ON CLOSED SETS OF POINTS AND CANTOR'S NUMBERS

By W. H. YOUNG.

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1. In a paper on "Sets of Intervals" (*Proc. London Math. Soc.*, Vol. xxxv., p. 245) we classified the points of a segment (A, B), in which a set of intervals is given, into (1) internal, (2) simple end-points, (3) semi-external, and (4) external points of a set of intervals, and proved certain properties of the external points, and that the semi-external points are at most countably infinite.

The set of external points by themselves is an unclosed set. The theory of unclosed sets is far more incomplete than that of closed sets. In the first part of the present paper I propose to deal exclusively with closed sets. It is an important fact that

By adding the semi-external points to the external points we get a closed set of points. (1)

For, if P is either an internal point or an ordinary end-point of intervals, a neighbourhood of P can be found so small that it contains no semi-external or external points, for such neighbourhood can be taken entirely within either one or two of the intervals; hence P cannot be a limiting point of the set of external and semi-external points; this set is therefore closed.

Similarly, of course,

The set obtained by adding all the end-points of the black intervals to the external points is closed. (2)

The extra points added are isolated points, whose limiting points were included before.*

The set (1) is none other than the first derived set of the set (2).

2. *Closed Sets and their Complementary Intervals.*

The latter theorem is for many purposes the more valuable of the two, since it has an important converse, which is an immediate result of the investigations into sets of intervals.

* *Loc. cit.*, p. 253.

It may be expressed as follows:—*Any set of non-overlapping intervals* δ_1, δ_2, ... *on the straight line defines a closed set of points, viz., all those points which are not internal to any of the intervals.* The converse is that any closed set of points, which does not consist merely of the whole segment (A, B) under discussion, defines a set of intervals such that the whole segment (A, B) consists of the internal points of those intervals and the points of the given set. This may be expressed more shortly by saying *a closed set of points is always complementary to a set of intervals*—open intervals, of course. This close connexion between closed sets of points and sets of intervals suggests that the theory of sets of intervals may be used to develop the theory of sets of points.

Certain classifications can be made at once. It is evident that, *if a set of intervals is dense everywhere, the complementary points are dense nowhere, and vice versa;* while, if the set of intervals be not dense everywhere, the complementary points certainly fill up some partial segment, and *vice versa.* If the given set of points contains any such whole segment, its potency is evidently c, and the process of derivation leaves that segment unaltered. Hence, from the point of view of the succeeding articles, the case when the intervals are dense everywhere and the complementary points dense nowhere is the only interesting case, and it will in future be assumed that this is the case unless the contrary is expressly stated.

3. *Potency of a Closed Set of Points.*

The first question to be answered is as to the potency of a set of points. The theory of sets of intervals enables us to answer this question fully for a closed set of points. *A closed set of points is either finite or countably infinite or has the potency of the linear continuum.*

This follows from the last section, and *loc. cit.*, Theorem 4 and § 28.

Two cases can be at once disposed of in detail:

If the set of complementary intervals consist of a finite number only, the closed set is finite, and vice versa. (*Loc. cit.*, Theorem 2.)

If the content of the set of intervals be less than l,* *the set of points has the potency* c. (*Loc. cit.*, Theorem 3′.)

There remains over the case when the set of complementary intervals is not finite, but has the content l. In this case the potency of the points may be a or c, and it depends entirely on the "ultimate set" (*loc. cit.*, Theorem 4) which of these will be the case. We shall return to the discussion of this ultimate set in § 6.

* l being the length of (A, B).

4. *Content of a Closed Set of Points.*

I propose now to give the following definition of the content of a closed set of points :—*The content I_P of a closed set of points is defined as $l-I_\delta$, where I_δ is the content of the complementary intervals.*

We notice that this definition agrees with our fundamental notions when the set of points is dense everywhere in a partial segment. The investigations of the paper on "Sets of Intervals" show that *the content, so defined, is always* $\geqslant 0$ *and* $\leqslant l$, and that, while it cannot be 0 if the set be dense everywhere in any partial segment, it is possible that the content of a set which is nowhere dense may be greater than zero ; indeed, it is possible to construct such sets with content as near the maximum as possible. This maximum l, the length of the segment (A, B) in which we are operating, can only be attained when the set consists of *all* the points of the segment (A, B). *The content of a finite or countably infinite closed set of points is, by Theorem 4, always zero ;* but these are not the only sets of content zero: for instance, H. J. S. Smith's sets of the first kind have zero content.

The following is the fundamental property of the content and might have been taken as the definition and the above deduced from it: historically, this is what was done :—*If we determine a finite number of non-overlapping intervals, such that every point of a given closed set is internal to one of the intervals, the content of these intervals is always greater than I_P, but may be made as near as we please to I_P, by taking the intervals small enough.*

First, it is obvious from our theory how to constuct such intervals. We only have to take any small quantity ϵ, and determine the finite number k of black intervals $> \epsilon$. These leave over a finite number of complementary intervals, and, if we lengthen each of these at each end by as small a quantity as we please, we get such a finite set of intervals as we are in search of.

