XI. *Investigation of a New Series for the Computation of Logarithms; with a New Investigation of a Series for the Rectification of the Circle.* By JAMES THOMSON, *LL.D., Professor of Mathematics in the University of Glasgow.*

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I.

The series $l(1+x) = \text{M}(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \text{\&c.})$, discovered by MERCATOR, seems to be the origin from which, directly or indirectly, all the series may be derived which are usually employed in the computation of logarithms. A series, which affords remarkable facilities for such computations, and which lately occurred to me, may be investigated in the following manner.

In MERCATOR's series, change x successively into $\frac{n}{x}$ and $-\frac{n}{x}$; then, by adding $l\,x$ to each of the results, we get

$$l(x+n) = l\,x + \text{M}\left(\frac{n}{x} - \frac{1}{2}\frac{n^2}{x^2} + \frac{1}{3}\frac{n^3}{x^3} - \frac{1}{4}\frac{n^4}{x^4} + \text{\&c.}\right) \ldots\ldots\ldots\ldots\ldots(1)$$

$$l(x-n) = l\,x + \text{M}\left(-\frac{n}{x} - \frac{1}{2}\frac{n^2}{x^2} - \frac{1}{3}\frac{n^3}{x^3} - \frac{1}{4}\frac{n^4}{x^4} - \text{\&c.}\right) \ldots\ldots\ldots\ldots(2)$$

Take half the sum and half the difference of these; then

$$\frac{l(x+n) + l(x-n)}{2} = l\,x - \text{M}\left(\frac{1}{2}\frac{n^2}{x^2} + \frac{1}{4}\frac{n^4}{x^4} + \frac{1}{6}\frac{n^6}{x^6} + \text{\&c.}\right) \ldots\ldots\ldots(3)$$

$$\frac{l(x+n) - l(x-n)}{2} = \text{M}\left(\frac{n}{x} + \frac{1}{3}\frac{n^3}{x^3} + \frac{1}{5}\frac{n^5}{x^5} + \text{\&c.}\right) \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

By multiplying the latter by n, and dividing the product by $2\,x$, we get

$$\frac{n\{l(x+n) - l(x-n)\}}{4\,x} = \text{M}\left(\frac{1}{2}\frac{n^2}{x^2} + \frac{1}{2.3}\frac{n^4}{x^4} + \frac{1}{2.5}\frac{n^6}{x^6} + \text{\&c.}\right) \ldots\ldots(5)$$

Adding this and (3), and by transposition, we obtain

$$l\,x = \frac{l(x+n) + l(x-n)}{2} + \frac{n\{l(x+n) - l(x-n)\}}{4\,x} + \text{M}\left(\frac{1}{3.4}\frac{n^4}{x^4} + \frac{2}{5.6}\frac{n^6}{x^6} + \frac{3}{7.8}\frac{n^8}{x^8} + \text{\&c.}\right)(6)$$

If $n = 1$, this becomes

$$l\,x = \frac{l(x+1) + l(x-1)}{2} + \frac{l(x+1) - l(x-1)}{4\,x} + \text{M}\left(\frac{1}{3.4}\frac{1}{x^4} + \frac{2}{5.6}\frac{1}{x^6} + \frac{3}{7.8}\frac{1}{x^8} + \text{\&c.}\right) \ldots (7)$$

The m^{th}, or general term of this series, is evidently $\text{M}\frac{m}{(2m+1)(2m+2)} \cdot \left(\frac{1}{x}\right)^{2m+2}$.

The last two series, besides the simplicity and elegance of their form, are remarkably convergent, when x is large, compared with n or 1. The latter of them gives, with great facility, the logarithm of a whole number from the logarithms of the two numbers immediately preceding and following it, when the number is considerable: and this, as we shall presently see, is a case of continual occurrence in the computation of logarithmic tables.

To exemplify the use of formula (7), suppose that the common logarithm of 2 has been computed by any of the known methods:* then, by doubling and trebling it, the logarithms of 4 and 8 are obtained; while that of 5 is found by subtracting it from 1, the logarithm of 10. From the logarithms of 8 and 10, the logarithm of 9 is obtained by means of series (7), as, by taking $x = 9$, that formula gives

$$l9 = \frac{l10 + l8}{2} + \frac{l10 - l8}{36} + \mathrm{M}\left(\frac{1}{3.4}\cdot\frac{1}{9^4} + \frac{2}{5.6}\cdot\frac{1}{9^6} + \&c.\right)$$

In this the convergence is so rapid, that to find the logarithm true for seven decimals, it is not necessary to proceed beyond the first term in the vinculum; and by employing additional terms, any assigned degree of accuracy is easily obtained. By halving the logarithm of 9, we get that of 3; from which, and from the logarithm of 2, that of 6 is found. Then, by series (7),

$$l7 = \frac{l8 + l6}{2} + \frac{l8 - l6}{28} + \mathrm{M}\left(\frac{1}{3.4}\cdot\frac{1}{7^4} + \frac{2}{5.6}\cdot\frac{1}{7^6} + \&c.\right);$$

—a series of rapid convergence.

