

Hence, finally, in (75), (76), or (77), P and P' are the same functions of the unaccented and accented second and higher derivatives respectively.

16. Any discussion of the multiplicity of pure reciprocants P , whose existence is here conditioned, must find place in a possible future communication. The investigation will naturally be on the lines, and by aid of the results, of my papers on pure cyclicants. I here note only the first specimen,

$$(x_{20}y_{11} - x_{11}y_{20})(x_{11}y_{02} - x_{02}y_{11}) - (x_{20}y_{02} - x_{02}y_{20})^2 \dots\dots\dots(90),$$

the eliminant of the first pair of quantics in (88). It may also be written

$$4(x_{20}x_{02} - \frac{1}{4}x_{11}^2)(y_{20}y_{02} - \frac{1}{4}y_{11}^2) - (x_{20}y_{02} + x_{02}y_{20} - \frac{1}{2}x_{11}y_{11})^2 \dots\dots\dots(91),$$

in which form its connexion with the first pure cyclicant $x_{20}x_{02} - \frac{1}{4}x_{11}^2$ is exhibited.

On Newton's Classification of Cubic Curves.

By MR. W. W. ROUSE BALL.

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CONTENTS.	Articles.
I. <i>Subject-matter of Newton's Tract</i>	1-5
II. <i>Reduction of Equation to One of Four Canonical Forms</i> ...	7-14
(i.) Newton's Original Method (Art. 10).	
(ii.) Newton's Second Method (Art. 11).	
(iii.) Newton's Third Method as published (Arts. 13, 14).	
III. <i>Classification of Cubics into 14 Genera and 78 Species</i> ...	15-38
IV. <i>Discrimination of Genus from General Equation</i>	39-43
V. <i>Generation of Cubics by Shadows</i>	44-53
VI. <i>Bibliography and Subsequent History of the Subject</i>	54-81
Appendix I. <i>Extracts from Newton's MS. entitled "Fragments"</i> ..	83-99
Appendix II. <i>Extracts from Newton's earliest holograph MS.</i> ...	100-107

I. *Subject-matter of Newton's Tract.*

1. The *Enumeratio linearum tertii ordinis*, first published by Newton in 1704 as an appendix to his *Opticks*, contains the earliest attempt to apply the methods of analytical geometry to an exhaustive classification of cubic curves in a manner analogous to that in which conics are divided into three species. I propose to give an outline of Newton's argument so far as it is concerned with cubics, to compare it with his earlier treatment of the subject as indicated by manuscripts in the Portsmouth collection, and to conclude with a few notes on the history and bibliography of the subject.

2. The essay on cubic curves is the first appendix to the *Opticks*; the second appendix being the tract *De Quadratura Curvarum*. In the advertisement, which is prefixed to the book, Newton explains the reason of his appending to a work on optics propositions on the quadrature of curves and on the subject of cubics by saying: "Some Years ago I lent out a Manuscript containing such Theorems, and having since met with some Things copied out of it, I have on this Occasion made it publick. . . . And I have joined with it another small Tract concerning the Curvilinear Figures of the Second Kind, which was also written many Years ago, and made known to some Friends, who have solicited the making it publick."

It is not known at what time the earliest of his manuscripts on cubic curves was written, but there is some reason for thinking that it was before 1676. It is probable that the manuscript from which the edition was printed was written in or after 1695.

3. The essay as published is a catalogue of results, and it is possible that it was issued rather with the view of establishing the priority of Newton's investigations than as an exposition of the subject. No proofs of the propositions are given, and such explanatory remarks as are occasionally made with a view of showing how the results may be demonstrated are very concise. In fact, the tract occupies only twenty-four pages.

4. Newton divides cubics into fourteen genera. These genera comprise seventy-two species, to which we ought to add six additional ones which are left out from his enumeration. In the tract, a diagram of a curve of every species is carefully drawn, and these diagrams (except for the six species omitted) present to the eye the figures of all the possible forms of cubic curves, when degenerate forms are not taken into account.

The method of classification is uniform, simple, and intelligible to any one acquainted with the elements of analytical geometry and of the theory of equations. These are great merits. Moreover, the arrangement into genera is not only easy, but leads at once to the characteristic properties of the curves. Thus, apart from its historical interest as being the earliest systematic discussion of a group of curves of a degree higher than the second, the essay is well worthy of careful study.

5. The tract is divided into seven sections. In the first, which is on the orders of lines, Newton explains that the orders of lines are measured by the degrees of their respective equations; but (since he does not reckon a straight line as a curve) he describes conics either as lines of the second order, or as curves of the first genus. Similarly, cubics are described either as lines of the third order or as curves of the second genus.

In the second section he treats of properties of cubics analogous to those of conics.

In the third section he shows that the equation of a cubic always can be written in one of four canonical forms: namely,

$$xy^2 + ey = ax^3 + bx^2 + cx + d \dots\dots\dots(i.),$$

$$xy = ax^3 + bx^2 + cx + d \dots\dots\dots(ii.),$$

$$y^2 = ax^3 + bx^2 + cx + d \dots\dots\dots(iii.),$$

or $y = ax^3 + bx^2 + cx + d \dots\dots\dots(iv.).$

These equations are hereafter alluded to as forms (i.), (ii.), (iii.), and (iv.). In this section he also describes his nomenclature. He designates the various classes of curves discussed as hyperbolas and parabolas, and distinguishes them from the conic sections by prefixing descriptive adjectives. This terminology, though it still remains in use, is neither clear nor convenient.

In the fourth section he discusses the nature of the curves indicated by these several equations, and he divides the curves into species by considering the maximum and minimum values of x .

In the fifth section he enunciates the remarkable proposition that, just as the shadow of a circle cast by a luminous point gives rise to all the conics, so the shadow cast by a luminous point of the curves represented by the equation $y^2 = ax^3 + bx^2 + cx + d$ gives rise to all

the cubics. He also refers to double points, both in the plane and at infinity.

In the sixth section he treats of the description of curves, especially of cubics satisfying given conditions.

In the seventh and last section he deals with the graphical solution of problems by the use of curves.

6. The subject of this paper is an account of the earlier investigations of Newton on cubics, as contained in his unpublished manuscripts and memoranda; a description of the methods used in the third, fourth, and fifth sections of the tract; the results arrived at; and the subsequent history of the subject, so far as it is concerned with Newton's method.

II. *Reduction of Cubic Curves to One of Four Canonical Forms.*

7. The first problem in the classification of cubic curves is the reduction of their equations to four canonical forms. To effect this, Newton originally (in the first rough draft of the essay, which is described later) changed the origin and axes, and he has left two independent proofs of this method. In the tract as published, he solved this problem by a process which depended on finding the diameter conjugate to an asymptote.

8. The first of these methods is by transformation of coordinates. Before describing the details of the procedure adopted by Newton, I may, for convenience of future reference, indicate briefly the necessary steps. The general equation of a cubic curve in cartesian coordinates (which for simplicity we will suppose to be rectangular) is

$$u_3 + u_2 + u_1 + u_0 = 0.$$

The cubic equation $u_3 = 0$ determines the directions in which the curve is cut by the line at infinity. At least one root of this cubic equation is real. If there is only one real root, the line determined by that root is taken as the axis of y ; similarly, if there are three real roots which are not all equal, the line determined by that root which is not equal to the other two is taken as the axis of y . In either of these cases x will be a factor of u_3 , but x^3 will not be a factor of u_3 ; hence u_3 will take the form $x(Ax^2 + Bxy + Cy^2)$, where $C \neq 0$. In the case where the three roots of $u_3 = 0$ are all equal, the direction given by the roots is taken as the axis of y , and u_3 will then take the

form Ax^3 . Thus the equation of the curve can be always written in the form

$$x(Ax^3 + Bxy + Cy^3) + Ex^3 + Fxy + Gy^3 + Hx + Ky + L = 0,$$

where either $C \neq 0$, or $B = 0$ and $C = 0$.

First, suppose $C \neq 0$; then, if the origin is changed to a point (p, q) , we can choose p and q to make the coefficient of xy (namely, $2Bp + 2Cq + F$), and the coefficient of y^3 (namely, $Cp + G$) vanish; next, if the axis of x is turned through an angle $\arctan(-B/2C)$, and the axis of y is kept unaltered, the coefficient of x^3y will vanish, and the equation takes the form (i.).

Next, suppose $B = 0$ and $C = 0$; then we have three cases:—(a) If $G \neq 0$, we can, by a transformation similar to that described in the last paragraph, make the coefficients of y and of xy vanish, and the equation takes the form (iii.). (β) But, if $G = 0$ and $F \neq 0$, we can, by changing the origin to $(-K/F, 0)$, express the equation in the form (ii.). (γ) Lastly, if $G = 0$ and $F = 0$, the equation takes the form (iv.).

9. Newton's original method is substantially the same as that given above; but the algebraic analysis involved is more complicated, since, instead of first taking the axis of y parallel to an asymptote, then changing the origin, and finally turning the axis of x through some convenient angle, he takes the general equation of the third degree, and changes the origin and the direction of the axis in one step.

I had intended at one time to print his manuscript as it stands, but, on reading it again carefully, I think that the following summary of his method brings out sufficiently the points of interest in it.