Secondly, suppose we have in any way determined such a finite set of intervals $d_1, d_2, \ldots$. Then we must be able to fix a limit ϵ, such that all the black intervals $\leqslant \epsilon$ of the given set lie inside the d-intervals, and all the black intervals $> \epsilon$ lie entirely or partially outside them.* Suppose any one of the intervals d_r to overlap at one or both ends into a black interval $> \epsilon$. Then, without freeing any but internal points of this black interval, we can ensure, by curtailing d_r if desirable, that the distance

* For otherwise we could determine a sequence of the black intervals having a single limiting point, external to all the intervals d, or an end-point of one of them, which is inconsistent with the hypothesis, since such a point must be a point of the given set.

of the end-point of the overlapped piece from the overlapping end-point of d_r should be less than $\frac{\sigma}{4r+4}$. The content of the overlapped parts being now less than $\sum_1^\infty \frac{\sigma}{2r+2}$, *i.e.*, less than $\frac{1}{2}\sigma$, and, the content of the black intervals $\leqslant \epsilon$ being denoted by $R(\epsilon)$, the content of those parts of the black intervals that lie outside the intervals d_r will now lie between $I_\delta - R(\epsilon)$ and $I_\delta - R(\epsilon) - \frac{1}{2}\sigma$. Hence the content of the intervals d_r will lie between $I_p + R(\epsilon)$ and $I_P + R(\epsilon) + \frac{1}{2}\sigma$.

Now, if we diminish the length of each interval d_r, ϵ must decrease without limit.* $R(\epsilon)$ can therefore be made as small as we please by sufficiently decreasing the lengths of the intervals d_r. Hence we can choose the lengths of the in ervals d_r so that the sum of them, though always greater than I_P, may differ by less than any assigned quantity from I_P. Q. E. D.

It may be remarked in this connexion that it is not possible to determine an infinite set of non-overlapping intervals such that each point of a closed set is internal to one of the intervals, and no interval is free of points of the set. For an infinite set of intervals has always at least one external or semi-external limiting point, and such a point would by the second assumption be a limiting point of the set and therefore belong to it, contrary to the first assumption.

Another form of the preceding, used by Hankel and Cantor as the basis of their original definitions, and frequently very useful, is the following:— *If we enclose each point of a closed set of points in a small interval of which it is the middle point, the limit of the sum of the finite number of non-overlapping intervals filled up by these small overlapping intervals, when the lengths of these latter are indefinitely decreased, is the content I_P of the set of points.*

In a note on "Sets of Overlapping Intervals" † I showed, in fact, that such a set of overlapping intervals enclosing all the points of a closed set can be replaced by a finite number of them; so that this theorem is seen to be merely a different form of the preceding.

5. *Limiting Points.*

A limiting point of a set of points may be either a limit for the set when approached from one side only or from both sides. We may express this difference by speaking of "limiting points on one side" and "limit-

* For, if not, let η be the limit of ϵ; then, however we construct our intervals d_r, there is always a black interval $\geqslant \eta$ inside them, which is obviously nonsense.

† *Proc. London Math. Soc.*, Vol. XXXV., p. 387.

ing points on both sides." If a limiting point be a limit on one side only, it is evidently an end point of a black interval, and, if it is a limit on both sides, it is external to the black intervals. Hence we see that *the limiting points of a closed set on one side only are the same as the semi-external points of the black intervals and are therefore, at most, countably infinite in number.* The limits on both sides are identical with the external points of the black intervals.* The potency of these latter, if not zero or a natural number, is a or c, and may be any one of these when the set is dense nowhere; when it is dense everywhere in any partial segment the potency is evidently always c.

6. *Derived Sets, Deduced Sets, and the Nucleus.*

We shall denote by E a set of points, and by E_1, E_2, ... the successive derived sets of E. If E is closed, E_1 is contained in E and those points of E which are not points of E_1 are end-points of abutting black intervals of E. The connexion between E and E_1 is, indeed, precisely that of the sets (1) and (2) of § 1. [Whether E is closed or not, E_1 and therefore all the successive derived sets are closed.] In passing from the black intervals of E to those of E_1 therefore, we amalgamate all abutting intervals of E. Starting with any particular black interval δ_r of E, the amalgamation process is only arrested by the first limiting points to which we reach on either side of δ_r, and we blacken up to these inclusive; let us denote this black interval of E_1 by δ'_r.

Now we saw (*loc. cit.*, p. 266) how it was possible, systematically and uniquely, to obtain from a given set of intervals an ultimate set of *non-abutting* intervals having the same content, and, with the exception of at most a countably infinite set of points, the same external points. The ultimate set is then either such a set as was contemplated in § 28, *loc. cit.*, or consists of the whole continuum,† and that, if these external points were not more than countable in number, the ultimate set of intervals would consist of the whole infinite straight line.