Now, by adding the logarithm of 2 to the logarithms of 6, 7, 8, 9, and 10, we get those of the even numbers 12, 14, 16, 18, and 20; and the logarithm of 15 is the sum of the logarithms of 3 and 5. We should then find with great ease, by means of (7), the logarithms of the prime numbers 11, 13, 17, and 19. By adding the logarithm of 2 to the logarithms of 11, 12, 13, ... 20, we should have those of the even numbers from 20 up to 40; and those of the primes between the same limits would be computed by means of (7). In a similar manner, we should first obtain the logarithms of the even numbers from 40 up to 80, and then those of the intermediate primes; and thus we might proceed as far as we please, the computations for the primes becoming easier and easier, as the numbers become larger. The logarithm of any whole number, indeed, from 40 upwards, would be obtained by (7), true for seven or more places of decimals, merely by means of the logarithms of the two numbers immediately preceding and follow-

* If the modulus of the common logarithms be supposed to be known, the common logarithm of 2 may be computed with great ease by finding, by MERCATOR's series, the logarithm of $1 + 0.024$; by adding to the result 3, the logarithm of 1000, and thus finding the logarithm of 1024; and, lastly, by dividing by 10, because 1024 is the tenth power of 2.

ing it, without employing any of the terms in the vinculum, and consequently *without any trouble with the modulus.*

The facility of the process by means of formula (7) will appear from the following example, in which the common logarithm of 61 is computed from those of 60 and 62.

$$
\begin{array}{rl}
l\,62 = & 1.792391689 \\
l\,60 = & 1.778151250 \\
\hline
& 2)3.570542939 \\
\hline
\text{Half sum} = & 1.785271469 \\
\hline
\text{Difference} = & 0.014240439
\end{array}
$$

Now $61 \times 4 = 244$, and dividing the difference by this, we get 0.000058362; the sum of which and of the half sum, found above, is 1.785329831, the logarithm of 61. This is true in all its figures except the last, which ought to be 5.

It may be proper to remark, that when x is large, its logarithm will be obtained very readily by means of formula (3); as, by taking $n = 1$, and transposing, we get

$$l\,x = \frac{l(x+1) + l(x-1)}{2} + \mathrm{M}\left(\frac{1}{2}\frac{1}{x^2} + \frac{1}{4}\frac{1}{x^4} + \frac{1}{6}\frac{1}{x^6} + \&c.\right);$$

—a formula which will give the logarithms of whole numbers above 2000, true for seven or more decimals, by means of the logarithms of the two numbers immediately preceding and following, without any term of the series.

II.

A series which gives the rectification of the circle with greater ease than any other with which I am acquainted, occurred to me some time ago, and I then believed it to be new. I have lately found, however, that the same series was discovered by EULER, and that it appeared in the eleventh volume (1793) of the *Nova Acta* of the Petersburgh Academy, with two investigations by that distinguished writer. My investigation is altogether different from those given by him, and is very simple—perhaps more so than either of his. It is obtained, also, by means of a method of integration which may be employed with advantage in many other instances: and though, as might be expected, several things in my paper are anticipated in EULER'S, yet mine contains others which are not to be found in his. For these reasons, I shall present the paper in almost exactly the same state in which it was before I saw the article by EULER.

If we put $\tan^{-1} x$ to denote the circular arc, whose tangent is x, we have, by the formula for the differential of the arc in terms of its tangent to the radius 1,

$$d\tan^{-1} x = \frac{dx}{1+x^2}, \text{ and therefore } \tan^{-1} x = \int\frac{dx}{1+x^2}.$$

The integral of the second member of this, in the form that will suit our purpose, will be obtained in perhaps the easiest manner by means of the formula,

$$d\left(\frac{u}{v}\right)=\frac{v\,du-u\,dv}{v^2}=\frac{du}{v}-\frac{u}{v}\cdot\frac{dv}{v};$$

which, by integration and transposition, gives

$$\int\frac{du}{v}=\frac{u}{v}+\int\frac{u}{v}\cdot\frac{dv}{v}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (8)$$