10. After a brief description of the meaning of coordinates, and their use in determining curves by means of equations, he says that the most general equation of a cubic curve referred to rectangular axes is

$$av^3 + bv^2v + cz^2v + dz^3 + ev^3 + fzv + gz^2 + hv + kz + l = 0,$$

(z, v) being the coordinates of a point on the curve. He then changes the origin to a point (r, s) , and takes as axes of ξ and η lines making angles α and β with the axis of z . He thus obtains two relations equivalent to the following:—

$$z = r + \xi \cos \alpha + \eta \cos \beta, \quad v = s + \xi \sin \alpha + \eta \sin \beta \dots\dots\dots(a).$$

[As a matter of fact, he does not use the trigonometrical ratios, but constructs triangles having one side equal to the unit of length, and having one angle equal to α or β , as the case may be. It is only to save circumlocution that I here use the trigonometrical functions.] But, instead of employing these values of z and v , he puts

$$\xi \cos \alpha = x \quad \text{and} \quad \eta \sin \beta = y,$$

and thus obtains

$$z = r + x + y \cot \beta = (\text{say}) r + x + qy,$$

and

$$v = s + x \tan \alpha + y = (\text{say}) s + px + y.$$

This is, he says, more convenient, because he can give p and q any values that he likes, while, if he used the formulæ (a), he would have to consider whether α and β were so chosen as to make each of the ratios $\sin \alpha$, $\sin \beta$, $\cos \alpha$, and $\cos \beta$ numerically less than unity. Moreover, he remarks, that, if he knows the relation between x and y , he can at any time deduce the corresponding relation between ξ and η , which is the equation of the curve; and the form of the equation will not be altered by the change.

Next, he substitutes these values of z and v in the given equation and obtains an equation in x and y which has no less than 84 terms. He takes for q one of the roots of

$$a + bq + cq^3 + dq^3 = 0,$$

and then in terms of this root he determines values of p , r , and s (and the expressions for them are neither short nor simple) which make certain other coefficients vanish. He thus obtains the four canonical forms.

11. The algebraic analysis above described involves considerable labour, but the details of the work are not given in the manuscript. Newton's second proof by transformation of axes is much more ingenious. The analysis is printed (I believe for the first time) in Appendix I., Arts. 89-95, and is very simple. The following is a summary of it:—

Suppose that the given equation of a cubic (referred to $A\xi$ and $A\eta$ as coordinate axes) is

$$a\eta^3 + b\xi\eta^2 + c\xi^2\eta + d\xi^3 + e\eta^2 + f\xi\eta + g\xi^2 + h\eta + k\xi + l = 0,$$

where

$$AB = \xi \quad \text{and} \quad BC = \eta.$$

Therefore $x' = nx''$ and $y' = y'' - DE = y'' - \frac{c'}{2b'}nx''$.

Substitute these values of x' and y' in the equation at the end of the last paragraph, and we obtain an equation (in x'' and y'' as current coordinates) of the form

$$b''x''y''^2 + d''x''^3 + e''y''^3 + f''x''y'' + g''x''^3 + h''y'' + k''x'' + l'' = 0.$$

Next, change the origin to $\left(-\frac{e''}{b''}, -\frac{f''}{2b''}\right)$, that is, on EA produced take a point H , so that

$$AH : 1 = e'' : b'',$$

and on CE produced take a point K , so that

$$EK : 1 = f'' : 2b'';$$

and complete the parallelogram $EHGK$. Hence, if

$$GK = x \text{ and } KO = y,$$

we have $x'' = x - \frac{e''}{b''}$, and $y'' = y - \frac{f''}{2b''}$.

Substitute these values of x'' and y'' , and the equation reduces to the form numbered (i.) in Art. 5.

(β) If $b' = 0$, then, if $c' \neq 0$, take (with a notation analogous to the above)

$$DE : AD = d' : c', \quad AH : 1 = f'' : 2c'', \quad EK : 1 = g'' : c'',$$

and the equation, when referred to GH and GK as axes of x and y , reduces to the form (i.).

(γ) If $b' = 0$ and $c' = 0$, then, if $e' \neq 0$, take

$$DE : AD = f' : 2e', \quad AH : 1 = g'' : 3d'', \quad EK : 1 = h'' : 2e'',$$

and the equation, when referred to GK and GH as axes of x and y , reduces to the form (iii.).

If, however, $b' = 0$, $c' = 0$, and $e' = 0$, we must slightly alter the

above construction. On DA produced take a point H (Fig. 2), defined as mentioned below. On CD produced take points E and K ,

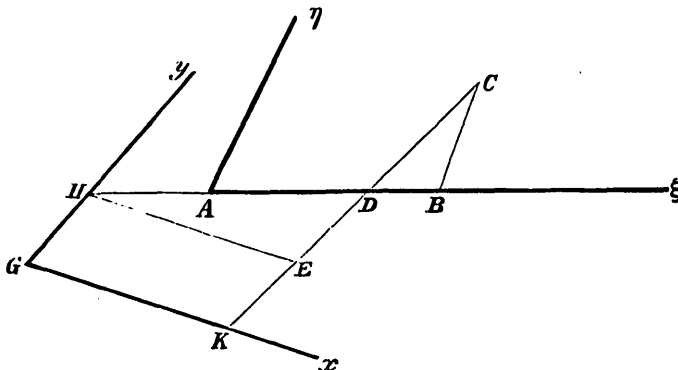


FIG. 2.

defined as mentioned below. Complete the parallelogram $EHKG$, and refer the equation to GK and GH as axes of x and y .

(δ) If $b' = 0$, $c' = 0$, $e' = 0$, then, if $f' \neq 0$, take

$$AH : 1 = h' : f', \quad DE : HD = g'' : f'', \quad EK : 1 = k'' : f'',$$

and the equation reduces to the form (ii.).

(ϵ) Lastly, if $b' = 0$, $c' = 0$, $e' = 0$, and $f' = 0$, take

$$AH : 1 = g' : 3d', \quad DE : HD = k'' : h'', \quad EK : 1 = l'' : h'',$$

and the equation reduces to the form (iv.).

In the cases (γ), (δ), and (ϵ) the equation is expressed finally in a somewhat simpler form than is necessary to show merely that it is included in one of the four canonical forms given in Art. 5.

12. Another method of reduction is given in Stirling's *Lineae Tertii Ordinis Newtonianae* (Prop. xv., p. 83). There is some reason for thinking that Stirling consulted Newton before publishing this work, and therefore I refer to it here in passing.

13. In the later manuscript copies of the tract, and as ultimately published, the reduction to the canonical forms was effected by a quasi-geometrical proof. Briefly sketched out, it is as follows. Draw a series of chords parallel to the real asymptote. If each chord cuts the curve in two points, the locus of their middle points will be a

hyperbola, and if the asymptotes of this hyperbola are taken as axes of coordinates, the equation takes the form (i.). But if each chord only meets the curve in one point, then, taking the asymptote as the axis of y , the equation takes the form (ii.); in fact, the asymptote is a tangent at a point of inflexion at infinity. Next, if the cubic is touched by the line at infinity (*si crura illa opposita parabolici sint generis*), then, if any chord meets the parabolic branch in two points, the locus of the point midway between them will be a straight line, and if this line is taken as the axis of x and a line parallel to the chord as the axis of y , the equation takes the form (iii.). But if the chord meets the curve in only one point, the equation similarly takes the form (iv.).

14. This may be clearer if it is expressed in analytical language. If a line parallel to the real asymptote is taken as the axis of y , in the manner explained in Art. 8, the equation takes the form

$$x(Ax^2 + Bxy + Cy^2) + Ex^3 + Fxy + Gy^2 + Hx + Ky + L = 0,$$

where either $C \neq 0$, or $B = 0$ and $C = 0$.

This equation may be written

$$y^2(Cx + G) + y(Bx^2 + Fx + K) + Ax^3 + Ex^2 + Hx + L = 0.$$

The locus of the middle points of chords parallel to the asymptote is

$$2y(Cx + G) + Bx^2 + Fx + K = 0;$$

and if the axes of this hyperbola are taken as coordinate axes, the equation of the cubic takes the form (i.). This covers the case when $C \neq 0$. If, however, $C = 0$, then also $B = 0$, and the locus of the middle points of chords parallel to the asymptote is a straight line: namely,

$$2Gy + Fx + K = 0.$$

If $G \neq 0$, this line is taken as the axis of x ; this is case (iii.). If $G = 0$ and $F \neq 0$, this line is taken as the axis of y ; this is case (ii.). Lastly, if $G = 0$ and $F = 0$, this locus is the line at infinity; and the equation is in the form (iv.).

III. Classification of Cubics.

15. Having thus, in Section III. of the tract, proved that every cubic equation is reducible to one of four canonical forms, Newton proceeds in Section IV. to classify the curves represented by these four equations into genera and species—degenerate forms being left out of account.

In this classification, the division into classes and genera (as well as that into the four canonical forms) depends on the nature of the infinite branches.

16. The separation into species of the curves which constitute any genus or subdivision is effected by solving the equation of the curve for y , and thus determining the maximum and minimum values of x .

The equations numbered (ii.), (iii.), (iv.) in Art. 5 give y as an explicit function of x . In the equation numbered (i.) this is also true if $e = 0$; and, if $e \neq 0$, Newton discusses the equation in the form

$$(xy + \tfrac{1}{2}e)^2 = ax^4 + bx^3 + cx^2 + dx + \tfrac{1}{4}e^2.$$

Thus, in all the groups hereafter described, the criterion which determines the species of the curves constituting any genus will be dependent on the nature and position of the roots of one of the two following equations: namely,

$$ax^4 + bx^3 + cx^2 + dx + \tfrac{1}{4}e^2 = 0, \quad \text{where } e \neq 0;$$

and

$$ax^3 + bx^2 + cx + d = 0.$$

17. The four canonical forms or cases are divided into seven (or six) classes. These classes are subdivided into fourteen (or thirteen) genera, which contain the seventy-eight species, from which however the degenerate forms of a conic and a straight line and of three straight lines are excluded. It will be convenient to present here, in a tabular form, the final result of the classification; it should be added that in his text Newton describes every division (whether a class, genus, or species) by the word "species."