The closed set of points consisting of the end-points and external points of the ultimate set of intervals we shall call the "Nucleus of E," and denote by E_Ω. The Nucleus is evidently perfect‡ (except when E is count-

* *Loc. cit.*, p. 253.

† In § 7, *loc. cit.*, it was pointed out that it was sometimes convenient to regard the part of the straight line exterior to (A, B) as black; from our present point of view this is so, and this interval must be considered as one of the black intervals.

‡ Dense in itself and closed.

able, when it evanesces), since, having no abutting black intervals, it has no isolated points. Let us denote the black interval of E_Ω which contains any particular interval δ_r of E by δ_r^Ω. Then, since in amalgamating the intervals of E with δ_r so as to form δ'_r we blacken out at most two limiting points, it is evident that δ'_r *will also lie inside* δ_r^Ω, though it may, of course, coincide with it.

Similarly the corresponding interval δ_r^n *of* E_n *lies inside* δ_r^Ω *for all values of* n.

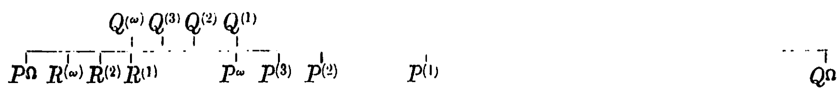

FIG. 1.

Let $P^{(1)}, P^{(2)}, \ldots, P^{(n)}$ be the left-hand end-points of $\delta_r^{(1)}, \delta_r^{(2)}, \ldots, \delta_r^{(n)}$. Then either from and after a definite integer m all the $P^{(n)}$s coincide, or else they define a limiting point to the left of all of them. In either case let us denote the point so obtained by $P^{(\omega)}$.

Now $P^{(n+1)}, P^{(n+2)}, \ldots$ are all points of E_n, and therefore $P^{(\omega)}$ is a limiting point of E_n, that is, since E_n is closed, a point of E_n. *Hence* $P^{(\omega)}$ *is a point of* E_n *for every value of* n.

The process of forming from any number of sets the largest set which is contained in all of them is usually called "the process of finding the H.C.F. of the sets." But, as the process is a very important one, we require a simpler term for expressing it; and, as the term has not been otherwise appropriated, I propose to call it "deduction." Thus: *Given any finite or infinite number of sets of points, that set which contains all the points which belong to every set, and no other points, is called the deduced set.* If we denote the given sets by $E_1, E_2 \ldots, E_n, \ldots$, the deduced set will usually be denoted by E_ω.

We notice that the process of deduction applied to any *finite* number of derived sets $E_1, E_2, \ldots, E_n$ gives us E_n and nothing new. The deduced set of an infinite number of derived sets* is, however, not necessarily identical with any one of them. It is convenient to give here the following example, due to Cantor, which we shall have occasion subsequently to refer to.

7. *Cantor's Classical Example.*

Let T denote the set of points obtained by bisecting any segment, and then bisecting the right-hand segment, and so on.

*It can easily be shown that such a set always exists, since $E_1, E_2, \ldots, E_n, \ldots$ are closed sets.

We start with T in the segment (0, 1). The corresponding binary fractions are 0, 1, ·1, ·11, $\cdot 1^2 1$, ..., $1^n 1$, ...

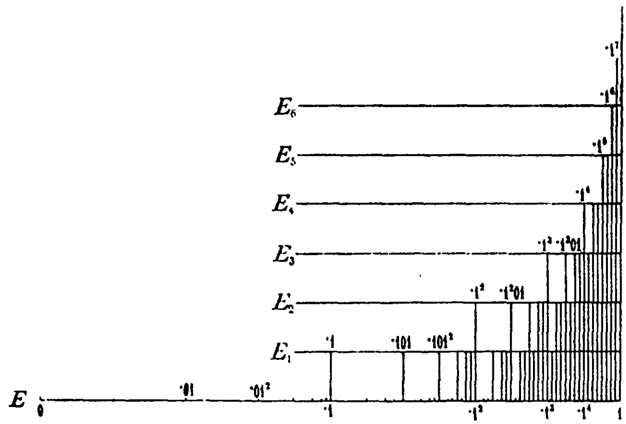

FIG. 2.

Now erect at each point $\cdot 1^r$ of T an ordinate of length r. In each of the black intervals of T place a set of points similar to T, and at all these points in any segment ($\cdot 1^r$, $\cdot 1^{r+1}$) erect ordinates of length r. In the largest segment these will only be dots. This largest segment we now leave undisturbed, and in each of the segments between each consecutive pair of points already marked we insert a set similar to T, and erect ordinates of length r in any segment ($\cdot 1^{r+1}$, $\cdot 1^{r+2}$). In the segment (·1, ·11) these will only be dots; this segment is subsequently left undisturbed.

Thus we go on, leaving undisturbed segments in which we have already inserted dots, and placing in all the rest sets of the type T. At each stage we erect perpendiculars of length one less than those erected in the same segment at the preceding stage of proceedings.