By taking in this $u=x$, and $v=1+x^2$, the expression found above becomes

$$\tan^{-1}x=\frac{x}{1+x^2}+\int\frac{x}{1+x^2}\cdot\frac{2x\,dx}{1+x^2},\ \text{or}\ \tan^{-1}x=\frac{x}{1+x^2}+\int\frac{2x^2\,dx}{(1+x^2)^2}.$$

The integral of the last term of this is obtained in a similar manner, from formula (8), by taking $du=2x^2\,dx$, and $v=(1+x^2)^2$, and is found to be

$$\frac{2}{3}\cdot\frac{x^3}{(1+x^2)^2}+\frac{2.4}{3}\int\frac{x^4\,dx}{(1+x^2)^3}.$$

It is plain that this process may be continued without limit; and, the law of continuation being manifest, we obtain

$$\tan^{-1}x=\frac{x}{1+x^2}+\frac{2}{3}\frac{x^3}{(1+x^2)^2}+\frac{2.4}{3.5}\frac{x^5}{(1+x^2)^3}+\frac{2.4.6}{3.5.7}\frac{x^7}{(1+x^2)^4}+\text{\&c.}\ldots(9)$$

This is the series proposed to be investigated; and, for giving an arc in the first quadrant, it requires the addition of no constant quantity.

When x is a fraction $\frac{p}{q}$, the foregoing series may be exhibited, after some modifications, in the convenient form,

$$\tan^{-1}\frac{p}{q}=\frac{pq}{p^2+q^2}\left\{1+\frac{2}{3}\frac{p^2}{p^2+q^2}+\frac{2.4}{3.5}\left(\frac{p^2}{p^2+q^2}\right)^2+\text{\&c.}\right\}\ldots\ldots\ldots(10)$$

By putting A, B, C, &c. to denote the successive terms of the last series, and k to denote the fraction $\frac{p^2}{p^2+q^2}$, we get the following expression, which answers best for the purposes of computation:—

$$\tan^{-1}\frac{p}{q}=\frac{pq}{p^2+q^2}+\frac{2}{3}k\text{A}+\frac{4}{5}k\text{B}+\frac{6}{7}k\text{C}+\text{\&c.}\ \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

We have thus obtained the means of computing a circular arc in terms of its tangent. The well known series,

$$\tan^{-1}x=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\text{\&c.}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

given, first by JAMES GREGORY, and afterwards by LEIBNITZ, serves the same purpose, but is far inferior in practice. Like (12), the series above investigated, converges the more rapidly, the smaller the tangent is in comparison of the radius.

Yet, even in the very unfavourable case in which $x = 1$, and the arc $= 45°$, we should have, by series (9),

$$\frac{1}{4}\pi = \frac{1}{2} + \frac{2}{3}\left(\frac{1}{2}\right)^2 + \frac{2.4}{3.5}\left(\frac{1}{2}\right)^3 + \text{\&c.};$$

less than twenty terms of which would give the circumference true for six places of decimals; while many thousand terms of the series,

$$\frac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{\&c.}$$

derived from (12), would be required to effect the same object.

In the actual computation, however, of the circumference to a great degree of accuracy, the series found above is applied with most advantage in connexion with the curious and elegant principle first employed by MACHIN, and afterwards extended by EULER,—that of finding arcs whose tangents are rational, and are small known fractions, and the sum or difference of which arcs, or of their multiples, is a known part of the circumference. Such arcs are innumerable; and, by taking them sufficiently small, any degree of convergence whatever may be obtained. Rapidity of convergence, however, is far from being the sole important consideration. The convergence may be very great, and yet the fraction k may be of such a form as to render the computation laborious and difficult. No arc, indeed, answers well, unless $p^2 + q^2$ be of the form $\frac{10^m}{2^n}$, m and n being whole positive numbers; and even of arcs having this property, many are, in other respects, inconvenient. Of a great number of tangents which I have tried, those which seem to answer best are $\frac{1}{3}$, $\frac{2}{11}$, $\frac{1}{7}$, and $\frac{3}{79}$; which give respectively for the values of k, 0.1, 0.032, 0.02, and 0.00144: and, since it is easy to shew that $3\tan^{-1}\frac{1}{3} - \tan^{-1}\frac{2}{11} = \frac{1}{4}\pi$, we get, by quadrupling,

$$\pi = 12\tan^{-1}\frac{1}{3} - 4\tan^{-1}\frac{2}{11} \quad \text{.................(13)}$$

