=\left(\frac{d\lambda_1}{ds}\right)^2\Sigma\left\{\mathrm{A}\,\frac{\alpha-\lambda_2}{\alpha-\lambda_1}\right\}+\left(\frac{d\lambda_2}{ds}\right)^2\Sigma\left\{\mathrm{A}\,\frac{\alpha-\lambda_1}{\alpha-\lambda_2}\right\},$$

and
$$4\left(\frac{x'^2}{\alpha}+\frac{y'^2}{\beta}+\frac{z'^2}{\gamma}\right)$$
$$=\left(\frac{d\lambda_1}{ds}\right)^2\Sigma\left\{\frac{\mathrm{A}}{\alpha}\,\frac{\alpha-\lambda_2}{\alpha-\lambda_1}\right\}+\left(\frac{d\lambda_2}{ds}\right)^2\Sigma\left\{\frac{\mathrm{A}}{\alpha}\,\frac{\alpha-\lambda_1}{\alpha-\lambda_2}\right\}.$$

But, taking
$$pD=k^2,$$
we have
$$\frac{1}{D^2}=\frac{x'^2}{\alpha}+\frac{y'^2}{\beta}+\frac{z'^2}{\gamma},$$
$$\frac{1}{p^2}=\frac{\lambda_1\lambda_2}{\alpha\beta\gamma},$$
so that
$$\frac{x'^2}{\alpha}+\frac{y'^2}{\beta}+\frac{z'^2}{\gamma}=\frac{\alpha\beta\gamma}{\lambda_1\lambda_2k^4}=\frac{\delta}{\lambda_1\lambda_2},$$
say, where
$$\delta=\frac{\alpha\beta\gamma}{k^4};$$

and, in the case of an ellipsoid for which $\alpha>\beta>\gamma$,
$$\alpha>\delta>\gamma.$$

Thus the second equation is
$$\frac{4\delta}{\lambda_1\lambda_2}=\left(\frac{d\lambda_1}{ds}\right)^2\Sigma\left\{\frac{\mathrm{A}}{\alpha}\,\frac{\alpha-\lambda_2}{\alpha-\lambda_1}\right\}+\left(\frac{d\lambda_2}{ds}\right)^2\Sigma\left\{\frac{\mathrm{A}}{\alpha}\,\frac{\alpha-\lambda_1}{\alpha-\lambda_2}\right\}$$

Now

$$\Sigma A \frac{a-\lambda_2}{a-\lambda_1} = \frac{a(\beta-\gamma)}{\Box}\frac{a-\lambda_2}{a-\lambda_1} + \frac{\beta(\gamma-a)}{\Box}\frac{\beta-\lambda_2}{\beta-\lambda_1} + \frac{\gamma(a-\beta)}{\Box}\frac{\gamma-\lambda_2}{\gamma-\lambda_1}$$

$$= \frac{\lambda_1(\lambda_1-\lambda_2)}{(a-\lambda_1)(\beta-\lambda_1)(\gamma-\lambda_1)},$$

$$\Sigma A \frac{a-\lambda_1}{a-\lambda_2} = \frac{\lambda_2(\lambda_2-\lambda_1)}{(a-\lambda_1)(\beta-\lambda_1)(\gamma-\lambda_1)},$$

$$\Sigma \frac{A}{a}\frac{a-\lambda_2}{a-\lambda_1} = \frac{\lambda_1-\lambda_2}{(a-\lambda_1)(\beta-\lambda_1)(\gamma-\lambda_1)},$$

$$\Sigma \frac{A}{a}\frac{a-\lambda_1}{a-\lambda_2} = \frac{\lambda_2-\lambda_1}{(a-\lambda_2)(\beta-\lambda_2)(\gamma-\lambda_2)}$$

Hence, writing $2d\Lambda_1 = \left\{\frac{\lambda_1-\lambda_2}{(a-\lambda_1)(\beta-\lambda_1)(\gamma-\lambda_1)}\right\}^{\frac{1}{2}} d\lambda_1,$

$$2d\Lambda_2 = \left\{\frac{\lambda_2-\lambda_1}{(a-\lambda_2)(\beta-\lambda_2)(\gamma-\lambda_2)}\right\}^{\frac{1}{2}} d\lambda_2,$$

the equations are

$$\left.\begin{aligned}\lambda_1\left(\frac{d\Lambda_1}{ds}\right)^2+\lambda_2\left(\frac{d\Lambda_2}{ds}\right)^2 &= 1\\ \left(\frac{d\Lambda_1}{ds}\right)^2+\left(\frac{d\Lambda_2}{ds}\right)^2 &= \frac{\delta}{\lambda_1\lambda_2}\end{aligned}\right\}$$

Introducing a quantity $R\lambda$, defined for $\lambda=\lambda_1$ and $\lambda=\lambda_2$ by the equation

$$R\lambda = -\lambda(a-\lambda)(\beta-\lambda)(\gamma-\lambda)(\delta-\lambda),$$

we have, on solving these equations

$$1-\frac{\delta}{\lambda_2} = (\lambda_2-\lambda_1)\left(\frac{d\Lambda_2}{ds}\right)^2;$$

and therefore $\qquad \frac{1}{4}(\lambda_2-\lambda_1)^2\left(\frac{d\lambda_2}{ds}\right)^2 = \frac{1}{\lambda_2^2} R\lambda_2,$

so that $\qquad \frac{1}{2}(\lambda_2-\lambda_1)\frac{d\lambda_2}{ds} = \frac{1}{\lambda_2}\sqrt{R\lambda_2}.$

Similarly, $\qquad \frac{1}{2}(\lambda_1-\lambda_2)\frac{d\lambda_1}{ds} = \frac{1}{\lambda_1}\sqrt{R\lambda_1}.$

Consequently,

$$\left.\begin{aligned} \frac{\lambda_1 d\lambda_1}{2\sqrt{R\lambda_1}} + \frac{\lambda_2 d\lambda_2}{2\sqrt{R\lambda_2}} &= 0 \\ \frac{d\lambda_1}{2\sqrt{R\lambda_1}} + \frac{d\lambda_2}{2\sqrt{R\lambda_2}} &= du \\ -\frac{ds}{\lambda_1\lambda_2} &= du \end{aligned}\right\}$$

the final form of the differential equations; it agrees with the form given by Weierstrass* in 1861, obtained by other considerations.