We see that this process, carried on *ad infinitum*, gives us on the axis a closed countable set of points E. If we draw parallels to the axis at heights 1, 2, 3, ..., the first, at height 1, will be cut by the ordinates which we erected in the first derived set E_1, the second in E_2, and so on. Every successive parallel is cut by the ordinates more and more to the right, the points crowding up to the point $\cdot\dot{1}$ or 1, which alone is common to all the derived sets. Thus E_ω consists of this one point, $\cdot\dot{1}$ or 1, while E_n contains an infinite number of points.

The binary numbers of our set are evidently all those of the form $1^m 0 1^n 0 1^p \dots 0 1^z$, where $m, n, \dots, z$ are any integers (including the case when any of these integers are absent, and the corresponding power of 1 drops out), the number of zeroes being at most $(m+1)$.

8. Returning now to the discussion of § 6, we not unnaturally want to know whether the set E_ω is identical with E_Ω, and it is obvious from the above that this is not so. In the above example E_Ω (which is, of course, evanescent) is the first derived of E_ω, but it is easy to construct more complicated examples on the same principle as the above, taking as T a more complicated set, *e.g.*, the set constructed in the preceding article; in this case E_ω consist of the set T itself. If, therefore, we take the series of derived sets of E_ω, the deduced set will consist of the point 1 alone, and the first derived set of this will be E_Ω.

It is evident that here again we have not arrived at the most general case. For we might have taken as T the more complicated set of the last section, and we should then have arrived at a more complicated E_ω, which, however, by an extended process of derivation and deduction, could be reduced to E_Ω.

On this principle we can evidently set up whole classes of countable sets of points, in each of which E_Ω is obtained after a complicated series of repetitions of the operations of derivation and deduction from the derived sets.

Here, however, we are not necessarily at an end. For, denoting the point called P_ω, by Q_1 (Fig. 1), we get, after deriving E_ω and then deducing a new set, another point Q_2, which, by precisely the same reasoning as before, lies to the right of P^Ω, if it does not coincide with it. If we have not now come to an end, we obtain another point Q_3, and Q_4, and so on— a whole sequence of points which will have a limiting point if they do not all coincide after a time. Such a limiting point we may denote by Q_ω. It will evidently belong to all the sets which we have so far obtained by derivation and deduction; that is, to the set obtained by deduction from all these sets, which we may, if we prefer, regard as deduced from the deduced sets alone.

Starting afresh with this set, we may evidently derive and deduce exactly as before. Denoting by R_1 the limiting point just obtained and denoted by Q_ω, we may get a whole sequence of points R_1, R_2, ... leading to a limiting point R_ω, which will again lie to the right of P^Ω if it do not coincide with it, since by none of our processes of amalgamation have we blackened out a more than countable set of external points.

Furthermore, having realized to ourselves any particular series of the operations of derivation and deduction, we can easily, on the principles laid down in the examples, construct a set of points in which that series of derivations and deductions can actually be performed, and give a set which either is E_Ω, or which leads to E_Ω when we subject it itself to the operations of derivation and deduction.

9. Special as this mode of constructing examples may seem, it can now be proved quite easily, by the principles established in the paper on "Sets of Intervals," that this is quite a general theorem which may be enunciated as follows :—

The Nucleus E_Ω of a given closed set of points E can always be obtained by means of a finite or countably infinite series of the operations of derivation and deduction.

For let us consider a black interval (P^Ω, Q^Ω) of the ultimate set, and, starting with any interval δ_r of E inside it, let us form the successive intervals $\delta_r^{(1)}, \delta_r^{(2)}, \ldots$ of all the derived and deduced sets possible. The left-hand end-points $P_1, P_2, \ldots$ of these form a countable set of points with all sorts and varieties of limiting points as in our examples, and may need all the letters of the alphabet, and of many alphabets, and all the indices over and over again to characterize them in their natural order. But they have the two properties :—(i.) they lead always to the left, (ii.) they never pass P^Ω. There are, therefore, only two possibilities :—(*a*) we can assign a point P to the right of P^Ω such that there is no one of the points $P_1, P_2, \ldots$ to the left of P, while no such point can be assigned more to the right than P; or (*b*) P^Ω itself is the only point which can be so assigned. Whichever of these cases is true, we will first show that, as we amalgamate our intervals, we can assign a definite stage of proceedings at which the point P has actually been reached, and is no longer a mere limit to which we continually approach.

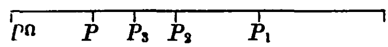

FIG 3.