Canonical Form or Case.	Class.	Genus.	Species.
I. $xy^2 + ey = ax^3 + bx^2 + cx + d$	Redundant Hyperbolas (a positive)	adiametral	9
		monodiametral	12 + 2
		tridiametral [asymptotes concurrent]	2 + 2 9
	Defective Hyperbolas (a negative)	adiametral	6
		monodiametral	7
	Parabolic Hyperbolas ($a = 0$)	adiametral monodiametral	7 4 + 2
II. $xy = ax^3 + bx^2 + cx + d.$	Hyberbolisms of Conics ($a = 0, b = 0$)	hyperbolic	4
		elliptic	3
		parabolic	2
III. $y^2 = ax^3 + bx^2 + cx + d.$	(The Trident)		1
IV. $y = ax^3 + bx^2 + cx + d.$	Diverging Parabolas (The Cubic Parabola)		5 1

18. I have given this table in the form which I understand to be indicated by Newton's language, but I should remark (i.) that the redundant and the defective hyperbolas would constitute more properly a single class of central cubic hyperbolas; and (ii.) that the hyperbolic and elliptic genera of the hyperbolisms would constitute more properly a single genus of hyperbolisms of central conics.

CASE I.

19. The first case comprises all curves whose equations can be expressed in the form

$$xy^3 + ey = ax^3 + bx^3 + cx + d.$$

These curves are divided into the following classes:—(i.) those in which a is positive, called *redundant hyperbolas*, or *hyperbolic hyperbolas*, or *hyperbolae triformes*; (ii.) those in which a is negative, called *defective hyperbolas* or *elliptic hyperbolas*; (iii.) those in which a is zero, which are further divided into (α) those in which b is not zero, called *parabolic hyperbolas*; and (β) those in which b is zero, called *hyperbolisms of conics*. It is usual to describe this last subdivision as a fourth class.

20. The curves in each class are further subdivided into genera according to the nature of their "absolute" diameters. If an asymptote cuts a cubic curve again in a point at a finite distance from the origin, the curve has no absolute diameter corresponding to that asymptote; but if an asymptote cuts the curve in three points at infinity (i.e., is a tangent at a point of inflexion at infinity), the curve has an absolute diameter corresponding to that asymptote. Thus to say that a curve is adiametral signifies merely that it has no point of inflexion at infinity.

An adiametral curve is described by Newton as *non-bifidus* or *circumflexus*, a monodiametral curve as *uno modo bifidus* or *directus*, and a tridiametral curve as *trifariam bifidus*.

21. CASE I. CLASS 1. *Redundant Hyperbolas*.—These are cubics whose equations can be expressed in the form

$$xy^2 + ey = ax^3 + bx^3 + cx + d,$$

where a is positive.

These curves have three real and distinct asymptotes: namely,

$$x = 0 \quad \text{and} \quad y = \pm \sqrt{a} (x + b/2a).$$

These asymptotes cut the cubic again in points determined by their respective intersections with the lines

$$y = d/e \quad \text{and} \quad x = (4ad \mp 2be\sqrt{a}) / (b^3 - 4ac \pm 4ae\sqrt{a}).$$

22. *First Subdivision. Adiametral redundant hyperbolas.*—Since a curve of this kind has no diameters, none of the three points on the curve at infinity is a point of inflexion there. Hence each asymptote cuts the curve again at a finite distance from the origin. Therefore

$$e \neq 0 \quad \text{and} \quad b^3 - 4ac \neq \mp 4ae\sqrt{a}.$$

These curves are divided into *nine species*, according as all, or some, or none of the roots of the equation $ax^4 + bx^3 + cx^2 + dx + \frac{1}{4}e^3 = 0$ are real or equal. In the typical case when all the roots are real, unequal, and of the same sign, there is a closed oval lying inside the triangle formed by the asymptotes. Nicole (*Mémoires de l'Académie des Sciences* for 1729, Paris, 1731, p. 217, Art. 31) asserted that there was in addition a tenth species formed by the oval coalescing with the ambigenous hyperbolic branch: this is not the case, the oval can coalesce with the circumscribing hyperbolic branch, but not with either of the others.

23. *Second Subdivision. Monodiametral redundant hyperbolas.*—Since a curve of this kind has only one diameter, one and only one of the three points on the curve at infinity is a point of inflexion there. If the asymptote at this point is taken as the axis of y , it will not cut the curve in any point at a finite distance from the origin. Therefore

$$e = 0 \quad \text{and} \quad b^3 \neq 4ac.$$

These curves are divided into *fourteen species* dependent on the nature and position of the roots of the equation $ax^3 + bx^2 + cx + d = 0$. Of these fourteen species, Newton enumerated only twelve; the remaining two being added by Stirling in 1717 (*Lineae tertii ordinis Newtonianae*, p. 99, species 11). In the *Ladies' Diary* for 1788 (pp. 44–46), and for 1789 (pp. 44–45) there is an attempt to show that there are three more species, making the total number in the subdivision to be seventeen. The examples given show curves of a slightly different form to those drawn by Newton, but they are in no sense new species.

24. *Third Subdivision. Tridiametral redundant hyperbolas.*—Since a curve of this kind has three diameters, each of the three points on the curve at infinity is a point of inflexion there. [Newton was aware that, if two of them are points of inflexion, the third is so also.]

Hence

$$e = 0 \text{ and } b^3 = 4ac.$$

These curves are divided into *four species*, dependent on the nature and position of the roots of the equation $x(2ax+b)^3 + 4ad = 0$. Of these four species, Newton enumerated only two; the other two being added by Stirling in 1717 (*Lineae tertii ordinis Newtonianae*, p. 102, species 24).

25. *Fourth Subdivision.*—Newton made a fourth subdivision, which comprises the particular cases of the preceding three subdivisions when the asymptotes are concurrent. He enumerated *nine species*; four of these being particular cases of the first subdivision, four of the second, and one of the third.

26. CASE I. CLASS 2. *Defective Hyperbolas.*—These are cubics whose equations can be expressed in the form

$$xy^2 + ey = ax^3 + bx^2 + cx + d,$$

where a is negative.

These curves have only one real asymptote: namely, $x = 0$. This asymptote cuts the cubic again in the point $(0, d/e)$. Hence the curve has no diameter if $e \neq 0$, and has one diameter if $e = 0$.

27. *First Subdivision. Adiametral defective hyperbolas.*—Since a curve of this kind has no diameter, the point on the curve at infinity is not a point of inflexion there. Hence the asymptote cuts the curve again at a finite distance from the origin. Therefore

$$e \neq 0.$$

These curves are divided into *six species*, dependent on the nature and position of the roots of the equation $ax^4 + bx^3 + cx^2 + dx + \frac{1}{4}e^3 = 0$.

28. *Second Subdivision. Monodiametral defective hyperbolas.*—Since a curve of this kind has one diameter, the point on the curve at infinity is a point of inflexion there. Hence the asymptote cannot

cut the curve again at a finite distance from the origin. Therefore

$$e = 0.$$

These curves are divided into *seven species*, dependent on the nature and position of the roots of the equation $ax^3 + bx^2 + cx + d = 0$.

29. CASE I. CLASS 3. *Parabolic Hyperbolas*.—These are cubics whose equations can be expressed in the form

$$xy^3 + ey = bx^3 + cx + d,$$

where $b \neq 0$.

These curves have one linear or proper asymptote and one parabolic asymptote. The proper asymptote is $x = 0$, and it cuts the cubic again in the point $(0, d/e)$. The parabolic asymptote is $y^3 = bx + c$, and it degenerates into two parallel straight lines if $b = 0$.

30. *First Subdivision. Adiametral parabolic hyperbolas*.—Since a curve of this kind has no diameter, the point on the curve at infinity is not a point of inflexion there. Hence the asymptote cuts the curve again at a finite distance from the origin. Therefore

$$e \neq 0.$$

These curves are divided into *seven species*, dependent on the nature and position of the roots of the equation $bx^3 + cx^2 + dx + \frac{1}{4}e^3 = 0$.

31. *Second Subdivision. Monodiametral hyperbolas*.—Since a curve of this kind has one diameter, the point on the curve at infinity is a point of inflexion there. Hence the asymptote does not cut the curve again at a finite distance from the origin. Therefore

$$e = 0.$$

These curves are divided into *six species*, dependent on the nature and position of the roots of the equation $bx^3 + cx + d = 0$. Of these six species, Newton enumerated only four. One of these omitted curves consists of two infinite branches and an oval, the other of two infinite branches and an acnode. Of these, the one with an oval was mentioned by Nicole in 1731 (*Mémoires de l'Académie des Sciences*, for 1731, Paris, 1733, p. 503, Fig. 10), and by Murdoch in 1740 (*Neutoni Genesis Curvarum per Umbras*, p. 77); the one with an acnode was mentioned by Nicholas Bernoulli (see Murdoch, p. 87). Both species were described fully by Edmund Stone in 1736 (*Philosophical Transactions*, 1840, vol. xli., pp. 318–320), and by De Gua De Malves in 1740 (*Usages de l'analyse de Descartes*, pp. 367, 368). It is probable that these discoveries of the two species not mentioned by Newton were made independently.