5. These have been deduced on the supposition that the two equations involving $\frac{d\Lambda_1}{ds}$, $\frac{d\Lambda_2}{ds}$ could be solved properly. If, however, the curve under consideration be a line of curvature, we have either

$$\lambda_1 = \text{constant} \quad \text{or} \quad \lambda_2 = \text{constant}.$$

When λ_1 is constant, $d\Lambda_1$ vanishes; and so $\delta = \lambda_1$. The length of the arc is given by

$$ds = \tfrac{1}{2}\left\{\frac{\lambda_2(\lambda_2-\delta)}{(\alpha-\lambda_2)(\beta-\lambda_2)(\gamma-\lambda_2)}\right\}^{\frac{1}{2}} d\lambda_2.$$

Similarly, when λ_2 is constant, $d\Lambda_2$ vanishes; and so $\delta = \lambda_2$. The length of the arc is given by

$$ds = \tfrac{1}{2}\left\{\frac{\lambda_1(\lambda_1-\delta)}{(\alpha-\lambda_1)(\beta-\lambda_1)(\gamma-\lambda_1)}\right\}^{\frac{1}{2}} d\lambda_1.$$

From the earlier investigation it appeared that the equation $pD =$ constant represents either a geodesic or a line of curvature; it consequently follows that *the proper equations of a geodesic are*

$$\left.\begin{aligned} \frac{\lambda_1 d\lambda_1}{2\sqrt{R\lambda_1}} + \frac{\lambda_2 d\lambda_2}{2\sqrt{R\lambda_2}} &= 0 \\ \frac{d\lambda_1}{2\sqrt{R\lambda_1}} + \frac{d\lambda_2}{2\sqrt{R\lambda_2}} &= du \\ -\frac{ds}{\lambda_1\lambda_2} &= du \end{aligned}\right\},$$

where $\quad R\lambda = -\lambda(\alpha-\lambda)(\beta-\lambda)(\gamma-\lambda)(\delta-\lambda),$

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* *Ges. Werke*, t. I., p. 262.

and λ_1, λ_2 are the (non-zero) roots of

$$\frac{x^2}{a-\theta}+\frac{y^2}{\beta-\theta}+\frac{z^2}{\gamma-\theta}=1.$$

6. When the given quadric is an *ellipsoid*, a, β, γ are all positive; take

$$a>\beta>\gamma>0.$$

Let λ_1 determine the confocal hyperboloid of two sheets, and λ_2 the confocal hyperboloid of one sheet; then we have

$$a>\lambda_1>\beta,\quad \beta>\lambda_2>\gamma.$$

Further, du must be real, and therefore both $R\lambda_1$ and $R\lambda_2$ must be positive. Taking account of the limits between which λ_1 and λ_2 must lie, we find that $R\lambda_1$ is positive if $\lambda_1>\delta$, and that $R\lambda_2$ is positive if $\lambda_2<\delta$; so that

$$\lambda_1>\delta>\lambda_2.$$

The only conditions other than these to which δ is subject are

$$a>\delta>\gamma.$$

They are covered by what precedes; hence the whole set of conditions is

$$a>\lambda_1>\left\{\begin{matrix}\beta\\ \delta\end{matrix}\right\}>\lambda_2>\gamma>0.$$

Three cases occur, according as

$$\text{(I.)}\quad \delta=\beta,$$

$$\text{(II.)}\quad \delta<\beta,$$

$$\text{(III.)}\quad \delta>\beta.$$

As regards the form of the curve, we have

$$p^2D^2=\frac{a\beta\gamma}{\delta}$$

In the first case, when $\delta=\beta$, we have

$$p^2D^2=a\gamma;$$

the geodesic passes through an umbilicus, and therefore also through the centrally opposite umbilicus.

In each of the other two cases, the geodesic touches a line of curvature. At any point on its course, we have

$$p^2 = \frac{\alpha\beta\gamma}{\lambda_1\lambda_2},$$

so that

$$\delta = \frac{\lambda_1\lambda_2}{D^2}.$$

When a geodesic touches a line of curvature on a hyperboloid of one sheet, D is the same at the point of contact as for the line of curvature, that is, $D^2 = \lambda_1$; and hence at that point

$$\delta = \lambda_2$$
$$< \beta.$$

Hence, in the second case, when $\delta < \beta$, the geodesic touches a line of curvature lying on the confocal one-sheeted hyperboloid; and it undulates between the two lines of curvature that constitute the complete intersection of the ellipsoid and the confocal quadric.

When a geodesic touches a line of curvature on a hyperboloid of two sheets, D is the same at the point of contact as for the line of curvature, that is, $D^2 = \lambda_2$; and hence at that point

$$\delta = \lambda_1$$
$$> \beta.$$

Hence, in the third case, when $\delta > \beta$, the geodesic touches a line of curvature lying on a confocal two-sheeted hyperboloid; and it undulates between the two lines of curvature that constitute the complete intersection of the ellipsoid and the confocal quadric.*

In the case of the oblate spheroid, for which $\alpha = \beta$, the first of the above classes gives rise to the meridians; the second of them gives rise to the non-meridianal geodesics, the course of which is well known; the third of them gives rise also to the meridians, as a limiting form.

Likewise for a prolate spheroid.

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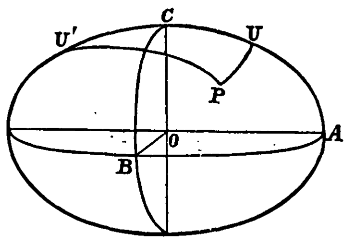
* *Cf.* Cayley, *Coll. Math. Papers*, Vol. VI., No. 425.