To show this, let P_1 be the left-hand end-point of $\delta_r^{(1)}$. Then either P_1 coincides with P or lies to the right of it. In the former case we have proved our point. In the latter case we bisect (P, P_1) at M_1. Then we can assign one of the derived or deduced sets, which we may denote by $E^{(2)}$, whose corresponding black interval $\delta_r^{(2)}$ extends beyond M_1 to the left. Let P_2 be its left-hand end-point, which either coincides with P or lies between P and M_1. We now reason with respect to P_2 as we did with respect to P_1. In this way either we determine a finite number of the sets $E_1, E^{(2)}, \ldots, E^{(n)}, \ldots$ chosen from among the derived and deduced sets on these principles, and such that the black interval $\delta_r^{(n)}$ has its left-hand end-point at P, which proves our theorem, or else we obtain a countable set $E_1, E^{(2)}, \ldots, E^{(n)}, \ldots$ and a corresponding set of points $P_1, P_2, \ldots, P_n, \ldots$ having P as sole and only limiting point, and such that P_n is the left-hand end-point of $\delta_r^{(n)}$, and therefore the first point

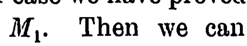

FIG. 4.

of $E^{(n)}$ at which we arrive in passing from P_1 towards P; so that P_n is certainly not a point of $E^{(n+1)}$. Let $E^{(\omega)}$ denote the deduced set of E_1, $E^{(2)}, \ldots, E^{(n)}, \ldots$. Then there is certainly no point of $E^{(\omega)}$ to the right of P; but we know that the black interval $\delta_r^{(\omega)}$ does not extend beyond P, hence P must be its left-hand end-point, so that we have determined a definite stage at which P is reached, as was asserted.

Now let us suppose that P lies to the right of P^{Ω}. Similarly we deter-

FIG. 5.

mine a point Q actually attained and never passed on the right of δ_r. Q may coincide with Q^{Ω}. Now, if from the black intervals of E lying in (P^{Ω}, P) we choose one and enlarge it by amalgamation as we did δ_r, we get a definite black interval (P_1, Q_1) actually attained, but never surpassed, during the processes of derivation and deduction.

Now P_1 may coincide with P^{Ω}, but Q_1 cannot coincide with P, because, if it did, one single derivation would amalgamate the abutting intervals (P_1, Q_1) and (P, Q), contrary to the hypothesis that P could not be passed starting from δ_r.

Continuing this process in the segments (P^{Ω}, P_1), (Q_1, P), (Q, Q^{Ω}), one at least of which must actually exist, viz., (Q_1, P), we ultimately get a set of non-abutting intervals, dense everywhere in (P^{Ω}, Q^{Ω}), since the intervals of E were dense everywhere in (P^{Ω}, Q^{Ω}). But by § 28, *loc. cit.*, we know that such a set of intervals has a more than countable set of external points in (P^{Ω}, Q^{Ω}). These external points are, however, certainly external points of E by our construction, and E has at most a countably infinite set of external points in (P^{Ω}, Q^{Ω}). (*Loc. cit.*, p. 267.)

Thus this is impossible; P must coincide with P^{Ω}, and Q with Q^{Ω}, and both P^{Ω} and Q^{Ω} are actually attained at a definite stage of proceedings.

We have still to show that we can assign a definite stage of proceedings at which *every* (P^{Ω}, Q^{Ω}) *has* been attained. If this were not possible, then, choosing any one of the sets obtained by derivation and deduction, and calling it E_1, we must be able to assign an interval of E_{Ω}, say $\delta_{1,\Omega}$, which has not not yet been actually attained. By what has been proved, however, we can assign a definite stage at which $\delta_{1,\Omega}$ has been attained. Let us denote by E_2 a definite set which has $\delta_{1,\Omega}$ as a black interval. E_2 is then certainly obtained from E_1 by derivation and deduction, since E_2 could not have occurred previously to E_1, by the hypothesis that $\delta_{1,\Omega}$ is not yet attained at the stage of E_1, and is attained at the stage of E_2.

So we go on, and form a simply infinite series of sets $E_1, E_2, \ldots, E_n$, from a corresponding series of the black intervals of E_Ω, viz., $\delta_{1,\Omega}, \delta_{2,\Omega}, \ldots, \delta_{n,\Omega}, \ldots$, where $\delta_{n,\Omega}$ does not belong to E_n, but does belong to E_{n+1}. Now it follows that, for all values of n, $\delta_{n,\Omega}$ will belong to the deduced set of $E_1, E_2 \ldots, E_n, \ldots$.

We might, then, have started with this set as $E_1^{(1)}$; since then every one of the countably infinite set just dealt with, $(\delta_{2,\Omega}, \delta_{3,\Omega}, \ldots)$, would already have been attained, we should have to seek a $\delta_{2,\Omega}^{(1)}$ among the remaining black intervals of E_Ω. At the end of another infinite set of processes we should get to another set, which we might have taken originally as $E_1^{(2)}$. Since the whole number of intervals of E_Ω is countable, we cannot go on putting aside countably infinite sets of intervals without exhausting them after at most a countably infinite series of steps. Thus the deduced set of $E_1^{(1)}, E_1^{(2)}, \ldots, E_1^{(n)}, \ldots$ must coincide with E_Ω.