32. CASE I. CLASS 4. *Hyperbolisms of Conics*.*—These are cubics whose equations can be expressed in the form

$$xy^2 + ey = cx + d.$$

They might be regarded as a subdivision of parabolic hyperbolas, being those parabolic hyperbolas whose asymptotic parabola degenerates into two parallel straight lines (namely, $y = \pm \sqrt{c}$); but Newton treated them as constituting a class by themselves.

They have three asymptotes: namely,

$$x = 0, \quad y = \sqrt{c}, \quad \text{and} \quad y = -\sqrt{c},$$

and they have also a double point at infinity.

Newton divided them into three subdivisions, according as this double point is a crunode, an acnode, or a spinode.

33. *First Subdivision. Hyperbolisms of hyperbolas*.—In this case

c is positive,

the double point at infinity is a crunode, and the three asymptotes are real. These curves are divided into *four species*, according as (i.) $e \neq 0$, $d^2 > ce^2$; (ii.) $e \neq 0$, $d \neq 0$, $d^2 < ce^2$; (iii.) $e \neq 0$, $d = 0$; (iv.) $e = 0$.

34. *Second Subdivision. Hyperbolisms of ellipses*.—In this case

c is negative,

the double point at infinity is an acnode, and only one asymptote is real. These curves are divided into *three species*, according as (i.) $e \neq 0$, $d \neq 0$; (ii.) $e \neq 0$, $d = 0$; (iii.) $e = 0$, $d \neq 0$. The last of these is the "Witch of Agnesi."

35. *Third Subdivision. Hyperbolisms of parabolas*.—In this case

$$c = 0,$$

the double point at infinity is a spinode, and the three asymptotes are real, though two of them are coincident. These curves are divided into *two species*, according as $e = 0$ or $e \neq 0$.

* If $y = \phi(x)$ is the equation of a curve, then $xy = \phi(x)$ is a *hyperbolism* of that curve.

Thus, since $y^2 + ey = cx^2 + dx$ is any conic, therefore $(xy)^2 + e(xy) = cx^2 + dx$ is a hyperbolism of a conic. In the text, the factor x is rejected.

CASE II.

36. The second case comprises all curves whose equations can be expressed in the form

$$xy = ax^3 + bx^2 + cx + d.$$

There is only *one species*, usually known as the *Trident*, but sometimes called the *Cartesian Parabola* (in memory of Descartes), and sometimes a *Parabolism of a hyperbola*.

CASE III.

37. The third case comprises all curves whose equations can be expressed in the form

$$y^2 = ax^3 + bx^2 + cx + d.$$

These curves are called *Diverging Cubic Parabolas*, or sometimes *Neilian Parabolas* (in memory of W. Neil, the earliest mathematician who succeeded in rectifying a curve). They are divided into *five species*, dependent on the nature and position of the roots of the equation $ax^3 + bx^2 + cx + d = 0$.

CASE IV.

38. The fourth case comprises all curves whose equations can be expressed in the form

$$y = ax^3 + bx^2 + cx + d.$$

There is only *one species*, usually known as the *Cubic Parabola*, but sometimes called the *Wallisian Parabola* (in memory of John Wallis, the Savilian Professor at Oxford).

IV. *Discrimination of Genera.*

39. In the earliest manuscript copy Newton proceeds next to give criteria by which it is possible to tell to which genus or subdivision a curve belongs, without reducing its equation to one of the canonical forms. I give Newton's discussion at length in Appendix II., Arts. 101-105, 107. It does not appear in any of the printed editions, and I am not aware that it has been published previously.

40. Stirling, in his *Lineae Tertii Ordinis Newtonianae* (pp. 126-128), mentions this problem; but seems to think that the discrimination should be effected by a reduction to the canonical form. This reduction appears to be the only way of discriminating between the species.

41. The rule that Newton gives is as follows. If the coefficient of y^3 is not zero, the equation can be written in the form

$$y^3 + bxy^2 + cx^2y + dx^3 + ey^2 + fxy + gx^2 + hy + kx + l = 0,$$

that is, $x^3(z^3 + bz^2 + cz + d) + x^2(ez^2 + fz + g) + x(hz + k) + l = 0$,

where z stands for y/x . We shall denote this by

$$ux^3 + vx^2 + wx + l = 0.$$

Let suffixes indicate differentiation to z , and let

$$s = u_2v^2 - 2u_1vv_1 + 2u_1^2w.$$

Then u , v , w , their differential coefficients, and s are functions of z .

(α) If the three roots of $u = 0$ are real and unequal, the curve will be a redundant hyperbola. The hyperbola will be adiametral, monodiametral, or tridiametral, according as the equation $s = 0$ is satisfied by none of these roots, by only one of them, or by all of them.

(β) If only one root of $u = 0$ is real, then (i.), if that root does not satisfy the equation $u_2^2 = 8u_1$, the curve will be a defective hyperbola; but (ii.), if it does satisfy the equation $u_2^2 = 8u_1$, the curve will be a hyperbolism of an ellipse; in both these cases the curve being adiametral or monodiametral according as this root does not or does satisfy the equation $s = 0$.

(γ) If $u = 0$ has three real roots and two of them are equal, then (i.), if the root which is not equal to the other two roots does not satisfy the equation $v = 0$, the curve will be a parabolic hyperbola; but (ii.), if it does satisfy the equation $v = 0$, but not the equation $v_1^2 = 2u_2w$, the curve will be a hyperbolism of a hyperbola; and (iii.), if it satisfies both $v = 0$ and $v_1^2 = 2u_2w$, the curve will be a hyperbolism of a parabola; in all these cases the curve being adiametral or monodiametral according as this value of z does not or does satisfy the equation $s = 0$.

(δ) Lastly, if $u = 0$ has three real roots and all of them are equal, then (i.), if this root does not satisfy the equation $v = 0$, the curve will be a diverging parabola; but (ii.), if it does satisfy $v = 0$ but does not satisfy $v_1 = 0$, the curve will be a trident; while (iii.), if it satisfies both $v = 0$ and $v_1 = 0$, the curve will be a cubic parabola.

Should the coefficient of y^3 in the original equation be zero, but not

that of x^3 , then the equation must be arranged in powers of x , we must put $z = x/y$, and corresponding tests can be applied.

If the coefficients of x^3 and of y^3 are both zero, then, if $b \neq 0$ and $c \neq 0$, the curve will be a redundant hyperbola; this is similar to case (α). But if either $b = 0$ or $c = 0$, then the curve will be a parabolic hyperbola or a hyperbolism of a hyperbola or of a parabola; this is similar to case (γ). The further determination of the genus may be effected in the way given above.

42. In this discussion Newton shows also how the values of u_1, u_2, v, v_1 and w , corresponding to a root of $u = 0$, enable us to give a geometrical construction for the corresponding asymptote (see Appendix II., Art. 106).

43. I may add here that Newton uses the binomial coefficients and not the differential coefficients, so that his formulæ are slightly different from those that I have given in Art. 41. Thus, if $u = z^n$, he uses $\frac{1}{2}n(n-1)z^{n-2}$ where I have used u_2 . This is an instance of John Bernoulli's "*regulam falsam*" (see Montucla's *Histoire des Mathématiques*, vol. III., p. 105, or H. Sloman's *Claim of Leibnitz to the Invention of the Differential Calculus*, Cambridge, 1860, p. 121).

V. Generation of Cubics by Shadows.

44. After having enumerated in Section IV. of the tract all the possible species of cubics, Newton begins Section V. by noting that every cubic curve may be regarded as the shadow of one of the five diverging parabolas cast by a luminous point on a plane properly situated.

His enunciation of this remarkable property is as follows:—If the shadows of curves caused by a luminous point are projected on an infinite plane, the shadows of conic sections will always be conic sections; those of curves of the second genus will always be curves of the second genus; those of curves of the third genus will always be curves of the third genus; and so on. And in the same manner as the circle, by projecting its shadow, generates all the conic sections, so the five divergent parabolas, by their shadows, generate all other curves of the second genus. And thus some of the more simple curves of other genera might be found which would form all curves of the same genus by the projection of their shadows on a plane.

45. No proof of this proposition is given in the printed essay, but among the Portsmouth collection of Newton's papers is a sheet of double foolscap containing rough holograph notes, enumerating the various species which arise from the section by a plane in some defined position of one of the five cones having these diverging parabolas for bases.

Thus, under the head of the second species of the diverging parabolas, he writes, "Si oculus infinite distet vel plana projectionis et projicientis parallela sint, projectio erit Parabola ejusdem speciei cum projiciente. . . . Si secat in nodo, projectio erit Hyperbola triformis bifida duas ex asymptotis parallelas habens"—i.e., the species which Newton terms number 64, and which is described above as the fourth species of the Hyperbolisms of a Hyperbola.

But no demonstrations are given, and the numerous erasures, corrections, and interlineations make the manuscript somewhat difficult to read.

46. I have little doubt that Newton had arrived at this remarkable result, which proved a puzzle to most of his contemporaries, by the method of projection indicated in the *Principia*, Bk. I., sect. 5, lemma XXII. (See also propositions XXV., XXVI., XXVII.)

47. It is not difficult to prove the property by analytical geometry (see, for example, Salmon's *Higher Plane Curves*, second edition, Arts. 195, 196); but to Newton's contemporaries, who relied largely on pure geometry, it was by no means easy to establish its truth, and, in fact, a quarter of a century elapsed before any one published a demonstration of it.

48. The earliest writer who gave a proof of it was Nicole, who communicated it to the French Academy on 1 Dec., 1731. (*Mémoires de l'Académie des Sciences*, for 1731, Paris, 1733, pp. 494-510.)