7. The differential relations of the geodesics can be replaced by expressions in terms of periodic functions.

(I.) In the first case, when $\delta = \beta$, the geodesics pass through the umbilici. As we take the lines of curvature from AB to UC, which lie on hyperboloids of one sheet, the quantity λ_2 increases; and as we



take the lines of curvature from OB to UA, which lie on hyperboloids of two sheets, the quantity λ_1 decreases. Hence at P, for the geodesic UP in the direction UP, we have

$$d\lambda_2 \text{ is negative,} \quad d\lambda_1 \text{ is positive.}$$

Also we take

$$\sqrt{R\lambda_1} = (\lambda_1 - \beta)\{\lambda_1(\alpha-\lambda_1)(\lambda_1-\gamma)\}^{\frac{1}{2}} = (\lambda_1-\beta)\sqrt{\Lambda_1},$$

$$\sqrt{R\lambda_2} = (\beta-\lambda_2)\{\lambda_2(\alpha-\lambda_2)(\lambda_2-\gamma)\}^{\frac{1}{2}} = (\beta-\lambda_2)\sqrt{\Lambda_2}.$$

Moreover at U we have $\lambda_1 = \beta$, $\lambda_2 = \beta$. Hence at P the equations of the geodesic UP in the direction UP are

$$\left.\begin{aligned} \int_\beta^{\lambda_1} \frac{\theta}{\theta-\beta}\frac{d\theta}{\sqrt{\Theta}} - \int_{\lambda_2}^{\beta} \frac{\theta}{\beta-\theta}\frac{d\theta}{\sqrt{\Theta}} &= 0 \\ \int_\beta^{\lambda_1} \frac{1}{\theta-\beta}\frac{d\theta}{\sqrt{\Theta}} - \int_{\lambda_2}^{\beta} \frac{1}{\beta-\theta}\frac{d\theta}{\sqrt{\Theta}} &= 2u \\ s &= \int_u^0 \frac{du}{\lambda_1\lambda_2} \end{aligned}\right\},$$

where u is chosen so as to vanish at U, and the arc s is measured from U.

The first two equations can be replaced by

$$\int_{\lambda_2}^{\lambda_1} \frac{\theta}{\theta-\beta}\frac{d\theta}{\sqrt{\Theta}} = 0,$$

$$2\beta u = -\int_{\lambda_2}^{\lambda_1} \frac{d\theta}{\sqrt{\Theta}},$$

where θ has continuous real values from λ_1 to λ_2, and in the former the principal value of the integral is to be taken. The first expresses the relation between λ_1 and λ_2 along the geodesic; for the explicit form of the relation, elliptic integrals of the third kind are necessary. In the second equation, the integral is elliptic of the first kind.

(II.) In the case when $\delta<\beta$ and the geodesic undulates between the two lines of curvature that are the complete intersection of the ellipsoid and a confocal hyperboloid of one sheet, the equations can be replaced by expressions involving hyperelliptic functions. We have

$$a>\lambda_1>\beta>\delta>\lambda_2>\gamma>0;$$

and we take

$$\left.\begin{aligned} a &= \int_\beta^{\lambda_1}+\int_\gamma^{\lambda_2}\frac{\theta\,d\theta}{2\sqrt{R\theta}} \\ u &= \int_\beta^{\lambda_1}+\int_\gamma^{\lambda_2}\frac{d\theta}{2\sqrt{R\theta}} \end{aligned}\right\},$$

where a is an arbitrary constant; it is unnecessary to associate an arbitrary constant with u. Now introduce two new quantities, viz.,

$$\left.\begin{aligned} a-\gamma u = u_1 &= \int_\beta^{\lambda_1}+\int_\gamma^{\lambda_2}\frac{\theta-\gamma}{2\sqrt{R\theta}}\,d\theta \\ a-\beta u = u_2 &= \int_\beta^{\lambda_1}+\int_\gamma^{\lambda_2}\frac{\theta-\beta}{2\sqrt{R\theta}}\,d\theta \end{aligned}\right\},$$

these quantities u_1 and u_2 being the arguments of the hyperelliptic functions in Weierstrass's theory.* We take

$$a_0,\ a_1,\ a_2,\ a_3,\ a_4 = a,\ \beta,\ \delta,\ \gamma,\ 0;$$

and then we have

$$\frac{x^2}{a}=\frac{(a-\lambda_1)(a-\lambda_2)}{(\beta-a)(\gamma-a)}=\frac{(a_0-\lambda_1)(a_0-\lambda_2)}{(a_1-a_0)(a_3-a_0)}=al_0^2(u_1,u_2),$$

$$\frac{y^2}{\beta}=\frac{(\beta-\lambda_1)(\beta-\lambda_2)}{(a-\beta)(\gamma-\beta)}=\frac{(a_1-\lambda_1)(a_1-\lambda_2)}{(a_0-a_1)(a_3-a_1)}=\beta\frac{\beta-\delta}{\beta-\gamma}al_1^2(u_1,u_2),$$

$$\frac{z^2}{\gamma}=\frac{(\gamma-\lambda_1)(\gamma-\lambda_2)}{(\gamma-a)(\gamma-\beta)}=\frac{(a_3-\lambda_1)(a_3-\lambda_2)}{(a_3-a_0)(a_3-a_1)}=\gamma\frac{\delta-\gamma}{\beta-\gamma}al_3^2(u_1,u_2).$$

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* *Ges. Werke*, t. I., pp. 133–152, pp. 297–355; the special case required is given by $n=2$.

Thus the equations of a geodesic are given by

$$\left.\begin{aligned} x &= \surd a \qquad al_0\,(a-\gamma u,\ a-\beta u) \\ y &= \beta\left(\frac{\beta-\delta}{\beta-\gamma}\right)^{\frac{1}{2}} al_1\,(a-\gamma u,\ a-\beta u) \\ z &= \gamma\left(\frac{\delta-\gamma}{\beta-\gamma}\right)^{\frac{1}{2}} al_3\,(a-\gamma u,\ a-\beta u) \end{aligned}\right\},$$

where a and δ are the arbitrary constants which can be determined by assigning any two points on the ellipsoid as points through which a geodesic is to be drawn and u is the parameter of the curve so drawn.

Again,
$$\begin{aligned} \frac{\lambda_1\lambda_2}{\beta\gamma} &= \frac{(a_4-\lambda_1)(a_4-\lambda_2)}{(a_4-a_1)(a_4-a_3)} \\ &= al_4^2\,(u_1,\ u_2) \\ &= 1+\frac{1}{a_1-a_4}\frac{\partial U}{\partial u_1}+\frac{1}{a_3-a_4}\frac{\partial U}{\partial u_2}, \end{aligned}$$

where U is the integral-function defined* by the equation

$$U = \int_\beta^{\lambda_1}+\int_\gamma^{\lambda_2}\frac{(\theta-\beta)(\theta-\gamma)}{2\sqrt{R\theta}}\,d\theta.$$

Thus $\qquad \lambda_1\lambda_2 = \beta\gamma+\gamma\frac{\partial U}{\partial u_1}+\beta\frac{\partial U}{\partial u_2}.$

But $\qquad dU = \frac{\partial U}{\partial u_1}du_1+\frac{\partial U}{\partial u_2}du_2,$ in general,

$$= -\left(\gamma\frac{\partial U}{\partial u_1}+\beta\frac{\partial U}{\partial u_2}\right)du, \text{ in the present case;}$$

and therefore

$$\begin{aligned} ds &= -\lambda_1\lambda_2\,du \\ &= -\beta\gamma\,du-\left(\gamma\frac{\partial U}{\partial u_1}+\beta\frac{\partial U}{\partial u_2}\right)du \\ &= dU-\beta\gamma\,du. \end{aligned}$$

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* Weierstrass, *l.c.*, pp. 337–346.