The theorem is now proved, and may be restated as follows:—

Any closed set of points E can be reduced by a finite, or at most countably infinite, series of the operations of derivation and deduction to one of two forms :—(i.) *no points at all,* (ii.) *a perfect set. This set, using the word in a somewhat extended sense if* (i.) *be the case, we call the Nucleus and denote it by* E_Ω. E_Ω *has the same content* (*in the first case zero*) *as E, and, with the possible exception of a finite, or at most countably infinite, set of points, the same external points as E.* (i.) *occurs if, and only if, the set E be countable.*

The following is also an immediate consequence of our reasoning :—

Given any finite or countably infinite series of the operations of derivation and deduction, a countable set of points can be constructed on which this series can actually be performed, giving at each stage of proceedings a new set of points, which evanesces at the last of the processes.

10. *Cantor's Transfinite Numbers of the First Potency.*

We notice that the ideas of derivation and deduction, and of the possible series of these operations which lead from closed sets of points to their Nuclei, gives us a conception of sequence which, though perfectly definite, goes beyond what we can qualify by means of the ordinal numbers first, second, third, and so on. Even if we use a whole alphabet, or many alphabets, to distinguish the new sets which arise, we find the symbols at our disposal inadequate, and the idea of sequence which we possess becomes hazy in a mist of symbols to which our minds are not accustomed as expressing sequence. Beyond the ordinary series of ordinal numbers, first, second, ..., or, which comes to the same thing, only is more restricted, the sequence of letters of the alphabet, our mind is accustomed to the sequence expressed by the points of a straight line in their natural order, or, which is the same thing, the sequence of all the numbers rational and irrational: *e.g.*, all the points of the segment (0, 1) of the x-axis, or the sequence of all the numbers from 0 to 1 expressed in any scale of notation, for instance, the ternary scale.

The potency of these numbers is c, and the question is whether we can employ some or all of these numbers, in an *ordinal* sense, to qualify the sets of points obtained from every conceivable E by derivation and deduction. It is, of course, quite easy to do so in any special case. Given, in fact, any finite or countably infinite series of the operations of derivation and deduction in their natural order, we know that we can, as in the examples, set up a countable set of points in the segment (0, 1) on which that series of operations can actually be effected. We might then choose the corresponding binary fractions as indices of the corresponding E's.

For instance, to be more precise, taking as basis the example of § 7 (see Fig. 2), we could denote the derived sets by $E_{\cdot 01}$, $E_{\cdot 01^2}$, $E_{\cdot 01^3}$, ...; the set deduced from these by $E_{\cdot 0\dot{1}}$ or $E_{\cdot 1}$; the derived sets of E_1 by E_{1001}, E_{1001^2}, ..., $E_{\cdot 1001^n}$, ...; and the set deduced from these by $E_{\cdot 101}$.

Proceeding on this principle, generally the derived sets of E_{101^m} will be denoted by $E_{\cdot 101^m 01}$, $E_{\cdot 101^m 01^2}$, ..., $E_{\cdot 101^m 01^n}$, ...; and the set deduced from these by $E_{\cdot 101^{m+1}}$.

Further, the set deduced from all these, or, which comes to the same thing, from $E_{\cdot 101}$, $E_{\cdot 101^2}$, ..., $E_{\cdot 101^n}$, ... will be denoted by $E_{\cdot 1^2}$.

As we proceed further, the general form of the index is $\cdot 1^m 01^n 0 \ldots 01^z$, where $m, n, \ldots, z$ are any integers (including zero, and, if zero, the corresponding figure is to be altogether omitted) and the number of zeros which occur between the 1's is at most $m+1$. These numbers present to our minds the same idea of sequence as the operations themselves, and, if we should speak of the $\cdot 1^3 0100111$-th set, we should know at once what was meant, and where the set occurred in the sequence under discussion.

But, although in any special case, however complicated, we can assign such a set of binary fractions, and the notation is a very convenient one for such purposes, the method does not permit us to discuss simultaneously two or more special cases, since sets which we recognize in our minds as the same, *i.e.*, having the same place in their respective sequences counted from the beginning, will in the two cases have different indices, while, on the other hand, the same index will sometimes refer to sets which we recognize as distinct.

The question as to whether it is possible to assign a system of fixed numerical indices to the E's so as to characterize the order of the whole set of them in every conceivable case has not been answered. A preliminary step would be to determine whether $\aleph_1$, the potency of all the E's, is greater, equal to, or less than that of the continuum. If it were greater than the continuum, the above question would certainly have to be answered in the negative. It has been proved* that the potency is not greater than c; it is still an open question whether or not it is equal to c.

Before proceeding to the proof of this theorem, however, I propose to describe the notation introduced by G. Cantor, and which for the earlier sets of the series is so convenient and obvious that it has met with universal acceptation.