In a similar manner, it would appear that

$$\pi = 8\tan^{-1}\frac{1}{3} + 4\tan^{-1}\frac{1}{7} \quad \text{.................(14)}$$

$$\pi = 10\tan^{-1}\frac{1}{3} - 2\tan^{-1}\frac{3}{79} \quad \text{.................(15)}$$

$$\pi = 8\tan^{-1}\frac{2}{11} + 12\tan^{-1}\frac{1}{7} \quad \text{.................(16)}$$

$$\pi = 20\tan^{-1}\frac{2}{11} - 12\tan^{-1}\frac{3}{79} \quad \text{.................(17)}$$

$$\pi = 20\tan^{-1}\frac{1}{7} + 8\tan^{-1}\frac{3}{79} \quad \text{.................(18)}$$

By means of any of these six formulæ, in connexion with series (11), the value of π may be computed with great despatch and facility. As an example, let us take formula (18), and putting successively $p=1$, $q=7$, and $p=3$, $q=79$, in formula (11), we get

$$\pi = \left\{ \begin{array}{l} \quad 2.8 + \tfrac{2}{3}\,\mathrm{A}\times 0.02 + \tfrac{4}{5}\,\mathrm{B}\times 0.02 + \tfrac{6}{7}\,\mathrm{C}\times 0.02 + \&c. \\ + 0.30336 + \tfrac{2}{3}\,\mathrm{A}'\times 0.00144 + \tfrac{4}{5}\,\mathrm{B}'\times 0.00144 + \&c. \end{array} \right\}$$

Hence, by carrying the decimals out to twelve places, the computation will stand thus:

First Series.

A =	2.800000000000
B =	37333333333
C =	597333333
D =	10240000
E =	182044
F =	3310
G =	61
$20\tan^{-1}\frac{1}{7}$ =	2.837941092081

Second Series.

A′ =	0.303360000000
B′ =	291225600
C′ =	335492
D′ =	414
$8\tan^{-1}\frac{3}{79}$ =	0.303651561506
$20\tan^{-1}\frac{1}{7}$ =	2.837941092081
π =	3.141592653587

This value of π is true in all its figures except the last. The computation of the terms is effected with great ease. Thus in the first series, B is found by doubling A, subtracting from the result one-third of itself, and rejecting the last two figures; C by doubling B, taking from the result one-fifth of itself, and rejecting two figures; and so on: while, in the second series, B′ is found by multiplying A′ by 144 (which is easily done on account of the repetition of the figure 4), by taking from the result one-third of itself, and rejecting five figures: and, in both the computations, various arithmetical contractions will suggest themselves as the work proceeds.

Numberless other expressions for π might be obtained, and of any degree of convergence whatever. Those given above, however, are preferable perhaps to any others, on account of the simple form of k, and the consequent facility with which it is managed in the computation. The following may be mentioned in addition to those already given:

$$\pi = 28\tan^{-1}\frac{3}{79} + 20\tan^{-1}\frac{29}{278} \quad \ldots\ldots\ldots\ldots (19)$$

$$\pi = 48\tan^{-1}\frac{3}{79} + 20\tan^{-1}\frac{1457}{22049} \quad \ldots\ldots\ldots\ldots (20)$$

$$\pi = 68\tan^{-1}\frac{3}{79} + 20\tan^{-1}\frac{24478}{873121} \quad \ldots\ldots\ldots\ldots (21)$$

$$\pi = 22\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{685601}{69049993} \quad \ldots\ldots\ldots\ldots (22)$$

$$\pi = 88 \tan^{-1}\frac{24478}{873121} + 68 \tan^{-1}\frac{685601}{69049993} \quad\text{.................(23)}$$

$$\pi = 88 \tan^{-1}\frac{3}{79} - 20 \tan^{-1}\frac{685601}{69049993} \quad\text{.................(24)}$$

These all converge rapidly, and more especially (20), (21), (23), and (24). Such, indeed, is the convergence in the last mentioned formula, that, in one of the series arising from it, each term is less than a seven-hundredth part, and in the other less than a ten-thousandth part, of the term preceding it! The value of k, however, for the tangent $\frac{685601}{69049993}$, is inconvenient in practice, being the unmanageable decimal 0.000098576363673563955 2: and though this inconvenience might be obviated in a considerable degree by tabulating the multiples of k by 2, 3, &c., up to 9, and thus facilitating its multiplication by the preceding terms of the series in employing formula (11), there is little doubt but the computation would be found to be easier by means of some of the other formulæ already given.