Consequently

$$s = \left[U - \beta\gamma u\right],$$

the right-hand side being taken between the values of u at two points on the geodesic, expresses the length of the arc between those points.

[*Added March* 16*th*, 1896.—The result can be obtained also as follows :—By the equations in § 4, we have

$$\frac{\dfrac{d\lambda_1}{2\sqrt{R\lambda_1}}}{\lambda_2} = \frac{\dfrac{d\lambda_2}{2\sqrt{R\lambda_2}}}{-\lambda_1} = \theta, \text{ say,}$$

so that $$du = (\lambda_2 - \lambda_1)\,\theta.$$

Now $$\begin{aligned} dU &= \frac{(\lambda_1-\beta)(\lambda_1-\gamma)}{2\sqrt{R\lambda_1}}\,d\lambda_1 + \frac{(\lambda_2-\beta)(\lambda_2-\gamma)}{2\sqrt{R\lambda_2}}\,d\lambda_2 \\ &= \theta\left\{\lambda_2(\lambda_1-\beta)(\lambda_1-\gamma) - \lambda_1(\lambda_2-\beta)(\lambda_2-\gamma)\right\} \\ &= \lambda_1\lambda_2(\lambda_1-\lambda_2)\,\theta + \beta\gamma(\lambda_2-\lambda_1)\,\theta \\ &= -\lambda_1\lambda_2\,du + \beta\gamma\,du \\ &= ds + \beta\gamma\,du, \end{aligned}$$

as before.]

(III.) In the case when $\delta > \beta$ and the geodesic undulates between the two lines of curvature that are the complete intersection of the ellipsoid and a confocal hyperboloid of two sheets, the result can similarly be expressed in terms of hyperelliptic functions. We now have

$$a > \lambda_1 > \delta > \beta > \lambda_2 > \gamma > 0,$$

and we take $$a,\ \delta,\ \beta,\ \gamma,\ 0 = a_0,\ a_1,\ a_2,\ a_3,\ a_4.$$

Then introducing

$$\left.\begin{aligned} u_1 &= \int_\delta^{\lambda_1} + \int_\gamma^{\lambda_2} \frac{\theta-\gamma}{2\sqrt{R\theta}}\,d\theta \\ u_2 &= \int_\delta^{\lambda_1} + \int_\gamma^{\lambda_2} \frac{\theta-\delta}{2\sqrt{R\theta}}\,d\theta \end{aligned}\right\},$$

so that $$u_1 = a - \gamma u, \quad u_2 = a - \delta u,$$

we easily find

$$\left.\begin{aligned} x &= \sqrt{a}\left(\frac{a-\delta}{a-\beta}\right)^{\frac{1}{2}} al_0(a-\gamma u,\ a-\delta u) \\ y &= \sqrt{\beta}\left(\frac{\delta-\beta}{a-\beta}\right)^{\frac{1}{2}} al_2(a-\gamma u,\ a-\delta u) \\ z &= \gamma al_3(a-\gamma u,\ a-\delta u) \end{aligned}\right\},$$

In these expressions a and δ are the two arbitrary constants; they can be determined by any two points through which the geodesic passes. And u is the current parameter of the geodesic.

To find the arc, we introduce the integral-function U, where

$$U = \int_{\epsilon}^{\lambda_1} + \int_{\gamma}^{\lambda_2} \frac{(\theta-\gamma)(\theta-\delta)}{2\sqrt{R\theta}}\, d\theta;$$

and then the arc between any two points is equal to

$$\left[U - \beta\gamma u\right],$$

between the limiting values of u that determine the two points.*

It has been assumed throughout that $a > \beta > \gamma$. Special cases arise when $a = \beta$, viz., an obl te spheroid, and when $\beta = \gamma$, viz., a prolate spheroid. The corresponding formulæ then belong to elliptic functions.†

8. If numerical approximations are desired, they can be obtained, as pointed out by Weierstrass in his paper already quoted, by using the double theta-functions. The Abelian functions, that occur in the preceding solution, are expressible as quotients of these theta-functions in forms substantially agreeing with results first given by Rosenhain;‡ and when once the parameters, being small quantities for a surface nearly spherical, are determined, expansions can be obtained to any degree of accuracy required.

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* For the umbilical geodesics, see a paper by Cayley, "On the Geodesics on an Ellipsoid," *Coll. Math. Papers*, Vol. VII., 478. For the general geodesics on an ellipsoid, the paper by Weierstrass, referred to in § 4, should be consulted; also two papers by Cayley, *Coll. Math. Papers*, Vol. VIII., 508, 511.

† For the case of an oblate spheroid, see a paper by the author, *Messenger of Mathematics*, Vol. XXV. (1896), pp. 81–124.

‡ "Mémoire sur les fonctions de deux variables et à quatre périodes," *Mém. des Savans Etr.*, t. XI., p. 361; the memoir is dated 1846.