The derived sets of E are denoted by the indices $1, 2, \dots, n, \dots$, the deduced set of these by ω; the derived sets of E_ω by $\omega+1$, $\omega+2, \dots, \omega+n, \dots$, and the deduced set of these by $\omega+\omega$ or $\omega . 2$; and so on. The set deduced from $E_\omega, E_{\omega 2}, \dots, E_{\omega n}$ has the index ω^2, that deduced from $E_{\omega^2}, E_{\omega^3}, \dots, E_{\omega^n}$ the index ω^ω, that deduced from the E's with indices $\omega^\omega, \omega^{2\omega}, \dots, \omega^{n\omega}$ the index ω^{ω^2}, and that deduced from the E's with indices $\omega^\omega, \omega^{\omega^2}, \omega^{\omega^3}, \dots$ the index ω^{ω^ω}. For the set deduced from the sets with indices $\omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \dots$ the notation breaks down, but the principle can be carried on *ad infinitum*. This principle, or more properly these *two* principles, are—(1) to every number a there is a *next* number, which shall be denoted by $a+1$; (2) to every infinite set of numbers there is a *next* number, which shall be called a "limiting number" (*Limeszahl*).

Following out these principles, if we have obtained any Cantor number β, then the set which is got by performing on E_β that operation which led from E to E_a, a being any Cantor number, will be denoted by $\beta+a$. We notice that, if a precedes β, $E_{\beta+a}$ is different from E_β, and from any set preceding E_β; but, if a follows β, then $E_{\beta+a}$ is the same as E_a. Hence the Cantor numbers do not follow the commutative law: that is, $a+\beta \neq \beta+a$.

Similarly, $2.\omega \neq \omega.2$. Since $E_{2.\omega}$ means that we derive E_2 twice, giving $E_{2.2}$, then twice more, giving $E_{2.3}$, and so on; finally $E_{2.\omega}$ we obtain by deduction from this series, but this is none other than E_ω, so that $2\omega = \omega$. Generally $a\beta \neq \beta a$.

The ideal symbols defined by these two principles are called Cantor's

* F. Bernstein, Inaug. Diss.: "Untersuchungen aus des Mengenlehre," Göttingen, 1901.

"transfinite numbers of the first potency." Cantor uses the notation $\aleph_1$ or *Aleph-eins* for the potency of his numbers of the first potency, while $\aleph_0$ or *Aleph-null* denotes the potency of a countable set of points; this notation we shall keep.

11. *Order represented by Diagrams. Ordinal Types.*

F. Bernstein has pointed out that the idea of *sequence*, or, to use Cantor's expression, of "ordinal type," is nothing more than a discontinuous function of two variables, and may be graphically represented by means of the diagram of a rectangular trellis, so familiar in the theory of numbers.

If we take any countably infinite set, whether it be of points on the straight line, or of the operations of derivation and deduction, or anything else, and arrange them in countable order, say $E_1, E_2, \ldots, E_n, \ldots$ (this may generally be done in a variety of ways, but we choose out one particular arrangement), the idea of the *natural order* of these E's is completely embodied by giving a law by which we can say whether or no E_i came after E_j originally. E_i could, of course, only coincide with E_j in the one case if it did in the other, *i.e.*, if $i \equiv j$. Unless we can give such a law, we cannot speak of a "natural order" at all; *vice versa*, given such a law, we can determine the position of any element E_i with respect to any other one E_j, and so can always say whether or no it lies between any two assigned elements: this is what we mean by saying we know the natural order, or the ordinal type.

The diagrammatic representation of the natural order depends on the customary representation of the pair of integers (i, j) by means of the cross points of a rectangular trellis, so that the point (i, j) is the point whose coordinates are i and j. Each such point (i, j) we mark with a black spot if E_j comes before E_i (in symbols, if $E_j < E_i$), and with a small circle if $j \equiv i$. It is obvious that in this way we have a diagram such that the reflection of any black spot in the line of small circles is a trellis point which has not been spotted. We may therefore, if we prefer, use only the wedge-shaped diagram bounded by the small circles, as it gives us all the information we require.

We may transform this diagram into a numerical representation by interpreting the black spots by the figure 1, and the unspotted trellis-points by 0, and reading the columns in order. Prefixing a point and interpreting in the binary scale, we get a binary fraction corresponding to each diagram, and *vice versa*. A given Cantor number will in general be represented by a variety of diagrams; hence $\aleph_1 \leqslant c$. Fig. 6 is one of the diagrams representing the natural order of the derived and deduced sets up to E_{ω^2} inclusive, the countable order being obtained from the diagram

ω^2	ω	1	2	3	4	5	...
		2ω	$\omega+1$	$\omega+2$	$\omega+3$	$\omega+4$	...
			3ω	$2\omega+1$	$2\omega+2$	$2\omega+3$	...
				4ω	$3\omega+1$	$3\omega+2$	...
					5ω	$4\omega+1$	...
						6ω	...

the columns being read in order, thus

$$\omega^2,\ \omega,\ 1,\ 2\omega,\ 2,\ \omega+1,\ 3\omega,\ 3,\ \omega+2,\ 2\omega+1,\ 4\omega,\ 4,\ \omega+3,\ \ldots.$$

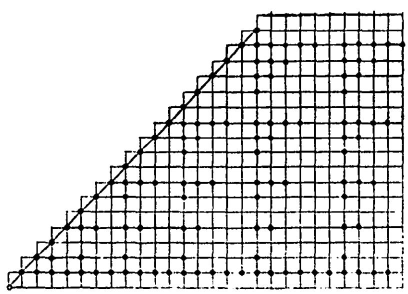

FIG. 6.