49. A few days later, on 12 Dec. 1731, Clairaut communicated another demonstration, in which the result is deduced from certain general theorems. This is contained in his paper entitled *Sur les courbes que l'on forme en coupant une surface courbe quelconque par un plan donné de position*: see the *Mémoires de l'Académie des Sciences* for 1731, Paris, 1733, pp. 490-493.

50. In 1740 another demonstration of this result was given by

P. Murdoch in his *Neutoni Genesis Curvarum per Umbras*, published at Leyden. This work was republished at London in 1746.

51. Lastly, in 1755, still another demonstration was given by Jacquier in the appendix to his *Elementi di prospettiva*.

52. This property enables us to divide cubics into five groups such that all the curves in each group have their fundamental characteristics (*i.e.*, those characteristics which are unaltered by projection) in common. This provides a better basis for classification than the one used by Newton. Chasles has made the interesting note that only five cubics have centres, and that these five central cubics can also by their shadows generate every form of a cubic. In fact, if the five cones described on the diverging parabolas as bases are cut by a certain plane, we obtain the five canonical central cubics. (See *Aperçu historique sur l'origine et le développement des méthodes en géométrie*, Brussels, 1837, note xx., p. 348.)

53. In more recent times this property has formed the subject of elaborate memoirs by Möbius in 1852 (*Die Grundformen der Linien der dritten Ordnung*, Collected works, vol. II., pp. 89–176), and by Cayley in 1864 (*On Cubic Cones and Curves*, *Cambridge Philosophical Transactions*, vol. XI., pp. 129–144).

VI. Bibliography.

54. I shall conclude this paper by enumerating the various original manuscripts and the editions of Newton's tract with which I am acquainted.

55. I have already mentioned that three holograph manuscripts of the tract are preserved at Cambridge in the Portsmouth collection (Section I., Class IV., Numbers 1, 2).^{*} There is also a fragment of a fourth manuscript (I., IV., 4), and some notes on the generation by shadows (I., IV., 3). The notes on the generation by shadows and the subsequent history of that theorem have been already mentioned. I shall describe the other manuscripts in what, I have no doubt, is their chronological order. Indeed the successive increases in the number

^{*} The two earliest manuscripts are catalogued in a single entry as "an early copy."

of species enumerated would serve alone to distinguish the order of production.

56. The earliest of these manuscripts is entitled *Enumeratio Curvarum Trium Dimensionum*. This is a rough draft of the tract as afterwards published, but it contains the substance of only the third and fourth sections. Moreover, the subdivision into species is not discussed, and the final result is nine divisions and sixteen genera, arranged and named as follows, the nomenclature being somewhat different to that subsequently used:—

- | | |
|--|--|
| 1. Hyberbola triformis | { sine diametro.
cum diametro.
cum tribus diametris. |
| 2. Cissoidalis vel Endea | { circumflexa.
directa. |
| 3. Parabola Cissoideos | { circumflexa.
directa. |
| 4. Hyperbola Hyperbolae | { sine diametro.
cum diametro. |
| 5. Conchoidalis vel Hyperbola Ellipseos | { circumflexa.
directa. |
| 6. Hyperbola Conchae vel Parabolae | { sine diametro.
cum diametro. |
| 7. Parabola reflexa vel recurvata. | |
| 8. Parabola fissa vel Cartesiana. | |
| 9. Parabola circumflexa sive Simplex cubica. | |

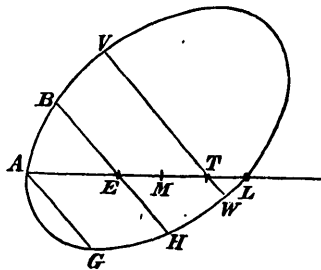
57. The language of the text is not exactly the same as that subsequently published, but the only material differences which I have noticed between this and the copy printed in the *Opticks* are (i.) the method by which the equation is reduced to the four canonical forms (see above, Art. 10); (ii.) a slightly fuller treatment of adiametral redundant hyperbolas, such as giving the coordinates of the points where the asymptote cuts the curve again; and (iii.) the addition of criteria for discriminating the genera (see above, Art. 41).

58. The next manuscript is that which is described in the Portsmouth catalogue as *Fragments concerning lines of the third order*.

This is intermediate in point of time between the first and second manuscripts. These fragments consist of two or three closely written foolscap sheets. Newton commences by proving some general properties of diameters and segments of chords, with special reference to conics and cubics (*Conicarum sectionum communes proprietates mutatis mutandis ad Curvas magis compositas applicari possunt*). These are a rough draft of what afterwards appeared as the first part of the second manuscript, and are the foundation of the results given in Section II. of the published tract. At the end of these he has written: "*Hæ sunt insigniores proprietates Conicarum sectionum, et videtis quod curvis etiam superioris ordinis conveniunt.*" These words are crossed out.

59. Next, he proceeds to give the geometrical interpretations of the general equations of the second and third degrees. These are as follows.

60. Suppose that AG , BH are two parallel chords of a conic, and that VW is any chord drawn parallel to them. Through A draw



another chord AL cutting BH in E , and VW in T . On AL take a point M so that

$$AM : EM = AG : EH - EB,$$

Let $TV = y$; then he asserts that y is determined by the equation

$$y^2 + \frac{AG}{AM} MT \cdot y + \frac{EB \cdot EH}{EA \cdot EL} AT \cdot TL = 0,$$

where the lines are regarded as directed magnitudes.

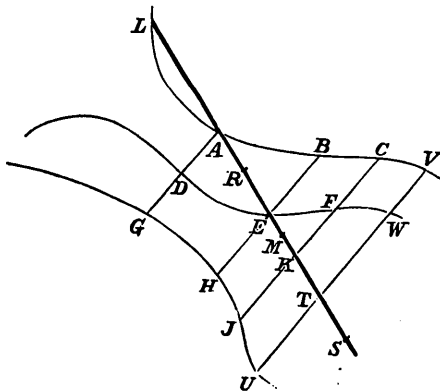
In fact, if AL and AG are taken as axes of x and y , the equation of any conic can be written in the form

$$y^2 - y(ax + b) + cx^2 + dx = 0.$$

In the above case,

$$AG = b, \quad AM = -\frac{b}{a}, \quad AL = -\frac{d}{c}, \quad \frac{EB \cdot EH}{EA \cdot EL} = -c.$$

61. Similarly, let ADG , BEH , CFJ be three parallel chords of a cubic, and VWU any chord drawn parallel to them. Join AE . Let



the chord AE cut the cubic again in L , the chord CJ in K , and the chord VU in T . On AE take a point M so that

$$AM : EM = AD + AG : EH - EB.$$

Also, on AE take points R and S , so that

$$\begin{aligned} AR \cdot AS : AD \cdot AG &= ER \cdot ES : EB \cdot EH \\ &= KR \cdot KS : KO \cdot KF - KO \cdot KJ - KF \cdot KJ. \end{aligned}$$

Let $TV = y$; then he asserts that y is determined by the equation

$$\begin{aligned} y^3 - \frac{AD + AG}{AM} TM \cdot y^2 - \frac{AD \cdot AG}{AR \cdot AS} TR \cdot TS \cdot y \\ - \frac{KO \cdot KF \cdot KJ}{KA \cdot KL \cdot KE} TA \cdot TL \cdot TE = 0, \end{aligned}$$

from which all the properties of every cubic may be deduced.

It is interesting to note that the geometrical properties ultimately

published in the second section of the tract are deducible from this relation, and were probably thus obtained.

62. The further discussion of this relation would involve the use of solid geometry; and (apparently for this reason) he abandons this geometrical method of investigating the properties of cubics, and commences afresh by taking the general equation of the third degree. His discussion of the reduction of that equation to the four canonical forms has been already mentioned (see above, Art. 11). He concludes by briefly indicating the principles of classification to be adopted, and gives a tabular analysis of the result. This table shows the division into classes, genera, and species; but only sixty species are enumerated. The missing species are—(i.) nine redundant hyperbolas, namely, two adiametral, six monodiametral, and two tridiametral; (ii.) two defective hyperbolas, namely, one of each subdivision; and (iii.) one diverging parabola. The more interesting parts of these *Fragments* are printed in Appendix I., Arts. 84–99.

63. The second manuscript is entitled, *Enumeratio Curvarum secundi Ordinis*. The original title was *Enumeratio Curvarum tertii Ordinis*. This was first corrected by replacing *Curvarum* by *Linearum*, and then changed again to the form given above. The numerous verbal corrections interlined in this manuscript alter the text to a form which corresponds closely with that in which it was published afterwards.

The geometrical propositions collected at the beginning of the *Fragments* are placed at the commencement of this manuscript, and an introductory paragraph is added. The geometrical equations of a conic and a cubic (see above, Arts. 60, 61) are omitted, and the reduction of cubic equations to the four canonical forms is effected in the way given in the printed edition, and which I described above in Art. 13. The nomenclature is also somewhat simplified, and is substantially the same as that afterwards printed. The division into species is carried out in considerable detail, but only sixty-nine species are enumerated, the missing species being three monodiametral redundant hyperbolas, of which one is the limiting form of the other two when their asymptotes are concurrent. The two paragraphs about the generation of cubics by means of the shadows of the diverging parabolas, which form part of Section V. of the printed edition, and the last paragraph in Section VII., are missing; and as the manuscript is closely and continuously written, and ends with “*Finis*,” we may assume that these paragraphs were not contained

in any pages which are now lost. Sections V. and VI. are written out twice. The manuscript is divided into thirty articles with marginal headings. The diagrams (if ever there were any) are lost. The loose sheet in which the manuscript is wrapped does not belong to the manuscript. There still remain numerous minute verbal differences between this and the text of the printed editions, but I do not think any material differences other than those described above.