9. When the given quadric is a *hyperboloid of one sheet*, we have

$$\alpha > \beta > 0 > \gamma.$$

The roots of the equation

$$\frac{x^2}{\alpha-\theta} + \frac{y^2}{\beta-\theta} + \frac{z^2}{\gamma-\theta} = 1$$

must correspond to an ellipsoid and a hyperboloid of two sheets. For the former, we have

$$\gamma > \lambda_2,$$

both of course being negative; for the latter, we have

$$\alpha > \lambda_1 > \beta.$$

In order to have real geodesics, both $R\lambda_1$ and $R\lambda_2$ must be positive. The former is positive if $\delta < \lambda_1$, the latter if $\delta > \lambda_2$; so that

$$\lambda_1 > \delta > \lambda_2.$$

Combining the inequalities, we have

$$\alpha > \lambda_1 > \left\{ \begin{matrix} \beta > 0 > \gamma \\ \delta \end{matrix} \right\} > \lambda_2.$$

There are seven cases, viz.,

(I.) $\delta > \beta > 0 > \gamma$,

(II.) $\delta = \beta > 0 > \gamma$,

(III.) $\beta > \delta > 0 > \gamma$,

(IV.) $\beta > \delta = 0 > \gamma$,

(V.) $\beta > 0 > \delta > \gamma$,

(VI.) $\beta > 0 > \delta = \gamma$,

(VII.) $\beta > 0 > \gamma > \delta$.

10. To discriminate these cases, we consider the configuration of the surface in the immediate vicinity of x, y, z, and compare it with the central section by a plane parallel to the tangent plane at the point. The generators are parallel to the asymptotes of the central section; the angles between the generators are bisected by the lines of curvature, which are parallel to the axes of the central section; and that angle between the generators in which the ellipsoidal line of curvature lies corresponds to that angle between the asymptotes in which the real part of the curve of the central section lies, say, the *internal* angle of the asymptotes.

Now, by § 4, we have

$$\frac{1}{D^2} = \frac{\delta}{\lambda_1 \lambda_2};$$

and in the present case λ_1 is positive, λ_2 is negative. Hence, when δ is positive, D^2 is negative; and the direction of the geodesic lies within the external angle of the generators. When δ is zero, D is infinite; and the direction of the geodesic is one of the generators. When δ is negative, D^2 is positive; and the direction of the geodesic lies within the internal angle of the generators.

If a geodesic can cross the principal section in the plane $z = 0$, we have there

$$\lambda_2 = \gamma.$$

Now, at any point,

$$-\frac{2}{\Gamma^{\frac{1}{2}}} \frac{dz}{ds} = \left(\frac{\gamma-\lambda_2}{\gamma-\lambda_1}\right)^{\frac{1}{2}} \frac{d\lambda_1}{ds} + \left(\frac{\gamma-\lambda_1}{\gamma-\lambda_2}\right)^{\frac{1}{2}} \frac{d\lambda_2}{ds},$$

and

$$\frac{d\lambda_1}{ds} = \frac{2\sqrt{R\lambda_1}}{\lambda_1(\lambda_1-\lambda_2)}, \quad \frac{d\lambda_2}{ds} = \frac{2\sqrt{R\lambda_2}}{\lambda_2(\lambda_2-\lambda_1)},$$

where the positive value has to be assigned to the real radicals $\sqrt{R\lambda_1}$ and $\sqrt{R\lambda_2}$, that is,

$$\sqrt{R\lambda_1} = \sqrt{\lambda_1(\alpha-\lambda_1)(\lambda_1-\beta)(\lambda_1-\gamma)(\lambda_1-\delta)},$$

$$\sqrt{R\lambda_2} = \sqrt{-\lambda_2(\alpha-\lambda_2)(\beta-\lambda_2)(\gamma-\lambda_2)(\delta-\lambda_2)}.$$

Substituting and then making $\lambda_2 = \gamma$, we have

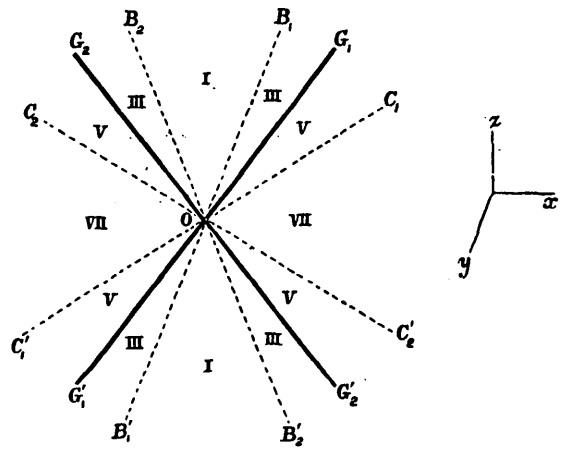
$$-\frac{dz}{ds} = \frac{\{\Gamma(\gamma-\lambda_1)\}^{\frac{1}{2}}}{\gamma(\gamma-\lambda_1)} \sqrt{-\gamma(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)}.$$

Now Γ is negative, as is also $\gamma-\lambda_1$; thus the first radical on the right-hand side is real. Again $-\gamma$, $\alpha-\gamma$, $\beta-\gamma$ are positive; hence, if $\delta > \gamma$, the value of $\frac{dz}{ds}$ is real. In this case, the geodesic crosses the principal section under consideration.

If $\delta = \gamma$, then $\frac{dz}{ds} = 0$ at the point; in this case the geodesic touches the principal section but does not cross it.

If $\delta < \gamma$, then $\frac{dz}{ds}$ is imaginary; that is, the geodesic cannot meet the principal section.

11. In the figure, G_1G_1' and G_2G_2' are the generators at the point O; they give the directions of the geodesics corresponding to $\delta = 0$. This is Case (IV.)



The lines B_1B_1' and B_2B_2' are lines equally inclined to the generators; they give the directions of the geodesics through O corresponding to $\delta = \beta$. This is Case (II.).

For any direction lying within the angles B_1OB_2 and $B_1'OB_2'$, we have $\delta > \beta$. Thus Case (I.) gives geodesics through O whose directions lie within one of the two regions marked (I.); one special line is the geodesic which touches the hyperboloidal line of curvature through O, the value of δ then being λ_1.

For any direction lying within one of the angles B_1OG_1, B_2OG_2, $B_2'OG_2'$, $B_1'OG_1'$, we have $\beta > \delta > 0$. Thus Case (III.) gives geodesics through O whose directions lie within one of the four regions marked (III.).

The lines C_1C_1' and C_2C_2' are lines equally inclined to the generators; they give the directions of the geodesics through O corresponding to $\delta = \gamma$. This is Case (VI.).

For any direction lying within one of the angles C_1OG_1, $C_1'OG_1'$, C_2OG_2, $C_2'OG_2'$, we have $O > \delta > \gamma$. Thus Case (V.) gives geodesics through O whose directions lie within one of the four regions marked (V.).