The number equivalent to the diagram is

$$\cdot 1\ 11\ 001\ 1011\ 01001\ 000001\ 1101011\ 01001001\ \ldots.$$

If we start with any generic binary fraction, or, which is the same thing, with any generic diagram, we shall not in general have a corresponding Cantor number. The only sets whose ordinal types are Cantor numbers are such as have a definite *first* element, and such that every element has a definite element immediately following it, as is seen by means of the sets of derived and deduced sets. Such sets are said to be "well ordered"; a set which corresponds to a generic binary fraction is said to be "simply ordered." For instance, the binary number $\cdot 1\ 11\ 111 \ldots$, or $\cdot\dot{1}$, or 1, represents the natural order of the negative integers $\ldots, -n, -n+1, \ldots, -2, -1$, in the countable order $-1, -2, \ldots, -n+1, -n, \ldots$, the corresponding diagram having dots at all the trellis points. This ordinal type is denoted by Cantor by ω^*. Thus the potency of the ordinal types of all simply ordered sets is less than or equal to c.

A simple proof of G. Cantor's enables us to go further and assert that this potency is c. Denoting by Π a set whose ordinal type is that of all the negative and positive integers, we take corresponding to any given binary fraction $.e_1 e_2 e_3 \ldots$, where the e's are either ones or zeros, the simply ordered set got by inserting a set Π between e_1 isolated points and e_2 isolated points, then another set Π, then e_3 isolated points and so on. It is easily proved that no set so obtained could be obtained in that order from any different ternary fraction. Hence c is less than or equal to the potency of the ordinal types of all the simply ordered sets. It follows then by a theorem of Cantor's proved by Bernstein and by Schröder, and later by Zermelo, *that c is the potency of the simply ordered sets.* This is the last word which has so far been said on this subject; it was to be hoped that a further investigation of the connexion between the set of simply ordered sets and the partial set of well ordered sets would lead to a determination of $\aleph_1$, but so far this is not the case.

12. *Contents of Closed Sets of Points.*

In amalgamating the black intervals of E to form those of E_1, E_2, ..., E_Ω, we blacken out at most a countably infinite set of limiting points; hence, by Theorem 3′ of the paper on "Sets of Intervals," all these sets of black intervals have the same content. Thus we see *the operations of derivation and deduction leave the content unaltered.*

In dealing with the content of closed sets of points, we need therefore only consider perfect sets, since E_Ω is always perfect, unless it evanesces, *i.e.*, unless the set E is countable. H. J. S. Smith's sets of points of the first kind give us examples of sets of points of potency c and content 0. They are not themselves perfect, but their first derived sets are perfect, *i.e.*, E_Ω is the first derived set. This set was set up in § 14, *loc. cit.*, and in the ternary case consists of all the binary fractions interpreted in the ternary scale; in the general case it consists of all the $(m-1)$-ary fractions interpreted in the m-ary scale. Cantor's perfect set of points with zero content (*loc. cit.*, § 23) consists of all the ternary fractions which do not involve the figure 1.

H. J. S. Smith's sets of the second kind have positive content, which always lies between $\frac{m-1}{m}$ and $\frac{m-2}{m-1}$, where m is the base of the scale adopted. The law of construction of the set E_Ω or E_1 for such a set was proved (*loc. cit.*, p. 260), and runs as follows in the case $m = 3$.

Let $.e_1 \mid e_2 e_3 \mid \ldots \mid e_{\frac{1}{2}[n(n-1)]+1} e_{\frac{1}{2}[n(n-1)]+2} \ldots e_{\frac{1}{2}[n(n+1)]} \mid$ denote any ternary fraction, and let A_n denote the conditions that for no positive integral value of $r \leqslant n$ all the figures from $e_{\frac{1}{2}[r(r-1)]+1}$ to $e_{\frac{1}{2}[r(r+1)]}$, both inclusive, should be 2's.

Then the set of numbers in question consists of all the ternary fractions which violate the conditions A_n for some value of n.

We have in this way got types of closed sets with contents ranging from 0 to l, the latter not inclusive. A closed set of content l is necessarily the whole continuum (A, B).

This follows at once from the definition, since a set of intervals of content zero is an impossibility, unless the whole set evanesces.

[*Note.*—A proof by Schoenflies of what I have called the "Theorem of the Nucleus," based upon the amalgamation of abutting intervals, which involves, however, the properties of well-ordered sets, but avoids the use of Cantor's numbers of the *third* class, appeared this year in the *Göttinger Nachrichten.* The idea of amalgamation as such, and of its application to the simplification of the proof in question, was, I believe, new when I communicated my paper on "Sets of Intervals" to this Society on November 13th, 1902. (See p. 246, lines 4–9, and p. 267 ; also §§ 14–16.) The present paper had, as there indicated, been written at the same time.]