64. The third manuscript is entitled, *Enumeratio Linearum tertii Ordinis*. The title was originally written as *Enumeratio Curvarum cubici generis*. This was then altered to the heading given in the second manuscript. Finally, the three last words were erased, and the title changed to the form given above. The first section of the printed tract, the proposition about generation by shadows, and the last paragraph of Section VII., all of which are absent from the second manuscript, are inserted here. The curves are classified into seventy-two species, but the diagrams are absent. The whole is carefully written out, and, except for a few verbal alterations, the text is the same as that subsequently published. There is however no reference to a division into sections, but the whole is divided into thirty-four articles.

65. With this manuscript are two loose sheets, used as scribbling paper, which contain, among other notes, two draft tabular analyses of the classification of cubics. These do not differ materially from those adopted in the text, but they are drawn up in a form closely resembling that which I have given in Art. 17, with the addition of descriptive adjectives for most of the species.

66. There is nothing to show the dates of the first two manuscripts or of the *Fragments*, but one of these loose scribbling sheets of the third manuscript is written on the back of a short business letter, dated "London, 6 June, '95," and directed to "Isaac Newton, Esq., a.M., Trinnity Collidge, Cambridge." It is probable, therefore, that the whole of this manuscript is of a date subsequent to this, but I see no reason to suppose that it was written later than the summer or autumn of 1695.

67. The following printed editions of the essay on cubic curves are in the library of Trinity College, Cambridge, and include, so far as I know, all that have been published.

68. The enumeration of cubic curves was originally published, in
VOL. XXII.—NO. 407.

1704, as an appendix to the first edition of the *Opticks*, the title-page of which runs as follows:—*Opticks: or, a Treatise of the Reflexions, Refractions, Inflexions, and Colours of Light. Also Two Treatises of the Species and Magnitude of Curvilinear Figures.*

In the advertisement to the second edition, which is dated July 16, 1717, Newton says: "In this Second Edition of these *Opticks* I have omitted the Mathematical Tracts published at the End of the former Edition, as not belonging to the Subject." The tracts were left out in all the subsequent English editions, and I believe that they were not added to any of the Continental editions of the *Opticks*.

69. In 1706 a Latin translation of the *Opticks*, prepared by Samuel Clarke and revised by Demoivre, was published in London. The two tracts above mentioned are appended to this, but they were omitted from subsequent editions.

70. In 1711 the essay on Cubic Curves was reprinted in the collection of Newton's tracts, which was edited by William Jones, and issued at London, under the title, *Analysis per Quantitatum Series, Fluxiones, ac Differentias: cum Enumeratione Linearum Tertii Ordinis.* The essay is reprinted without comment. Of all the editions this is the most pleasant to use, as the type, diagrams, and get-up are excellent. In 1723 this book was reprinted at Amsterdam, and was issued both by itself and also bound up with a reprint of the second edition of the *Principia*, which was published there in the same year.

71. In 1744 the essay was reprinted in the *Opuscula Mathematica* of Newton, edited by J. Castillioneus, and issued in three volumes at Lausanne.

72. In 1779 it was reprinted in vol. I. of the collected edition of Newton's works, issued by S. Horsley at London.

73. In 1861 a translation, with notes, was published by C. R. M. Talbot at London. The title-page of some of the earlier copies is dated 1860.

74. The chief commentary on the work is one by Stirling, which was published at Oxford in 1717, under the title, *Lineae Tertii Ordinis Newtonianae, sive Illustratio Tractatus D. Newtoni De Enumeratione Linearum Tertii Ordinis.* This contains demonstrations of most of the results which had been merely stated by Newton. A considerable part of the work is occupied by introductory matter, and the portion

devoted to the classification of cubics occupies less than forty pages. Stirling was an intimate friend of Newton, and there is reason to think that the proofs and comments were approved, and to some extent supplied, by Newton. This work was reprinted at Paris in 1797.

75. A commentary on part of the work, and especially on the algebraical reduction of a cubic to one of four canonical forms, was given by Nicole under the title *Traité des lignes du troisième ordre*, in the *Mémoires de l'Académie des Sciences*, for 1729, Paris, 1731, pp. 194-224.

76. Two objections have been urged against the system of classification adopted by Newton. In the first place, he nowhere defines what is meant by a species, and he is not consistent in his use of the term. In the second place, the curves in each group have not the same fundamental characteristics, that is, such as are unaltered by projection.

77. De Gua De Malves, in his *Usages de l'analyse de Descartes*, published at Paris in 1740, pointed out the first of these objections, and he proposed (pp. 365-367) to consider the maximum and minimum values of y as well as of x . This would increase largely the number of species, but would not materially improve the system of classification.

On the second of these objections, see above, Arts. 52, 53.

78. Cramer, in his *Introduction à l'analyse des lignes courbes algébriques*, published at Geneva in 1750, and Euler, in his *Analysis Infinitorum*, published at Lausanne in 1748, proposed to classify curves solely by reference to the character of their infinitely distant points. They both refused to recognise the various curves formed by the degeneration of an oval into an acnode, or its coalescence with other branches of the curve, as constituting distinct species. Cramer (p. 359-369) gives 14 species of cubics, and Euler (Bk. II., chap. ix., pp. 114-126) gives 16 species. This, after all, is not very different from the method used by Newton, except that Cramer and Euler would describe as species what I have termed genera.

79. In more recent times Plücker suggested a new system of classification of cubic curves, with special reference to critic centres. He divides cubics into 6 classes, 61 groups or genera, and 219 species (*System der analytischen Geometrie und der Theorie der Curven dritter Ordnung*, Berlin, 1835). His classification forms the subject of an exhaustive criticism by Cayley in the *Transactions of the Cambridge*

Philosophical Society for 1864, pp. 81-128, where the reader will find a comparison of the results with those of Newton.

80. Lastly, I may call attention to the classification of cubics given in Salmon's *Higher Plane Curves*, where a method not very different from that of Newton is adopted. See the second edition, Arts. 195-209.

81. In conclusion I may perhaps add that Waring applied the Newtonian method to quartic curves. Treated in this way, there are 12 characteristic forms of their equations, and not more than 84,551 species. (See his *Miscellanea Analytica*, Cambridge, 1762, Bk. II., chap. v.) Cramer divided quartics into 9 classes, but did not continue the subdivision into genera and species. Euler's method gives 8 classes subdivided into 146 genera. Salmon gives 10 classes.

APPENDICES.

82. The following extracts are literal transcripts of parts of two of the manuscripts alluded to in the text. No corrections are inserted, but (i.) words which are obviously redundant are placed in square brackets; (ii.) punctuation has been added; (iii.) the articles are numbered, for convenience of reference; and (iv.) the notation used to express a ratio has been altered from the obsolete form to that now current: that is, the ratio of a to b is printed here as $a:b$, while Newton and his contemporaries would have written it as $a.b$. Such notes as are given are in English and are placed in square brackets.

APPENDIX I.

Extracts from Newton's Manuscript entitled *Fragments concerning lines of the third order.*

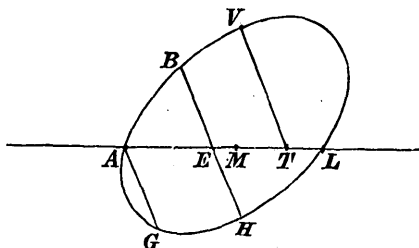
(See Articles 11, 58-62, above.)

83. [The manuscript commences with a discussion of geometrical properties (especially of diameters) of conics and other curves (see Art. 58). After which, Newton proceeds to enunciate the two propositions stated in Arts. 60, 61. The text is as follows.]

84. Cacterum cum hujusmodi proprietates eo præsertim spectent ut sciamus commode exprimere Curvas, eo ut de iisdem commode ratiocinemur dicam breviter qua ratione id fiat in Curvis cubicis aequè ac in Conicis sectionibus.

85. Sit primo ABC con. sectio. Eam secent rectæ duæ parallelæ AG, BH in punctis A, G, B, H ; et tertia quævis recta non parallela AL in puncto L .

Fiat $AG : EH - BE :: AM : EM$,
 et sit $d : e :: AG : AM$,
 et $b : e :: EB \times EH : EA \times EL$.

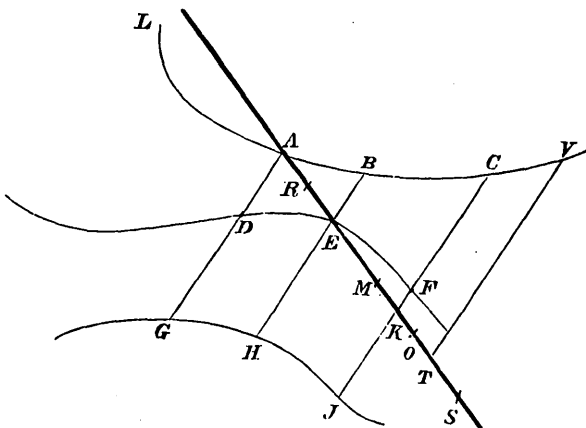


Et ad AE erecta quavis $TV \parallel BE$, et occurrente curvae in V , dicatur $AE = y$ [an obvious slip for $TV = y$], et erit

$$yy - \frac{d}{e} TM \times y - \frac{b}{e} TA \times TL = 0,$$

expressio conicae sectionis.