For any direction lying within the angles C_1OC_2' and C_2OC_1', we have $\delta < \gamma$. Thus Case (VII.) gives geodesics through O whose directions lie within one of the regions marked (VII.); one special line is the geodesic which touches the ellipsoidal line of curvature through O, the value of δ then being λ_2.

Geodesics through O whose directions lie within (but not on the boundary of) either of the angles C_1OC_2 and $C_1'OC_2'$ cross the principal elliptic section of the surface when they are continued.

The two geodesics through O whose directions are the lines C_1C_1' and C_2C_2' at that point touch, but do not cross, the principal elliptic section.

Geodesics through O whose directions lie within (but not on the boundary of) either of the angles C_1OC_2' and C_2OC_2' do not meet the principal elliptic section of the surface. Each of them touches an ellipsoidal line of curvature, determined by the value of δ; and extends, on either side of this point of contact, towards infinity away from the principal elliptic section. By this extension of the geodesic is implied a curve at every part of which the characteristic geodesic property is possessed; but the length of the arc of this curve between any two points of it is not necessarily the shortest surface-distance between the two points.

12. The course of the geodesic can be indicated by expressing the coordinates of any point on it in terms of a single parameter. The expressions in Cases (I.), (III.), (V.), (VII.) require hyper-elliptic functions as in two of the cases on the surface of the ellipsoid; in Cases (II.) and (VI.), elliptic functions and elliptic integrals of the third kind occur; in Case (IV.), the expressions are algebraical.

13. When the given quadric is a *hyperboloid of two sheets*, we have

$$\alpha > 0 > \beta > \gamma.$$

The roots, other than zero, of the equation

$$\frac{x^2}{\alpha-\theta}+\frac{y^2}{\beta-\theta}+\frac{z^2}{\gamma-\theta}=1$$

must correspond to an ellipsoid and a hyperboloid of one sheet. For the former, we have

$$\gamma > \lambda_2,$$

both of course being negative; for the latter, we have

$$\beta > \lambda_1 > \gamma.$$

In order to have real geodesics, we must have $R\lambda_1$ positive, a condition which is satisfied if $\delta < \lambda_1$; and we must have $R\lambda_2$ positive, a condition which is satisfied if $\delta > \lambda_2$, so that

$$\lambda_1 > \delta > \lambda_2.$$

Combining these inequalities, we have

$$\alpha > 0 > \beta > \lambda_1 > \begin{Bmatrix} \gamma \\ \delta \end{Bmatrix} > \lambda_2.$$

There are three cases, viz.,

$$\text{(I.)}\ \gamma = \delta,$$

$$\text{(II.)}\ \gamma > \delta,$$

$$\text{(III.)}\ \gamma < \delta.$$

14. The cases are similar to those that occur in the ellipsoid.

The first represents a geodesic passing through an umbilicus, but, with a single exception, not through the other umbilicus on the same sheet; beyond these points, it extends towards infinity.

The second represents a geodesic touching one ellipsoidal line of curvature and extending towards infinity in both directions.

The third represents a geodesic touching one line of curvature that lies upon a confocal hyperboloid of one sheet and extending towards infinity in both directions.

The last two require hyper-elliptic functions for the explicit expression of the variables along the course of the curve; the first, for the same purpose, requires elliptic integrals of the third kind.

15. It is unnecessary to consider, in any detail, geodesics on a *cone* or *cylinder;* their characteristic equation for such a surface can be deduced from the property that, when a developable surface is developed, the geodesic gives rise to a straight line on the developed surface. Thus, for instance, on a cone we should have

$$r \sin \phi = \text{constant};$$

where the constant is the parameter of the geodesic, r is the distance of any point on it from the vertex of the cone, and ϕ is the angle between the direction of the geodesic at the point and the generator through the point.

Non-Central Quadrics. §§ 16–22.

16. When the quadric is paraboloidal, its equation can be taken in the form

$$\frac{y^2}{a} + \frac{z^2}{c} = 4x.$$

When the paraboloid is *elliptic*, we have

$$a > c > 0;$$

when it is *hyperbolic*, we have

$$a > 0 > c.$$

The confocal paraboloids are given by

$$\frac{y^2}{a-k} + \frac{z^2}{c-k} = 4(x-k),$$

a cubic equation in k for each point x, y, z. One root is zero; let the others be k_1 and k_2, of which k_1 is assumed the greater. Then 0, k_1, k_2 are the roots of

$$4(a-k)(c-k)(x-k) - y^2(c-k) - z^2(a-k) = 0.$$

It is easily seen* that the roots are separated by ∞, a, c, $-\infty$. Hence in the case of the *elliptic paraboloid* we have

$$\infty > k_1 > a > k_2 > c > 0;$$

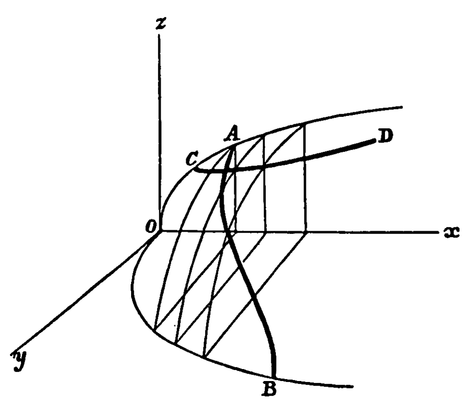
k_1 determines an elliptic paraboloid and k_2 a hyperbolic paraboloid. And in the case of the *hyperbolic paraboloid*, we have

$$\infty > k_1 > a > 0 > c > k_2;$$

k_1 and k_2 determine elliptic paraboloids.

17. The intersections of the confocal surfaces are lines of curvature on each of them.

Consider first the elliptic paraboloid.



* Frost's *Solid Geometry*, p. 138.

Its intersection with the confocal elliptic paraboloid is a curve one quarter of which is AB; when this curve is orthogonally projected on the plane of yz, it becomes the ellipse

$$\frac{y^2}{a(k_1-a)}+\frac{z^2}{c(k_1-c)}=4.$$

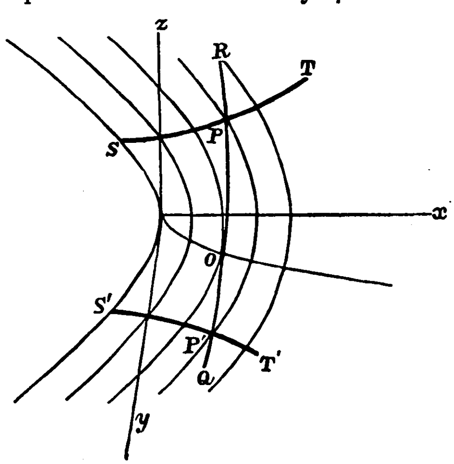
This curve is the whole of the real intersection with the confocal elliptic paraboloid.

The intersection with the confocal hyperbolic paraboloid consists of two curves. One half of one of them is CD, the other half of it being on the negative side of the plane zx; and the other curve is the reflexion of this curve in the plane of xy. When these curves are orthogonally projected on the plane of yz, they become the two branches of the hyperbola

$$\frac{y^2}{a(a-k_2)}-\frac{z^2}{c(k_2-c)}=4.$$

The two real curves constitute the whole intersection with the confocal hyperbolic paraboloid.

Now consider the hyperbolic paraboloid. Its intersection with the confocal elliptic paraboloid determined by k_1 consists of two curves;



one is $QOPR$ ..., and the other is the reflexion of this curve in the plane of xz. When these curves are orthogonally projected on the plane of yz, they become the two branches of the hyperbola

$$\frac{y^2}{a(k_1-a)}+\frac{z^2}{c(k_1-c)}=4.$$

These two (real) curves constitute the whole intersection.

The intersection with the confocal elliptic paraboloid determined by k_3 consists of two curves; $SPT\ldots$, $S'P'T'\ldots$ are halves of them, the other halves being their reflexion in the plane of xz. When these curves are orthogonally projected on the plane of yz, they become the two branches of the hyperbola

$$\frac{y^2}{a(a-k_3)}+\frac{z^2}{c(c-k_3)}=-4.$$

These two (real) curves constitute the whole intersection.

18. Take any point on a paraboloid and consider the geodesics through the point. If l, m, n denote the direction of the curve there, if p be the perpendicular from the vertex upon the tangent plane at the point, and if D denote the length of the chord through the vertex parallel to the geodesic direction, then* we have

$$\frac{lx^3}{p^2D}$$

constant along a curve. And, by an investigation similar to that contained in §§ 2 and 3, it can be proved—the analysis is not reproduced here—that the equation

$$\frac{d}{ds}\left(\frac{lx^3}{p^2D}\right)=0$$

determines upon the paraboloid either a geodesic or one of the lines of curvature through the point. If then the quantities k_1 and k_3 be introduced, the lines of curvature are given by

$$\frac{dk_1}{ds}=0,\quad \frac{dk_3}{ds}=0;$$

the equation

$$\frac{d}{ds}\left(\frac{lx^3}{p^2D}\right)=0,$$

or

$$\frac{lx^3}{p^2D}=\text{constant},$$

when transformed, will then represent a proper geodesic. Now

$$\frac{x^2}{p^2}=1+\frac{y^2}{4a^2}+\frac{z^2}{4c^2},$$

$$4\frac{l}{D}=\frac{m^2}{a}+\frac{n^2}{c};$$

* Frost's *Solid Geometry*, p. 320.

hence the equation characteristic of geodesics is

$$\left(\frac{m^2}{a}+\frac{n^2}{c}\right)\left(1+\frac{y^2}{4a^2}+\frac{z^2}{4c^2}\right) = \text{constant}$$

$$= \frac{1}{b}, \text{ say.}$$

Further, it is only upon the elliptic paraboloid that the umbilici are real. They are given by

$$x_1 = a-c, \quad y_1 = 0, \quad z_1 = 2\sqrt{c\,(a-c)};$$

also, for any direction in the tangent plane at an umbilicus, we have

$$-2l+\frac{nz_1}{c} = 0,$$

so that

$$l = n\sqrt{\frac{a-c}{c}}.$$

Thus

$$\frac{l^2}{a-c} = \frac{n^2}{c} = \frac{1-m^2}{a},$$

so that

$$\frac{m^2}{a}+\frac{n^2}{c} = \frac{1}{a}.$$

And

$$1+\frac{y_1}{4a^2}+\frac{z_1^2}{4c^2} = \frac{a}{c};$$

hence for a geodesic through an umbilicus the constant is

$$\frac{a}{c}\,\frac{1}{a} = \frac{1}{c}.$$

If therefore $b = c$, the geodesic passes through an umbilicus.

19. To use the parameters of the confocal paraboloids, we have

$$\frac{y^2}{a-k}+\frac{z^2}{c-k}-4\,(x-k) = 4\,\frac{k\,(k-k_1)(k-k_2)}{(a-k)(c-k)};$$

so that

$$y^2 = 4\,\frac{a\,(a-k_1)(a-k_2)}{c-a},$$

$$z^2 = 4\,\frac{c\,(c-k_1)(c-k_2)}{a-c};$$

and then $$x = \frac{y^2}{4a} + \frac{z^2}{4c}$$

$$= k_1 + k_2 - a - c.$$

With these values, we have

$$1 + \frac{y^2}{4a^2} + \frac{z^2}{4c^2} = \frac{k_1 k_2}{ac},$$

so that the equation of the geodesic is

$$\frac{m^2}{a} + \frac{n^2}{c} = \frac{ac}{b}\,\frac{1}{k_1 k_2} = \frac{f}{k_1 k_2},$$

where $$f = \frac{ac}{b};$$

and $f = a$ for a geodesic that passes through an umbilicus.