86. Eodem modo se res habet in curvis cubicis. Dentur enim positione tres rectae parallelae AG, BH, CJ secantes curvam propositam singulae in totidem datis punctis $A, D, G; B, E, H; C, F, J$. Et per extremum unius rectae punctum A et alterius medium



punctum E agatur quarta recta AE , eaque producta secet curvam in

L et rectam CJ in K . Praeterea in AE capiatur punctum M ita ut sit

$$AD + AG : EH - EB :: AM : EM,$$

et sit

$$d : e :: AD + AG : AM,$$

[ut] et

$$b : e :: KO \times KF \times KJ : KL \times KA \times KE.$$

Deinde, facto

$$KJ : KF :: KO - KJ : KM,$$

capiantur etiam in AE puncta R et S ita ut sint

$$[AD \times AG : AR \times AS :: EB \times EH : ER \times ES],$$

$$AD \times AG : AR \times AS :: EB \times EH : ER \times ES$$

$$:: KO \times KF - KO \times KJ - KF \times KJ : KR \times KS :: e : c.$$

Et, acta quavis recta TV ipsi AG parallela, et occurrens [probably a slip for occurrente] rectae AE in T et curvae ABO in V , si TV dicatur y erit

$$y^3 - \frac{d}{e} \times TM \times yy + \frac{c}{e} TR \times -TS \times y - \frac{b}{e} TA \times TL \times TE = 0.$$

Estque generalis regula exprimendi curvas omnes ordinis cubici, si modo signorum + et - mutatio pro diversa positione linearum probe fiat.

87. Puncta autem R et S per planam Geometriam investigantur. Scilicet faciendum est

$$\frac{AD \times AG \times EK}{AE} + \frac{EB \times EH \times KA}{AE}$$

$$+ KO \times KJ - KO \times KF + KJ \times KF : AK \times EK :: b : e,$$

et erit $b : e :: AD \times DG : AR \times AS :: BE \times EH : ER \times ES$.

Dantur igitur rectangula

$$AR \times AS \quad \text{et} \quad ER \times ES.$$

Sint ista

$$AE \times p \quad \text{et} \quad AE \times q,$$

et erit

$$p + q + AE = AR + AS.$$

Ejus summae dimidium esto AO et erunt

$$TR \quad \text{et} \quad TS = \sqrt{AT^2 - AE \times p},$$

ubi AT^2 semper major erit quam $AE \times p$, si modo recta AE non per

duo extrema linearum puncta A et B vel G et H , nec per media D et E , sed per extremum et medium uti A et E . Unde constat hanc regulam nulli exceptioni obnoxiam esse.

88. Caeterum cum haec sit aequatio cubica quae non nisi per solidam geometriam construi possit docebimus quomodo hae curvae redigendae sunt ad aequationes quae per planam Geometriam construi possunt.

Et in hunc finem sumamus plenam aliquam aequationem.

[Here follow the general equation of the third degree in x and y , and a discussion of the possible simplifications produced in it by transformation of coordinates. Much of this is crossed out, and possibly the whole should be cancelled as being merely rough notes of the calculations necessary for the following analysis, which I have already summarized in Art. 11.]

89. Sit basis $AB = z$,
ordinatim incedens $BC = v$.

Proponatur aequatio plena

$$av^3 + bv^2v + czv + dz^3 + evv + fzv + gzz + hv + kz + l = 0.$$

Construatur quomodocunque haec aequatio

$$a + bp + cpp + dp^3 = 0,$$

et invento p , cape BD ad BC ut p ad 1 [see Fig. 1], et agatur CD quae sit ad BC ut q ad 1. Et dicto

$$AD = r \text{ et } CD = s,$$

erit

$$BC = qs \text{ et } AB = r - pqs.$$

Quibus scriptis in aequatione pro v et z , orietur nova aequatio in qua terminus multiplicatus per s^3 deerit. Sit ista

$$brss + crss + dr^3 + ess + frs + grr + hs + kr + l = 0.$$

[The analysis requires that $AB = r + pqs$. Hence, if p is positive, D must be between A and B , and not as in Newton's figure.]

90. Et si terminus $brss$ non desit produc [Fig. 1] CD ad E ut sit $2b : c :: AD : DE$. Et propter datos angulos trianguli ADE simul dabitur ratio AD ad AE ; sit ista n ad 1. Et dictis

$$AE = t \text{ et } CE = w,$$

erit $AD = nt$ et $CD = w - \frac{cn}{2b}t$.

Quibus scriptis in aequatione pro r et s emerget aequatio sine

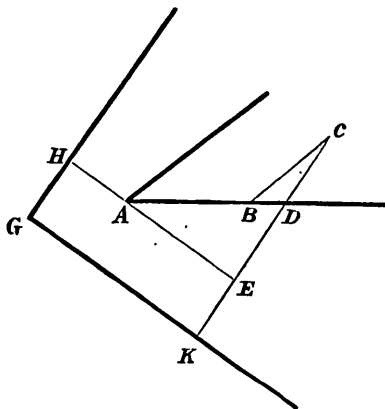


FIG. 1.

termino multiplicato per ttw . Sit ista

$$btww + eww + ftw + lw + dt^3 + gtt + kt + l = 0,$$

retentis denovo literis cum mutato significato. Et producat CE ad K ut sit

$$EK = \frac{f}{b}, \quad \left[\text{a slip for } \frac{f}{2b} \right],$$

et EA ad H ut sit

$$AH = \frac{e}{b},$$

et completo parallelogrammo $HEKG$, dic

$$GK = x \text{ et } OK = y,$$

et erit $AE = x - \frac{e}{b}$ et $CE = y - \frac{f}{b}$ [it should be $y - \frac{f}{2b}$].

Quibus scriptis pro t et w , orietur aequatio sine terminis yy et xy , exprimens relationem inter basem GK et ordinatam KO et non ultra reducibilis. Sit ista

$$bxyy + ky + dx^3 + gxx + kx + l = 0.$$

91. Quod, si terminus $brrs$ supra defuerit, non autem terminus $crrs$, fac $c : d :: AD : DE$, et pro r et s scriptis nt et $w - \frac{dn}{c}t$, emerget aequatio sine termino t^3 . Sit ista

$$eww + cttw + ftw + lw + gtt + kt + l = 0.$$

Et positis $EK = \frac{g}{c}$ et $AH = \frac{f}{2c}$,

et pro t et w scripto $x = \frac{f}{2c}$ et $y = \frac{g}{c}$,

oriatur aequatio sine terminis x^3, xyy, y^3, xx, xy , hujus formae

$$eyy + caxy + hy + kx + l = 0$$

[quae quidem].

92. Sin terminus etiam *crrs* defuerit non autem *ess*, fac

$$2e : f :: AD : DE,$$

et perinde, scriptis nt et $w = \frac{f}{2e} nt$ pro r et s , oriatur aequatio sine terminis w^3, tww, ttw, tw . [Sit.] Deinde, positis

$$EK = \frac{h}{2e} \quad \text{et} \quad AH = \frac{g}{3d},$$

et pro t et w perinde scriptis

$$x = \frac{g}{3d} \quad \text{et} \quad y = \frac{h}{2e},$$

oriatur aequatio hujus formae

$$eyy + dx^3 + kx + l = 0.$$

93. Quod si terminus *ess* etiam defuerit, non autem terminus *frs*, ...
produc [Fig. 2] DA ad H ut sit

$$AH = \frac{h}{f}.$$

Et dicto DH t , erit $AD = t - \frac{h}{f}$.

Quo scripto pro r oriatur aequatio hujus formae

$$fts + dt^3 + gtt + kt + l = 0.$$

Dein produc CD ad E et K ut sit

$$f : g :: HD : DE, \quad \text{et} \quad EK = \frac{k}{f}.$$

Et acta HE comple parallelogrammum $HEKG$, sitque

$$HD : DE :: n : 1,$$

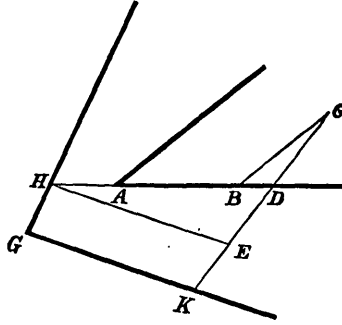


FIG. 2.

et GK vel $HE = x$ et $KO = y$,

erit $HD = nx$, et $CD = y - \frac{k}{f} - \frac{gnx}{f}$.

Quibus scriptis pro t et s emerget æquatio hujus formæ

$$fxy + dx^3 + l = 0.$$

94. Sin denique terminus etiam frs desit, fac

$$AH = \frac{g}{3d},$$

et scripto $t - \frac{g}{3d}$ pro r , orietur æquatio hujus formæ

$$hs + dt^3 + kt + l = 0.$$

Deinde posito $h : k :: HD : DE$ et $EK = \frac{l}{h}$,

scriptisque nx pro t et $y - \frac{l}{h} - \frac{k}{h}nx$ pro s , orietur æquatio

$$hy + dn^3x^3 = 0.$$

95. Atque ita videre est quod omnes æquationes ad hasce quatuor

formas reduci possint

$$bxyy = dx^3 + gxx + hy + kx + l [= 0],$$

$$eyy = dx^3 + kx + l [= 0],$$

$$fxy = dx^3 + l [= 0],$$

et $hy = dx^3 [= 0],$

ubi tamen signa + et - omnimodo variari possint et aliqui termini deesse. Atque hae formae non sunt ultra reducibiles.

96. [The remainder of the manuscript has been briefly described in Art. 62, and is as follows.]