Now $l, m, n, = \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, respectively; thus

$$l = \frac{dk_1}{ds} + \frac{dk_2}{ds},$$

$$-m = \sqrt{\frac{a}{c-a}} \left\{ \sqrt{\frac{a-k_2}{a-k_1}}\,\frac{dk_1}{ds} + \sqrt{\frac{a-k_1}{a-k_2}}\,\frac{dk_2}{ds} \right\},$$

$$-n = \sqrt{\frac{c}{a-c}} \left\{ \sqrt{\frac{c-k_2}{c-k_1}}\,\frac{dk_1}{ds} + \sqrt{\frac{c-k_1}{c-k_2}}\,\frac{dk_2}{ds} \right\}.$$

Substituting these values in

$$l^2 + m^2 + n^2 = 1,$$

we find $$\frac{k_1(k_1-k_2)}{(a-k_1)(c-k_1)} \left(\frac{dk_1}{ds}\right)^2 + \frac{k_2(k_2-k_1)}{(a-k_2)(c-k_2)} \left(\frac{dk_2}{ds}\right)^2 = 1;$$

and substituting them in

$$\frac{m^2}{a} + \frac{n^2}{c} = \frac{f}{k_1 k_2},$$

we find $$\frac{(k_1-k_2)}{(a-k_1)(c-k_1)} \left(\frac{dk_1}{ds}\right)^2 + \frac{(k_2-k_1)}{(a-k_2)(c-k_2)} \left(\frac{dk_2}{ds}\right)^2 = \frac{f}{k_1 k}.$$

Let $$\left.\begin{aligned} K_1 &= k_1(k_1-a)(k_1-c)(k_1-f) \\ K_2 &= k_2(k_2-a)(k_2-c)(k_2-f) \end{aligned}\right\};$$

then, when these equations are solved for $\left(\frac{dk_1}{ds}\right)^2$ and $\left(\frac{dk_2}{ds}\right)^2$, we have

$$\left.\begin{aligned} k_1(k_1-k_2)\frac{dk_1}{ds} &= \sqrt{K_1} \\ k_2(k_2-k_1)\frac{dk_2}{ds} &= \sqrt{K_2} \end{aligned}\right\}$$

Hence the equations of a geodesic upon a paraboloid are

$$\left.\begin{aligned} \frac{k_1\,dk_1}{\sqrt{K_1}}+\frac{k_2\,dk_2}{\sqrt{K_2}} &= 0 \\ \frac{dk_1}{\sqrt{K_1}}+\frac{dk}{\sqrt{K_2}} &= du \\ -\frac{ds}{k_1k_2} &= du \end{aligned}\right\},$$

which correspond in form to those obtained in § 5 for a central quadric.

It would have been possible to deduce these results from the results in the case of a central quadric by changing the origin to a vertex of the latter and then passing to the limiting case, in which two of the semi-axes are made to increase without limit subject to the customary conditions.

20. In the case of the *elliptic paraboloid*, we have

$$k_1 > a > k_2 > c > 0.$$

Hence, in order that the geodesics may be real, we must have

$$k_1 > f,$$

$$k_2 < f,$$

that is, $$k_1 > f > k_2;$$

and therefore the aggregate of conditions is

$$k_1 > \binom{a}{f} > k_2 > c > 0.$$

There are therefore three distinct classes to consider, viz.,

(I.) $f = a$,

(II.) $f < a$,

(III.) $f > a$.

They correspond to the three classes in the case of an unruled central quadric.

For the first of these classes, we have $f=a$; the geodesic passes through an umbilicus (but not through the other umbilicus) in the finite part of the surface.

To discriminate between the other classes, a simple method is to trace the course of a geodesic through the variations of k_1 and k_2. We have

$$\frac{k_1 dk_1}{\sqrt{K_1}} + \frac{k_2 dk_2}{\sqrt{K_2}} = 0,$$

$$\frac{dk_1}{\sqrt{K_1}} + \frac{dk_2}{\sqrt{K_2}} = du,$$

and therefore $\frac{dk_1}{du} = \frac{k_2}{k_2-k_1}\sqrt{K_1}, \quad \frac{dk_2}{du} = \frac{k_1}{k_1-k_2}\sqrt{K_2}.$

Thus k_1, for finite values of k_1, can be a maximum or a minimum, only when $K_1 = 0$; and, for ll other values, K_1 must be positive. The only possible roots of K_1 are

$$k_1 = a, \quad k_1 = f;$$

and, for values of k_1 that are not roots,

$$(k_1-a)(k_1-f)$$

must be positive.

Hence when $f>a$, the only possible root is $k_1 = f$; and all other admissible values of k_1 must be greater than f. When $f<a$, the only possible root is $k_1 = a$; and all other admissible values of k_1 must be greater than a.

Again, k_2 can be a maximum or a minimum only when $K_2 = 0$; and, for all other values, K_2 must be positive. The only possible roots of K_2 are

$$k_2 = c, \quad k_2 = f;$$

and, for values of k_2 that are not roots,

$$(k_2-c)(k_2-f)$$

must be negative.

Hence when $f>a$, the only possible root is $k_2 = c$; all other admissible values of k_2 must lie between c and a. When $f<a$, both $k_2 = c$, $k_2 = f$ are possible roots; all other admissible values of k_2 lie between c and a.

Moreover, $k_2 = c$ refers to the (confocal) parabola in the plane $z=0$, lines G_1G_1' and G_2G_2' are the generators through the point P; these give the geodesics corresponding to Case (IV.).

The lines $A_1'A_1$, A_2A_2' give directions through P on the surface that determine the geodesics corresponding to Case (II.).

The lines C_1C_1', C_2C_2' give directions through P on the surface that determine the geodesics corresponding to Case (VI.).

Every geodesic through P belonging to Class (I.) has its direction at P lying within (but not on the boundary of) one of the angles A_1PA_2, $A_1'PA_2'$.

Every geodesic through P belonging to Class (III.) has its direction at P lying within (but not on the boundary of) one of the angles A_1PG_1, A_2PG_2, $A_1'PG_1'$, $A_2'PG_2'$.

Every geodesic through P belonging to Class (V.) has its direction at P lying within (but not on the boundary of) one of the angles C_1PG_1, G_2PC_2, $C_1'PG_1'$, $G_2'PC_2'$.

Every geodesic through P belonging to Class (VII.) has its direction at P lying within (but not on the boundary of) one of the angles C_1PC_2', C_2PC_1'.

Every geodesic through P that has its direction at P lying within (but not on the boundary of) one of the angles C_1PC_2, $C_1'PC_2'$ will, when produced, cut and cross the principal section of the surface by the plane $z = 0$.

The two geodesics through P having C_1PC_1' and C_2PC_2' as their directions through P will, when produced, touch, but not cross, this principal section of the surface.

And, lastly, no geodesic through P having its direction at P lying within (but not on the boundary of) one of the angles C_1PC_2', C_2PC_1' can, however far it may be produced, meet this principal section of the surface.

The differential equations of Classes (I.), (III.), (V.), (VII.) require the introduction of elliptic integrals for their integration; those belonging to Classes (II.) and (III.) can be integrated by logarithmic and circular functions; and those belonging to Class (IV.) can be integrated by algebraical functions.