97. Jam vero in prima harum formarum, posito $bxyy$ affirmativo, si terminus dx^3 sit etiam affirmativus curva erit Hyperbola triformis cum tribus Asymptotis, quarum nullae erunt parallelae. Sin termini isti [probably a slip for terminus iste] dx^3 sit negativus duae ex Hyperbolis mutabuntur in Ellipsin, quae tamen non semper habet ovalem formam, sed aliquando conjungitur reliquae figurae Hyperbolicae, aliquando evadit punctum, et aliquando imaginaria est. Inter Hyperbolam et Ellipsin media existit Parabola cum Hyperbola conjugata ubi dx^3 nullus est dummodo terminus etiam gxx [non] non desit. Nam si iste etiam desit, figura rursum evadet omni ex parte Hyperbolica. Et quidem triplex Hyperbola cum tribus Asymptotis quarum duae sunt parallelae si terminus k affirmativus est. Ubi vero k evanescit una Hyperbola evanescit et parallelae Asymptoti coincidunt, et ex his hyperbolis alia evanescit ubi k fit negativus.

98. In secunda forma curva semper Parabolica est sine aliqua Hyperbola conjugata et aliquando habet Ellipsin conjugatam juxta verticem. In tertia forma exprimit Parabolam Cartesianam. Et quarta Parabolam simplicem cubicam.

99. Atque ita patet novem esse genera curvarum ordinis cubici. Sub hisce tamen multae comprehenduntur species. Nam aliae non habent aliquam diametrum bisectionis; aliae habent, quas bifidas nominare licebit. Aliquae nullam habent partem similem alii parti; aliae habent partes omnino similes et aequales a centro intermedio, hinc inde infinitum procurentes quas ideo aequicruras nominare licet. In aliquibus Asymptoti et omnes diametri bisectionis concurrunt ad idem punctum; in aliis non. Aliquae habent

punctum conjugatum, quas punctatas voco ; aliae acutè terminantur ad vertices ad modum Cissoidis, quas ideo Cissoidales vel cuspidatas voco ; et aliae se decussant ad modum crucis. Hæ sunt præcipuæ differentiæ harum curvarum. Et secundum has differentias possumus enumerare species harum curvarum juxta sequentem tabulam.

[Here follows a tabular analysis of cubic curves into 7 classes, 14 genera, and 60 species. See Art. 62.]

APPENDIX II.

Extract from Newton's holograph manuscript, entitled

Enumeratio Curvarum Trium Dimensionum.

(See Articles 39–43, 56, 57 above.)

100. [This extract is confined to the section on the Discrimination of Genera, which I described in Arts. 41–43 above. Newton's notes on the subject are written on a sheet of paper of a size slightly smaller than the rest of the manuscript, and with different ink. On the other hand, the absence of a division into articles, the absence of references, and the absence of marginal headings are points in which it is similar to this manuscript, and differs from all the others ; and, as it has been preserved with the first manuscript, I treat it as a part of it. It runs as follows.]

101. Enumeratis harum Curvarum speciebus et ostenso quo pacto aequationes quibus quomodocunque exprimuntur reduci possint ad formulas quarum subsidio definivimus ipsas et distinximus in species videor determinationem horum locorum ad finem* *perduxisse*. Sed quoniam reductio aequationum ad formulas istas saepenumero taedio esse possit, ex abundanti jam docebo quomodo speciem curvæ sine istius modi transmutatione diagnoscere liceat. Supponamus igitur quaestionem aliquam de hujus generis Loco ad aequationem *perductam* esse, sitque aequatio illa ut supra

$$y^3 + bxyy + cxy + dx^3 + eyy + fxy + gxx + hy + kx + l = 0,$$

in qua x denotat basem AB , y ordinatam BO , et $+b$, $+c$, $+d$, &c., datas quantitates cum suis signis affectas. Construatur haec aequatio

$$x^3 + bzz + cz + d = 0,$$

et ponatur

$$3[a]zz + 2bz + c = m, \quad 3z + b = n, \quad ez^2 + fz + g = p,$$

$$2ez + f = q \quad \text{et} \quad hz + k = r.$$

* A hole has been burnt in the paper; and the letters in italics are conjecturally restored.

Ubi, si in construenda praefata aequatione tres obvenerint reales radices z , illae dabunt totidem quantitates m, n, p, q, r . Et si forte d non habeatur tunc duae radicum ex aequatione

$$zz + bz + c = 0$$

elici debent, et 0 pro tertia usurpari; hoc est, pro m, n, p, q, r scribendum erit c, b, g, f et k , neglectis terminis in quibus z sive 0 reperitur. Eodem modo si c ac d simul desunt duae reputandae sunt radices et sibi mutuo et nihilo aequales et tertia radix erit $-b$. Et si omnes tres termini b, c, d desunt tres radices reputandae sunt aequales, et pro illis scribendum erit 0.

102. Jam igitur, si tres obvenerint inaequales radices, et nulla earum in colligendis quantitativibus m, n, p, q, r usurpata efficiat terminos $npp - qpm + rmm$ sese destruere, figura erit Hyperbola triformis non bifida. Sed si aliqua earum efficiat terminos istos $npp - qpm + rmm$ evanescere Hyperbola illa triformis erit uno modo bifida, et trifariam bifida si termini illi in unoquoque trium casuum evanescunt.

103. Quod si aequatio illa non nisi unicam habeat radicem z , caeteris duabus existentibus imaginariis, figura erit Hyperbola Elliptica modo non sit

$$nn - 2m = 0,$$

nam in hoc casu habebitur Hyperbolismus Ellipseos.

104. Ad haec si trium radicum z duae sunt aequales, et tertia usurpata pro z non efficiat

$$p = 0,$$

figura erit Hyperbola Parabolica. Sed si efficiatur

$$p = 0,$$

habebitur Hyperbolismus Hyperbolae, et Hyperbolismus Parabolae si insuper sit

$$qq - 4nr = 0.$$

Et in omnibus hisce casibus figura erit bifida si tertia illa radix z efficiat

$$npp - qpm + rmm = 0;$$

aliter erit non bifida.

105. Denique si omnes tres radices z aequales esse contigerit, et non sit

$$p = 0,$$

figura erit Parabola divergens. Sed si sit

$$p = 0,$$

habebitur Parabolismus Hyperbolæ nisi insuper sit

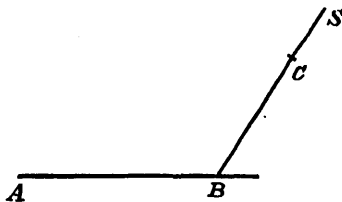
$$q = 0,$$

in quo casu figura erit Parabola Cubica.

106. Cognita specie curvae si præterea desideretur positio Asymptotorum et plaga ad quam crura Parabolica tendunt. In [a slip for , in] ordinata BC , quam y designat, cape

$$BS = zx - \frac{p}{m},$$

et punctum S attinget asymptoton si modo z non sit una duarum vel



trium aequalium radicum. Hoc igitur pacto tot invenies Asymptotos quot sunt radices z non aequales. Sed si z sit una duarum aequalium radicum cape

$$BS = zx - \frac{q}{2n} \pm \sqrt{\frac{qq - 4rn}{4nn}},$$

et habebis duas parallelas Asymptotos si figura sit Hyperbolismus Hyperbolæ [aut aut] aut si existente

$$qq - 4rn = 0,$$

figura sit Hyperbolismus Parabolæ, habebitur Asymptotos ponendo

$$BS = zx - \frac{q}{2n}.$$

Ac denique si tres sunt aequales radices z et figura sit Parabolismus Hyperbolæ, Asymptotos determinabitur ponendo

$$BS = zx - \frac{r}{q}.$$

Eodem modo ubi curva est de genere quovis Parabolico, capiend

$$BT = zx,$$

punctum T attingit lineam rectam quae tendat ad eandem plagam cum infinitis cruribus Parabolicis figurae, si modo pro z sumatur una aequalium radicum.

107. In praefatis determinationibus supposui primum terminum aequationis y^3 non deesse. Quamobrem si terminus ille desit et dx^3 non desit, debet y fieri basis figurae et x ordinata, et caetera peragi ut supra. Sed si uterque terminus simul desit sed b et c non desunt, figura erit Hyperbola triformis cujus una Asymptotos determinatur

capiendo
$$AB = -\frac{e}{b},$$

et ad B erigendo lineam parallelam ordinatis, altera determinatur

capiendo
$$BS = -\frac{g}{c}$$

et ducendo per S lineam parallelam basi. Nam istae parallelae erunt Asymptoti. Tertia attingetur a puncto S sumendo

$$BS = -\frac{cx}{b} + \frac{ec}{bb} - \frac{f}{b} + \frac{g}{c}.$$

Denique si terminorum etiam b et c alteruter puta c desit, figura vel Hyperbola Parabolica erit vel Hyperbolismus aliquis cujus determinationem supra satis explicuimus.

On the q -Series derived from the Elliptic and Zeta Functions of $\frac{1}{3}K$ and $\frac{1}{4}K$. By J. W. L. GLAISHER, Sc.D., F.R.S.

[Read Dec. 11th, 1890.]

1. In a paper on the function $H(n)$,* which denotes the excess of the number of divisors of n which $\equiv 1, \text{ mod. } 3$, over the number which $\equiv 2, \text{ mod. } 3$, it was shown that the q -series having $H(n)$ as the coefficient of its general term, n denoting any integer, was expressible by means of a zeta function of argument $\frac{1}{3}K$, and that the q -series in which $H(m)$ was the coefficient of the general term, m denoting any uneven integer, was expressible by means of an elliptic function of $\frac{1}{3}K$. These results suggest that it would be of interest to obtain the developments in ascending powers of q of the complete system of

* Vol. xxi., p. 395.